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The Number of Finite Groups Whose Element Orders is Given

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Abstract. The spectrum $\omega(G)$ of a finite group G is the set of element orders of G. If Ω is a non-empty subset of the set of natural numbers, $h(\Omega)$ stands for the number of isomorphism classes of finite groups G with $\omega(G) = \Omega$ and put $h(G) = h(\omega(G))$. We say that G is recognizable (by spectrum $\omega(G)$) if $h(G) = 1$. The group G is almost recognizable (resp. nonrecognizable) if $1 < h(G) < \infty$ (resp. $h(G) = \infty$). In the present paper, we focus our attention on the projective general linear groups $\mathrm{PGL}(2, p^n)$, where $p = 2^\alpha 3^\beta + 1$ is a prime, $\alpha \geq 0, \beta \geq 0$ and $n \geq 1$, and we show that these groups cannot be almost recognizable, in other words $h(\mathrm{PGL}(2, p^n)) \in \{1, \infty\}$. It is also shown that the projective general linear groups $\mathrm{PGL}(2, 7)$ and $\mathrm{PGL}(2, 9)$ are nonrecognizable. In this paper a computer program has also been presented in order to find out the primitive prime divisors of $a^n - 1$.

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1. Introduction

Throughout the paper, all the groups under consideration are finite and simple groups are non-Abelian. For a group G, we denote the set of orders of all elements in G by $\omega(G)$ which has been recently called the *spectrum* of G. Obviously $\omega(G)$ is a subset of the set $\mathbb{N}$ of natural numbers, and it is closed and partially ordered by divisibility, hence, it is uniquely determined by $\mu(G)$, the subset of its maximal elements.

One of the most interesting concepts in finite group theory which has recently attracted several researchers is the problem of characterizing finite groups by element orders. Let Ω be a non-empty subset of $\mathbb{N}$. Now, we can put forward the following questions:

Is there any group G with $\omega(G) = \Omega$? If the answer is affirmative then how many non-isomorphic groups exist with the above set of element orders?

Certainly, if there exists such a group, Ω must contain 1 and furthermore Ω must be closed and partially ordered under the divisibility relation. These conditions are necessary but not sufficient, for example if $\Omega = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, then there does not exist any group G with $\omega(G) = \Omega$. In fact, R. Brandl and W. J. Shi in [1] have classified all groups whose element orders are consecutive integers and in that paper they have shown that if $\omega(G) = \{1, 2, 3, \ldots, n\}$, for some group G, then $n \leq 8$.

For a set Ω of natural numbers, define $h(\Omega)$ to be the number of isomorphism classes of groups G such that $\omega(G) = \Omega$, and put $h(G) = h(\omega(G))$. Evidently, $h(G) \geq 1$. Now we give a "new classification" for groups using the h function. A group G is called *recognizable* (resp. *almost recognizable* or *nonrecognizable*) if $h(G) = 1$ (resp. $1 < h(G) < \infty$ or $h(G) = \infty$). Some list of simple groups that are presently known to be recognizable, almost recognizable or nonrecognizable is given in [13]. In particular, it was previously known that the projective general linear groups $\mathrm{PGL}(2, 2^n)$ with $n \geq 2$ are recognizable and $\mathrm{PGL}(2, 2) \cong S_3$ is nonrecognizable (see [16], Theorem 2). In [12], V. D. Mazurov proved the following result: Let P be a field which is the union of an ascending series of finite fields of orders $2^{m_i}, m_i > 1, i \in \mathbb{N}$. If there exists a natural number s such that 2^s does not divide m_i for any $i \in \mathbb{N}$ then $h(\mathrm{PGL}(2, P)) = 1$. In all other cases $h(\mathrm{PGL}(2, P)) = \infty$. Also he proved the following result in [11]: If p, r are odd primes, $p-1$ is divisible by r but not by r^2, and s is a natural number non-divisible by r, then $h(\mathrm{PGL}(r, p^s)) = \infty$.

Let $q = p^n$ where p is a prime. In this paper, we focus our attention on the projective general linear groups $\mathrm{PGL}(2, q)$. The structure of $\mathrm{Aut}(L_2(q))$ is well known, it is isomorphic to the semidirect product of $\mathrm{PGL}(2, q)$ by a cyclic group of order n. On the other hand we know $\mu(L_2(q)) = \{\frac{q-1}{\epsilon}, p, \frac{q+1}{\epsilon}\}$, $\epsilon = (2, q-1)$, and $\mu(\mathrm{PGL}(2, q)) = \{q-1, p, q+1\}$.

A group G is called *C_{pp}-group* if p is a prime divisor of $|G|$ and the centralizer of any non-trivial p-element in G is a p-group. Evidently, the projective general linear groups $\mathrm{PGL}(2, q)$ where $q = p^n$, are C_{pp}-groups. In [3], the second author has classified the simple C_{pp}-groups, where p is prime and $p = 2^\alpha 3^\beta + 1$, $\alpha \geq 0$, $\beta \geq 0$ (see Lemma 8 and Table 1). Using these results, we prove the following theorem.

Theorem 1. *Let $p = 2^\alpha 3^\beta + 1$ $(\alpha \geq 0, \beta \geq 0)$ be a prime. Then the projective general linear groups $PGL(2, p^n)$ cannot be almost recognizable. In other words,*

$h(PGL(2,p^n)) \in \{1,\infty\}$.

In 1994, R. Brandl and W. J. Shi in [2] showed that all projective special linear groups $L_2(q)$ with $q \neq 9$ are recognizable and $L_2(9)$ is nonrecognizable.

Here, we similarly prove that:

Theorem 2. *The projective general linear group $PGL(2,9)$ is nonrecognizable.*

For us it was interesting to face with some groups G such that $\mu(G)$ contain three consecutive natural numbers in the form $\{p-1, p, p+1\}$ where $p \geq 5$ is a prime. Such sets appear for almost simple groups $\mathrm{PGL}(2,p)$, where $p \geq 5$ is a prime, in fact we proved in [14] that $h(\mathrm{PGL}(2,p)) \in \{1,\infty\}$. For ∞ we have found an example. It has been proved in [1] that $\mathrm{PGL}(2,5) \cong S_5$ has ∞ for its h function. Here we also give another example of groups of type $\mathrm{PGL}(2,p)$ with value ∞ for its h function.

Theorem 3. *There exists an extension G of a 7-group by $2.S_4$ such that $\mu(G) = \mu(PGL(2,7)) = \{6,7,8\}$. In particular, the projective general linear group PGL $(2,7)$ is nonrecognizable.*

Notation. Our notation and terminology are standard (see [4]). Given a group G, denote by $\pi(G)$ the set of all prime divisors of the order of G. If m and n are natural numbers and p is a prime, then we let $\pi(n)$ be the set of all primes dividing n, and $r_{[n]}$ the largest prime not exceeding n. Note that $\pi(G) = \pi(|G|)$. The notation $p^m \parallel n$ means that $p^m | n$ and $p^{m+1} \nmid n$. The expression $G = K : C$ denotes the split extension of a normal subgroup K of G by a complement C.

2. Some preliminary results

First, we collect some results from Elementary Number Theory which will be useful tools for our further investigations in this paper. We start with a famous theorem due to Zsigmondy.

Zsigmondy's Theorem (see [19]). *Let a and n be integers greater than 1. Then there exists a primitive prime divisor of $a^n - 1$, that is a prime s dividing $a^n - 1$ and not dividing $a^i - 1$ for $1 \leq i \leq n-1$, except if*

(1) *$a = 2$ and $n = 6$, or*

(2) *a is a Merssene prime and $n = 2$.*

We denote by a_n one of these primitive prime divisors of $a^n - 1$. Evidently, if a_n is a primitive prime divisor of $a^n - 1$, then a has order n modulo a_n and so $a_n \equiv 1 \pmod{n}$. Thus $a_n \geq n+1$.

The next elementary result will be needed later.

Lemma 1. *Let p and q be two primes and m be a natural number, where p, q and m satisfying one of the following conditions. Then, for every $n \geq m$, there exists a primitive prime divisor $p_n > q$.*

(1) $p = 7$, $m = 5$ *and* $q = 13$,
(2) $p = 13$, $m = 5$ *and* $q = 19$,
(3) $p = 17$, $m = 4$ *and* $q = 19$,
(4) $p = 19$, $m = 7$ *and* $q = 37$,
(5) $p = 37$, $m = 7$ *and* $q = 109$,
(6) $p = 73$, $m = 5$ *and* $q = 127$.

Proof. In all cases, if $n \leq q$, the result is straightforward. Therefore, we may assume that $n > q$. Since

$$\pi(q!) \subseteq \pi(p\prod_{i=1}^{q-1}(p^i - 1)) \subset \pi(p\prod_{i=1}^{n}(p^i - 1)),$$

by Zsigmondy's theorem we deduce that there exists a primitive prime divisor $p_n > q$, completing the proof. □

Function for finding the primitive prime divisors. In the following we submit a GAP program [5], which determines all the primitive prime divisors in the sequence $a^i - 1$ ($i = 1, 2, \ldots, n$) for some a and n.

```
gap> PPD:=function(a,n)
    local b,i,j,s1,s2,s;
     for i in [1..n] do
     s1:=Set(Factors(a^i-1));
    s2:=[];
        for j in [1..(i-1)] do
            b:=Set(Factors(a^j-1));
            Append(s2,b);
        od;
        s:=Difference(s1,s2);
        Print(i," ",s,"\n");
    od;
end;
```

Using this programme we list all primitive prime divisors p_n for $p = 7$, 13, 17 and $2 \leq n \leq 19$, in Table 1. Using Table 1, the reader can easily check the proof of Lemma 1 (1)–(3) for $n \leq q$.

Lemma 2. *Let p and q be two primes and m, n be natural numbers such that $p^m = q^n + 1$. Then one of the following holds:*

(1) *$n = 1$, m is a prime number, $p = 2$ and $q = 2^m - 1$ is a Mersenne prime;*

(2) *$m = 1$, n is a power of 2, $q = 2$ and $p = 2^n + 1$ is a Fermat prime;*

(3) *$p = n = 3$ and $q = m = 2$.*

Proof. Well known exercise using the Zsigmondy's theorem. □

The set $\omega(G)$ defines the prime graph $\mathrm{GK}(G)$ of G whose vertex-set is $\pi(G)$ and two primes p and q in $\pi(G)$ are adjacent (we write $p \sim q$) if and only if $pq \in \omega(G)$.

n	7_n	13_n	17_n
2	–	7	3
3	19	61	307
4	5	5, 17	5, 29
5	2801	30941	88741
6	43	157	7, 13
7	29, 4733	5229043	25646167
8	1201	14281	41761
9	37, 1063	1609669	19, 1270657
10	11, 191	11, 2411	11, 71, 101
11	1123, 293459	23, 419, 859, 18041	2141993519227
12	13, 181	28393	83233
13	16148168401	53, 264031 1803647	212057, 2919196853
14	113, 911	29, 22079	22796593
15	31, 159871	4651, 161971	6566760001
16	17, 169553	407865361	18913, 184417
17	14009 2767631689	103, 443, 15798461357509	10949, 1749233, 2699538733
18	117307	19, 271, 937	1423, 5653
19	419 4534166740403	12865927, 9468940004449	229, 1103, 202607147, 291973723

Table 1. The primitive prime divisors p_n where $p \in \{7, 13, 17\}$ and $2 \leq n \leq 19$.

The number of connected components of GK(G) is denoted by $t(G)$, and the connected components are denoted by $\pi_i = \pi_i(G)$, $i = 1, 2, \ldots, t(G)$. If $2 \in \pi(G)$ we always assume $2 \in \pi_1$. Denote by $\mu_i(G)$ the set of all $n \in \mu(G)$ such that $\pi(n) \subseteq \pi_i$.

The Gruenberg-Kegel Theorem (see [18]). *If G is a group with disconnected graph $GK(G)$ then one of the following holds:*

(1) *$t(G) = 2$, G is Frobenius or 2-Frobenius.*

(2) *G is an extension of a $\pi_1(G)$-group N by a group G_1, where $S \leq G_1 \leq Aut(S)$, S is a simple group and G_1/S is a $\pi_1(G)$-group. Moreover $t(S) \geq t(G)$ and for every i, $2 \leq i \leq t(G)$, there exists j, $2 \leq j \leq t(S)$ such that $\mu_j(S) = \mu_i(G)$.*

Lemma 3. *Let S be a simple group with disconnected prime graph $GK(S)$. Then $|\mu_i(S)| = 1$ for $2 \leq i \leq t(S)$. Let $n_i(S)$ be a unique element of $\mu_i(S)$ for $i \geq 2$. Then value for S, $\pi_1(S)$ and $n_i(S)$ for $2 \leq i \leq t(S)$ are the same as in Tables* 2a–2c *of [13].*

Proof. The simple groups S and the sets of $\pi_i(S)$ are described in [18] and [7]; the rest is proved in Lemma 4 of [8]. The values of the numbers $n_i(S)$, $i \geq 2$ are listed in Table 2a–2c of [13]. □

We also use the following lemma (see [11], Lemma 1).

Lemma 4. *If a group G contains a soluble minimal normal subgroup then G is nonrecognizable. In particular, if G is a soluble group then G is nonrecognizable.*

The following result of V. D. Mazurov will be used several times.

Lemma 5. (see [10]) *Let G be a group, N a normal subgroup of G, and G/N a Frobenius group with Frobenius kernel F and cyclic complement C. If $(|F|, |N|) = 1$ and F is not contained in $NC_G(N)/N$, then $p|C| \in \omega(G)$ for some prime divisor p of $|N|$.*

The following lemma is taken from [16], Theorem 2.

Lemma 6. *Let G be a group such that*

$$\mu(G) = \mu(PGL(2, 2^n)) = \{2^n - 1, 2, 2^n + 1\}.$$

Then, the following statements hold:

(1) *If $n \geq 2$, then $G \cong PGL(2, 2^n)$.*
(2) *If $n = 1$, then $G \cong S_3$ has ∞ for its h function.*

We are now ready to prove the following lemma.

Lemma 7. *Let G be a group such that*

$$\mu(G) = \mu(PGL(2, p^n)) = \{p^n - 1, p, p^n + 1\},$$

where p is an odd prime, $n \geq 2$. Then, the following statements hold:

(1) *If $(p, n) \neq (3, 2)$, then item (2) of the Gruenberg-Kegel theorem holds. Moreover, S is isomorphic to none of the following simple groups:*
   (a) *alternating groups on $n \geq 5$ letters,*
   (b) *sporadic simple groups,*
   (c) *$L_2(p^k)$ where $k \neq n$, or*
   (d) *$L_2(2p^m \pm 1)$, $m \geq 1$, where $2p^m \pm 1$ is a prime.*
(2) *If $(p, n) = (3, 2)$, then there exists a soluble group G such that $\mu(G) = \mu(PGL(2, 3^2))$.*

Proof. (1) First of all, we show that G is insoluble. Assume the contrary. If $\pi(p^n-1)=\{2\}$, then by Lemma 2 we obtain $(p,n)=(3,2)$ which is a contradiction. Hence, there exists a prime $2\neq r\in\pi(p^n-1)$. On the other hand, we consider the primitive prime divisor $s=p_{2n}$. Now assume that H is a $\{p,r,s\}$-Hall subgroup of G. Since G has no elements of order pr, ps and rs, it follows that H is a soluble group all of whose elements are of prime power orders. By [6], Theorem 1, we must have $|\pi(H)|\leq 2$, which is a contradiction.

Since $t(G)=2$, G satisfies the conditions of the Gruenberg-Kegel theorem. Now we show that G is neither Frobenius nor 2-Frobenius. Evidently, G can not be a 2-Frobenius group, because G is insoluble. Suppose $G=KC$ is a Frobenius group with kernel K and complement C. Clearly C is insoluble, $\pi(C)=\pi_1(G)=\pi(p^{2n}-1)$, $\pi(K)=\pi_2(G)=\{p\}$ and by [15], Theorem 18.6 C has a normal subgroup C_0 of index ≤ 2 such that $C_0\cong SL(2,5)\times Z$, where every Sylow subgroup of Z is cyclic and $\pi(Z)\cap\pi(30)=\emptyset$. Therefore GK$(C)$ can be obtained from the complete graph on $\pi(C)$ by deleting the edge $\{3,5\}$. On the other hand, if there exist primes $2\neq r\in\pi(p^n-1)$ and $2\neq s\in\pi(p^n+1)$, then since $rs\notin\omega(G)$ it follows that $rs\notin\omega(C)$. Hence, we must have $Z=1$ and $\pi(p^{2n}-1)=\pi(SL_2(5))=\{2,3,5\}$ and since $\{2,3,5\}\subset\pi(p^4-1)$, by Zsigmondy's theorem we obtain that $n=2$. Now, it is easy to see that $\pi(p^2-1)=\{2,3\}$ and $\pi(p^2+1)=\{2,5\}$. From $\pi(p^2-1)=\{2,3\}$, we infer that p is a Mersenne prime or a Fermat prime. In the first case we obtain $p=7$, and in the latter case $p=17$. If $p=17$, then $29\in\pi(p^2+1)$, a contradiction. If $p=7$, then C contains an element of order 16, which is a contradiction.

Therefore, by the Gruenberg-Kegel theorem, G is an extension of a $\pi_1(G)$-group N by a group G_1, where $S\leq G_1\leq \mathrm{Aut}(S)$, S is a simple group and G_1/S is a $\pi_1(G)$-group. Now, we show that S is not isomorphic to an alternating group, a sporadic simple group, a linear group $L_2(p^k)$ where $k\neq n$ or $L_2(2p^m\pm1)$, $m\geq 1$, where $2p^m\pm 1$ is a prime.

Before beginning we recall that in the prime graph of G the connected component $\pi_1(G)$ consists of the primes in $\pi(p^n-1)$ which form a complete subsection and also the primes in $\pi(p^n+1)$ which forms another complete subsection. Moreover, every odd vertex in $\pi(p^n-1)$ is not joined to any odd vertex in $\pi(p^n+1)$.

(a) Assume that $S\cong A_m$, $m\geq 5$. By Lemma 3, $m=p,p+1,p+2$. Suppose $S\cong A_p$, $p\geq 5$. We have that in the prime graph GK(A_p) the vertex 3 is joined to $2,5,7,\ldots,r_{[p-3]}$. If 3 divides $p-1$, then by the remark mentioned in the previous section, we conclude that $2,3,5,\ldots,r_{[p-3]}$ belong to $\pi(p^n-1)$. Now, if there exists a prime $s\in\pi(p^n+1)\backslash\pi(A_p)$ then $s\in\pi(N)$, because $A_p\cong S\leq G/N\leq \mathrm{Aut}(S)\cong S_p$. On the other hand, $A_4=2^2:3\leq A_p$ and by Lemma 5 it follows that $s\sim 3$ which is a contradiction. Hence, $\pi(p^n+1)\subseteq\pi(A_p)$. As $(p^n-1,p^n+1)=2$ and $2,3,5,7,\ldots,r_{[p-3]}\in\pi(p^n-1)$, the only possible cases are: $\pi(p^n+1)=\{2\}$ or $\pi(p^n+1)=\{2,p-2\}$ in which in the latter case $p-2$ is a prime. Evidently, the first case will never occur. So, we consider the case $\pi(p^n+1)=\{2,p-2\}$, i.e., $p^n+1=2^l(p-2)^k$. Now, if $k>1$ then since $(p-2)^k\in\omega(G)$ and $(p-2)^k\notin\omega(\mathrm{Aut}(S))=\omega(S_p)$ we obtain $(p-2)\in\pi(N)$ and again since A_p contains a Frobenius subgroup of shape $2^2:3$ by Lemma 5, we get $p-2 \sim 3$ which is a contradiction. Finally, we have $k=1$ and $l>1$. Moreover $2 \parallel p^n-1$ which implies that n must be odd. But in this case we have $p^n+1=(p+1)(p^{n-1}-p^{n-2}+\cdots-p+1)=2^l(p-2)$ for which it follows that $p^{n-1}-p^{n-2}+\cdots-p+1=p-2$, giving no solution for $p \geq 5$. This final contradiction shows that $S \ncong A_p$. The case when 3 divides $p+1$, is similar. The other cases are settled similarly.

(b) Suppose S is isomorphic to one of the sporadic simple groups, for instance $S \cong J_2$. Since $p \in \pi_2(G)$, by Lemma 3 it follows that $p=7$. If $n \geq 5$, then we choose the primitive prime divisors $7_n, 7_{2n}$ in $\pi(G)$. Evidently, $7_{2n} \in \pi(p^n+1)$, and so G does not contain an element of order $7_n.7_{2n}$. On the other hand since $\pi(\mathrm{Aut}(S))=\{2,3,5,7\} \subset \pi(7\prod_{i=1}^{4}(7^i-1))$, it follows that $7_n, 7_{2n} \notin \pi(\mathrm{Aut}(S))$. Therefore $7_n, 7_{2n} \in \pi(N)$, and since N is nilpotent we obtain that $7_n.7_{2n} \in \omega(N)$, which is a contradiction. Thus $n \leq 4$. If $n=4$, then $\mu(G)=\{2^5.3.5^2, 7, 2.1201\}$. Because, there does not exist any element of order 1201 in $\mathrm{Aut}(S)$, 1201 divides the order of N. Without loss of generality we may assume that $N \neq 1$ is an elementary Abelian 1201-group. Now since S contains the Frobenius group $A_4=2^2:3$, from Lemma 5 we infer that G contains an element of order 1201.3, which is a contradiction. If $n=2$ or 3, then $5 \in \pi(S) \backslash \pi(G)$, which is impossible.

The other sporadic simple groups are examined similarly.

(c) Assume that $S \cong L_2(p^k)$, where $k \neq n$. In this case we must have $k<n$, since otherwise by Zsigmondy's Theorem we get $p_{2k} \in \pi(S) \backslash \pi(G)$, which is a contradiction. Now, we choose the primitive prime divisors p_n and p_{2n} in $\pi(G)$. Since $p_{2n} \in \pi(p^n+1)$, G does not contain an element of order $p_n.p_{2n}$. On the other hand, since $p_n>n>k$ we have $p_n, p_{2n} \notin \pi(\mathrm{Aut}(S))=\pi(\mathrm{PGL}(2,p^k) \rtimes Z_k)$, and so $p_n, p_{2n} \in \pi(N)$. Now, since N is nilpotent we obtain that $p_n.p_{2n} \in \omega(N) \subset \omega(G)$, which is a contradiction.

(d) Suppose that $S \cong L_2(2p^m \pm 1)$, $m \geq 1$, where $2p^m \pm 1$ is a prime. By the structure of $\mu(G)$, we see that $p \in \omega(G)$ and $p^2 \notin \omega(G)$. So, if $S \cong L_2(2p^m \pm 1)$, where $2p^m \pm 1$ is a prime and $m \geq 1$, then we deduce $m=1$, because in this case $p^m \in \omega(L_2(2p^m \pm 1))=\omega(S) \subseteq \omega(G)$. On the other hand, we know $|\mathrm{Aut}(S)|=2^2p(p \pm 1)(2p \pm 1)$, where $2p \pm 1$ is a prime, and so $\pi(\mathrm{Aut}(S))=\{p, 2p \pm 1\} \cup \pi(p \pm 1)$. If $n=2$, then $(2p \pm 1, |G|)=1$, which is a contradiction. Therefore $n \geq 3$. Now, we consider the primitive prime divisors p_n and p_{2n}. Since $(p_n, p_{2n})=(p_n, p \pm 1)=(p_{2n}, p \pm 1)=1$, it follows that $p_n \notin \pi(\mathrm{Aut}(S))$ or $p_{2n} \notin \pi(\mathrm{Aut}(S))$, thus we may assume N is a p_n-subgroup or a p_{2n}-subgroup. First, we assume that $2p+1$ is a prime. Let P be a Sylow $(2p+1)$-subgroup of S, then $N_S(P)$, the normalizer of P in S, is a Frobenius group of order $(2p+1)p$, with cyclic complement of order p. Now, by Lemma 5, we deduce that $p_n \sim p$ or $p_{2n} \sim p$, which is a contradiction. Next, we assume that $2p-1$ is a prime. In this case, if there exists a prime $s \in \pi(p^n+1) \backslash \pi(\mathrm{Aut}(S))$ then $s \in \pi(N)$, because $G/N \leq \mathrm{Aut}(S)$. Moreover, if Q is a Sylow $(2p-1)$-subgroup of S, then $N_S(Q)$ is a Frobenius group of order $(2p-1)(p-1)$, with cyclic complement of order $p-1$. Now, as previous case we get $s.(p-1) \in \omega(G)$, which is a contradiction. Hence, $\pi(p^n+1) \subseteq \pi(\mathrm{Aut}(S))$. As $(p^n-1, p^n+1)=2$, the only possible case is $\pi(p^n+1)=\{2, 2p-1\}$, i.e., $p^n+1=2^l(2p-1)^k$ for some l and k in $\mathbb{N}$.

Now, if $k > 1$ then since $(2p-1)^k \in \omega(G)$ and $(2p-1)^k \notin \omega(\mathrm{Aut}(S))$ we obtain $(2p-1) \in \pi(N)$. On the other hand, it is easy to see that $p_n \in \pi(G)\backslash\pi(\mathrm{Aut}(S))$, and so $p_n \in \pi(N)$. Since N is nilpotent, we deduce that $p_n \sim (2p-1)$, which is a contradiction. Finally, we have $k = 1$ and since $(p, n) \neq (3, 2)$, we obtain that $l > 1$. Moreover $2 \parallel p^n - 1$ which implies that n must be odd. But in this case we have $p^n + 1 = (p+1)(p^{n-1} - p^{n-2} + \cdots - p + 1) = 2^l(2p-1)$ for which it follows that $p^{n-1} - p^{n-2} + \cdots - p + 1 = 2p - 1$, giving no solution for $p \geq 3$. This final contradiction shows that $S \ncong L_2(2p^m \pm 1)$.

(2) Consider the group $H = \langle a, b | a^8 = b^5 = 1, ba = ab^2\rangle \cong Z_5 : Z_8$. For this group we have $\mu(H) = \{8, 10\}$. Now, we assume that G is an extension of elementary Abelian 3-group K of order 3^{40l} by H, and the generators a, b of H act on K cyclically. Then G is a soluble group and $\omega(G) = \omega(\mathrm{PGL}(2, 3^2)) = \{1, 2, 3, 4, 5, 8, 10\}$. □

The following lemma gives a classification of simple C_{pp}-groups, where p is a prime of form $p = 2^\alpha 3^\beta + 1$, $\alpha \geq 0, \beta \geq 0$.

Lemma 8. (see [3]) *Let p be a prime and $p = 2^\alpha 3^\beta + 1$, $\alpha \geq 0, \beta \geq 0$. Then any simple C_{pp}-group is given by Table 2.*

The next lemma gives the maximal odd factors set $\psi(F_4(q))$ of $\mu(F_4(q))$, $q = 2^e$.

Lemma 9. *Let $S \cong F_4(q)$, where $q = 2^e$, $e \geq 1$. Then $\psi(S) = \{q^4-1, q^4+1, q^4-q^2+1, (q-1)(q^3+1), (q+1)(q^3-1)\}$.*

Proof. The $2'$-elements of S is contained in the maximal tori of S. From [17] we see that $\mu(F_4(q))$ contains 25 maximal tori $H(1)$, $H(2)$, …, $H(25)$. Since $(q-1, q^3+1) = 1, (q+1, q^3-1) = 1$, $H(13)$ and $H(15)$ are all cyclic. The conclusion holds. □

3. Main results

In this section we prove the statement of Theorems 1, 2 and 3.

Proof of Theorem 1. Let G be a group and

$$\mu(G) = \mu(\mathrm{PGL}(2, p^n)) = \{p^n - 1, p, p^n + 1\},$$

where $p = 2^\alpha 3^\beta + 1$ is a prime, and n is a natural numbers. If $\alpha = \beta = 0$, then $p = 2$ and the result is correct by Lemma 6. Also for $n = 1$, the result holds by [14], and so from now on we assume that p is an odd prime and $n \geq 2$. Then $t(G) = 2$, in fact we have

$$\pi_1(G) = \pi(p^{2n} - 1) \quad \text{and} \quad \pi_2(G) = \{p\}.$$

Lemma 6(1) shows that G is an extension of a $\pi_1(G)$-group N by a group G_1, where $S \leq G_1 \leq \mathrm{Aut}(S)$, S is a simple group of Lie type (except $L_2(p^k)$, $k \neq n$ and $L_2(2p^m \pm 1)$ where $m \geq 1$ and $2p^m \pm 1$ is a prime) and G_1/S is a $\pi_1(G)$-group. Moreover, there exists $2 \leq j \leq t(S)$ such that $\mu_j(S) = \{p\}$, in fact S is a simple C_{pp}-group. Using the results summarized in Table 2, we will show that S is isomorphic to $L_2(p^n)$.

Step 1. *$S \cong L_2(q)$, $q = p^n$, $n \geq 2$.*

In the following case by case analysis we assume that $S \ncong L_2(p^n)$ and try to obtain a contradiction. Moreover, as S is always a C_{pp}-group for some appropriate prime p, we make use of the results summarized in Table 2 and Lemma 7 and omit the details of the argument.

Case 1. *$q = 3^n$, $n \geq 2$.*

In this case S can only be isomorphic to one of the following simple groups: $L_2(2^3)$, $L_3(2^2)$. Since G does not contain an element of order 9, S can not be isomorphic to $L_2(2^3)$. If $S \cong L_3(2^2)$, then since $7 \in \pi(S)$ we obtain that $n \geq 6$. Assume first that $n = 6$. In this case we have $\pi(G) = \{2, 3, 5, 7, 13, 73\}$. Evidently $13, 73 \notin \pi(\mathrm{Aut}(S))$ and $13 \nsim 73$. Hence $\{13, 73\} \subseteq \pi(N)$, and since N is nilpotent we get $13.73 \in \omega(N)$, which is a contradiction. Next we suppose that $n \geq 7$.

Now we choose the primitive prime divisors 3_n and 3_{2n} in $\pi(G)$. Evidently $3_{2n} \in \pi(3^n + 1)$, and so $3_n \nsim 3_{2n}$. Moreover, since $\{2, 3, 5, 7, 11, 13\} = \pi(3\prod_{i=1}^{6}(3^i - 1))$, $3_n, 3_{2n} \notin \pi(\mathrm{Aut}(S))$, and hence $3_n, 3_{2n} \in \pi(N)$. Again since N is nilpotent, N contains an element of order $3_n.3_{2n}$, which is of course impossible.

Case 2. *$q = 5^n$, $n \geq 2$.*

In this case we see that S can only be isomorphic to one of the following simple groups: $L_2(7^2)$, $L_3(2^2)$, $S_4(3)$, $S_4(7)$, $U_4(3)$, $Sz(2^3)$ or $Sz(2^5)$. Since G has no element of order 25, S can not be isomorphic to $L_2(7^2)$ or $Sz(2^5)$. If S is isomorphic to one of the simple groups: $L_3(2^2)$, $S_4(7)$, $U_4(3)$, or $Sz(2^3)$, then $7 \in \pi(S)$ and so we must have $n \geq 6$. Also note that

$$\pi(S) \subset \{2, 3, 5, 7, 11, 13\} \subset \pi(5\prod_{i=1}^{6}(5^i - 1)).$$

If $n = 6$, then $31,601 \in \pi(G)\backslash\pi(\mathrm{Aut}(S))$ and thus $31,601 \in \pi(N)$. Therefore N contains an element of order 31.601, which is a contradiction as $31.601 \notin \omega(G)$. For case $n \geq 7$, since by Zsigmondy's Theorem $5_n, 5_{2n} > 13$, a similar argument with the primitive prime divisors $5_n, 5_{2n} \in \pi(G)$ also leads to a contradiction. Similarly, S can not be isomorphic to $S_4(3)$.

Case 3. *$q = 7^n$, $n \geq 2$.*

p	simple C_{pp}-groups
2	A_5, A_6, $L_2(q)$ where q is a Fermat prime, a Mersenne prime or $q=2^m$, $m\geq 3$, $L_3(2^2)$, $Sz(2^{2m+1})$, $m\geq 1$.
3	A_5, A_6, $L_2(q)$, $q=2^3$, 3^m or $2.3^m\pm 1$, which is a prime, $m\geq 1$, $L_3(2^2)$
5	A_5, A_6, A_7, M_{11}, M_{22}, $L_2(q)$, $q=7^2$, 5^m or $2.5^m\pm 1$, which is a prime, $m\geq 1$, $L_3(2^2)$, $S_4(q)$, $q=3,7$, $U_4(3)$, $Sz(q)$, $q=2^3,2^5$.
7	A_7, A_8, A_9, M_{22}, J_1, J_2, HS, $L_2(q)$, $q=2^3$, 7^m or 2.7^m-1, which is a prime, $m\geq 1$, $L_3(2^2)$, $S_6(2)$, $O_8^+(2)$, $G_2(q)$, $q=3,19$, $U_3(q)$, $q=3,5,19$, $U_4(3)$, $U_6(2)$, $Sz(2^3)$.
13	A_{13}, A_{14}, A_{15}, Suz, Fi_{22}, $L_2(q)$, $q=3^3$, 5^2, 13^m or 2.13^m-1, which is a prime, $m\geq 1$, $L_3(3)$, $L_4(3)$, $O_7(3)$, $S_4(5)$, $S_6(3)$, $O_8^+(3)$, $G_2(q)$, $q=2^2,3$, $F_4(2)$, $U_3(q)$, $q=2^2,23$, $Sz(2^3)$, ${}^3D_4(2)$, ${}^2E_6(2)$, ${}^2F_4(2)'$.
17	A_{17}, A_{18}, A_{19}, J_3, He, Fi_{23}, Fi'_{24}, $L_2(q)$, $q=2^4,17^m$ or $2.17^m\pm 1$, which is a prime $,m\geq 1$, $S_4(4)$, $S_8(2)$, $F_4(2)$, $O_8^-(2)$, $O_{10}^-(2)$, ${}^2E_6(2)$.
19	A_{19}, A_{20}, A_{21}, J_1, J_3, $O'N$, Th, HN, $L_2(q)$, $q=19^m$ or 2.19^m-1, which is a prime, $m\geq 1$, $L_3(7)$, $U_3(2^3)$, $R(3^3)$, ${}^2E_6(2)$.
37	A_{37}, A_{38}, A_{39}, J_4, Ly, $L_2(q)$, $q=37^m$ or 2.37^m-1, which is a prime, $m\geq 1$, $U_3(11)$, $R(3^3)$, ${}^2F_4(2^3)$.
73	A_{73}, A_{74}, A_{75}, $L_2(q)$, $q=73^m$ or 2.73^m-1, which is a prime, $m\geq 1$, $L_3(2^3)$, $S_6(2^3)$, $G_2(q)$, $q=2^3$, 3^2, $F_4(3)$, $E_6(2)$, $E_7(2)$, $U_3(3^2)$, ${}^3D_4(3)$.
109	A_{109}, A_{110}, A_{111}, $L_2(q)$, $q=109^m$ or 2.109^m-1, which is a prime, $m\geq 1$, ${}^2F_4(2^3)$.
$p=2^m+1$, $m=2^s$	A_p, A_{p+1}, A_{p+2}, $L_2(q)$, $q=2^m,p^k$, $2p^k\pm 1$, which is a prime, $k\geq 1$, $S_a(2^b)$, $a=2^{c+1}$ and $b=2^d$, $c\geq 1$, $c+d=s$, $F_4(2^e)$, $e\geq 1$, $4e=2^s$, $O^-_{2(m+1)}(2)$, $s\geq 2$, $O_a^-(2^b)$, $a=2^{c+1}$ and $b=2^d$, $c\geq 2$, $c+d=s$.
Other	A_p, A_{p+1}, A_{p+2}, $L_2(q)$, $q=p^m$ or $2p^m-1$, which is a prime, $m\geq 1$.

Table 2. Simple C_{pp}-groups, $p=2^\alpha 3^\beta+1$, $\alpha\geq 0,\beta\geq 0$.

In this case the possibilities for S are: $L_2(2^3)$, $L_3(2^2)$, $S_6(2)$, $O_8^+(2)$, $G_2(3)$, $G_2(19)$, $U_3(3)$, $U_3(5)$, $U_3(19)$, $U_4(3)$, $U_6(2)$ or $Sz(2^3)$. First of all, since G has no element of order 49, $S \not\cong G_2(19)$ or $U_3(19)$. Next, we note that $\pi(S) \subset \pi(13!)$ and by Lemma 1 we see that for every $n \geq 5$ there exists a primitive prime divisor $7_n \geq 13$. Therefore for $n \geq 5$, as previous cases a similar argument with the primitive prime divisors 7_n and 7_{2n}, leads to a contradiction.

If $n = 4$, then $\mu(G) = \{2^5.3.5^2, 7, 2.1201\}$. In this case S can only be $L_2(2^3)$, $L_3(2^2)$, $S_6(2)$, $O_8^+(2)$, $U_3(3)$, $U_3(5)$, or $U_4(3)$ by checking their prime divisors sets. On the other hand, since the simple groups $L_2(2^3)$, $S_6(2)$, $O_8^+(2)$ and $U_4(3)$ contain an element of order 9 and $9 \notin \omega(G)$, S can only be $L_3(2^2)$, $U_3(3)$ or $U_3(5)$. Moreover, since there does not exist any element of order 1201 in $\mathrm{Aut}(S)$, 1201 divides the order of N. Without loss of generality we may assume that $N \neq 1$ is an elementary Abelian 1201-group. Because $A_4 = 2^2 : 3 < A_6 < L_3(2^2)$, $7 : 3 < L_2(7) < U_3(3)$ and $A_4 = 2^2 : 3 < A_7 < U_3(5)$, in all cases S contains a Frobenius group of shape $2^2 : 3$ or $7 : 3$, and so G contains an element of order 1201.3 by Lemma 5, which is a contradiction.

If $n = 3$, then $\mu(G) = \{2.3^2.19, 7, 2^3.43\}$. In this case, S can only be $L_2(2^3)$ by checking their element orders sets. As $43 \notin \pi(\mathrm{Aut}(S))$ we have $43 \in \pi(N)$. Now, we may assume that $N \neq 1$ is an elementary Abelian 43-group. Since $2^3 : 7 < L_2(2^3)$ we get $43.7 \in \omega(G)$ by Lemma 5, which is a contradiction.

If $n = 2$, then $\mu(G) = \{2^4.3, 7, 2.5^2\}$. In this case, by checking element orders S can only be $L_3(2^2)$, $U_3(3)$ or $U_3(5)$. If $S \cong L_3(2^2)$ or $U_3(5)$, then 5 divides the order of N since $25 \notin \omega(\mathrm{Aut}(S))$. Without loss of generality we may assume that $N \neq 1$ is an elementary Abelian 5-group. Since S contains a Frobenius subgroup of shape $2^2 : 3$ (in fact we have $A_4 = 2^2 : 3 \leq A_6 \leq L_3(4)$ and $A_4 = 2^2 : 3 \leq A_7 \leq U_3(5)$), we get $5.3 \in \omega(G)$ by Lemma 5, a contradiction. If $S \cong U_3(3)$, then $5 \in \pi(N)$, because $5 \notin \pi(\mathrm{Aut}(S))$. Again, since $7 : 3 \leq L_2(7) \leq U_3(3)$ we get $5.3 \in \omega(G)$ by Lemma 5, a contradiction.

Case 4. $q = 13^n$, $n \geq 2$.

In this case S can only be isomorphic to one of the following simple groups: $L_2(3^3)$, $L_2(5^2)$, $L_3(3)$, $L_4(3)$, $O_7(3)$, $S_4(5)$, $S_6(3)$, $O_8^+(3)$, $G_2(2^2)$, $G_2(3)$, $F_4(2)$, $U_3(2^2)$, $U_3(23)$, $Sz(2^3)$, ${}^3D_4(2)$, ${}^2E_6(2)$ or ${}^2F_4(2)'$. Since $13^2 \notin \omega(G)$, and $U_3(23)$ contains an element of order 13^2, $S \not\cong U_3(23)$. Moreover, we have $\pi(S) \subseteq \pi(19!)$. Now since, by Lemma 1, for every $n \geq 5$, there exists a primitive prime divisor $13_n > 19$. we can consider the primitive prime divisors 13_n and 13_{2n}, and we get a contradiction as before cases. Henceforth, we may assume that $n \leq 4$.

If $n = 4$, then $\mu(G) = \{2^4.3.5.7.17, 13, 2.14281\}$. In this case, by comparing element orders, we conclude that S can only be $L_2(3^3)$, $L_2(5^2)$, $L_3(3)$, $L_4(3)$, $O_7(3)$, $S_4(5)$, $O_8^+(3)$, $G_2(2^2)$, $G_2(3)$, $F_4(2)$, $U_3(2^2)$, $Sz(2^3)$, ${}^3D_4(2)$, or ${}^2F_4(2)'$. In all above cases, except $S \cong F_4(2)$, since $17, 14281 \notin \pi(\mathrm{Aut}(S))$, we have $17, 14281 \in \pi(N)$, and so $17.14281 \in \omega(N)$, which is a contradiction. If $S \cong F_4(2)$, then 14281 divides the order of N, and since S contains a Frobenius group $2^2 : 3$ (note that $2^2 : 3 = A_4 < S_{10} < S_8(2) < F_4(2)$), G must contain an element of order 14281.3, by Lemma 5, which is not possible. If $n = 3$, then $\mu(G) = \{2^2.3^2.61, 13, 2.7.157\}$.

In this case we have $61, 157 \notin \pi(\mathrm{Aut}(S))$ and so $61, 157 \in \pi(N)$, hence we get $61.157 \in \omega(N) \subset \omega(G)$, which is impossible.

Case 5. $q = 17^n$, $n \geq 2$.

In this case S can only be isomorphic to one of the following simple groups: $L_2(2^4)$, $S_4(4)$, $S_8(2)$, $F_4(2)$, $O_8^-(2)$, $O_{10}^-(2)$ or ${}^2E_6(2)$. First of all, since $5 \in \pi(S)$, we deduce $n \geq 4$. Moreover, we have $\pi(S) \subseteq \pi(19!)$. From Lemma 1, for every $n \geq 4$, there exists a primitive prime divisor $17_n > 19$. Now, for the primitive prime divisors 17_n and 17_{2n}, a similar argument as before leads to a contradiction.

Case 6. $q = 19^n$, $n \geq 2$.

In this case S can only be isomorphic to one of the following simple groups: $L_3(7)$, $U_3(2^3)$, $R(3^3)$ or ${}^2E_6(2)$. Evidently $\pi(S) \subseteq \pi(37!)$. Since $7 \in \pi(S)$, $3|n$. If $n > 7$, then by Lemma 1 there exists a primitive prime divisor $19_n > 37$. Now we consider the primes 19_n and 19_{2n}, and we get a contradiction as previous cases. If $n = 6$, then we have

$$\mu(G) = \{2^3.3^3.5.7.127, 19, 2.13^2.181.769\}.$$

In this case we consider the primes $127, 769 \in \pi(G)$, and we obtain a contradiction as before. If $n = 3$, then $\mu(G) = \{2.3^3.127, 19, 2^2.5.7^3\}$. In this case S can be only $L_3(7)$ or $U_3(2^3)$, and since $5, 127 \notin \pi(\mathrm{Aut}(S))$, we get a contradiction.

Case 7. $q = 37^n$, $n \geq 2$.

In this case S can only be isomorphic to one of the following simple groups: $U_3(11)$, $R(3^3)$ or ${}^2F_4(2^3)$. Evidently, $\pi(S) \subseteq \{2, 3, 5, 7, 11, 13, 19, 37, 73, 109\}$. If $n \geq 7$, then by Lemma 1 there exists a primitive prime divisors $37_n > 109$, and hence we consider the primes $37_n, 37_{2n} \in \pi(G)$, and we get a contradiction as before. Therefore we may assume that $n \leq 6$. Since

$$\begin{aligned}
\pi(G) &= \{2, 3, 5, 7, 13, 19, 31, 37, 43, 67, 137, 144061\}, & n &= 6,\\
\pi(G) &= \{2, 3, 11, 19, 37, 41, 4271, 1824841\}, & n &= 5,\\
\pi(G) &= \{2, 3, 5, 19, 37, 89, 137, 10529\}, & n &= 4,\\
\pi(G) &= \{2, 3, 7, 19, 31, 37, 43, 67\}, & n &= 3,\\
\pi(G) &= \{2, 3, 5, 19, 37, 137\}, & n &= 2,
\end{aligned}$$

it is easy to see that $109 \notin \pi(G)$, and so $S \ncong {}^2F_4(2^3)$. Moreover, since $55 \in \omega(U_3(11))\backslash\omega(G))$, $S \ncong U_3(11)$. Finally, if $S \cong R(3^3)$, since $13 \in \pi(R(3^3))$, we must have $n = 6$. Yet, in this case, we can choose the primes $67, 144061 \in \pi(G)\backslash\pi(\mathrm{Aut}(S))$, and we get a contradiction as before (note that $67.144061 \notin \omega(G)$).

Case 8. $q = 73^n$, $n \geq 2$.

In this case S can only be $L_2(73^n)$, $L_3(2^3)$, $S_6(2^3)$, $G_2(2^3)$, $G_2(3^2)$, $F_4(3)$, $E_6(2)$, $E_7(2)$, $U_3(3^2)$ or ${}^3D_4(3)$. We assume that $S \ncong L_2(73^n)$. It is not difficult to see that $\pi(S) \subseteq \pi(19!) \cup \{31, 41, 43, 73, 127\}$. Let $n \geq 5$. Then by Lemma 1(6), $73_n, 73_{2n} > 127$. Evidently $73_n.73_{2n} \notin \omega(G)$, as $73_{2n} \in \pi(73^n + 1)$. On the other hand, since $73_n, 73_{2n} \notin \pi(\mathrm{Aut}(S))$, $73_n, 73_{2n} \in \pi(N)$ which implies that $73_n, 73_{2n} \in \omega(N) \subseteq \omega(G)$, a contradiction. Hence $n \leq 4$. Because

$\omega(G) = \{2^5.3^2.5.13.37.41, 73, 2.14199121\}$, $n = 4$,
$\omega(G) = \{2^3.3^3.1801, 73, 2.7.37.751\}$, $n = 3$,
$\omega(G) = \{2^4.3^2.37, 73, 2.5.13.41\}$, $n = 2$,

by checking the sets of element orders for each simple group, the only possibility for S is $U_3(3^2)$, when $n = 4$. In this case, we consider the primes 41 and 14199121 in $\pi(G)$. Since $41 \in \pi(73^4 - 1)$ and $14199121 \in \pi(73^4 + 1)$, $41 \nsim 14199121$ and also $41, 14199121 \notin \pi(\mathrm{Aut}(S))$, which implies that $41, 14199121 \in \pi(N)$. Now by the nilpotency of N, we obtain that $41.14199121 \in \omega(N) \subset \omega(G)$, which is a contradiction.

Case 9. *$q = 109^n$, $n \geq 2$.*

The proof of this case follows immediately from Lemmas 7(1) and 8.

Case 10. *$q = (2^m + 1)^n$, where $2^m + 1$ is a prime and $n \geq 2$.*

In this case S can only be isomorphic to: $L_2(2^m)$, $S_a(2^b)$, $a = 2^{c+1}$, $c \geq 1$, and $b = 2^d$, $c + d = s$, $F_4(2^e)$, $e \geq 1$, $4e = 2^s$, $O^-_{2(m+1)}(2)$, $s > 1$, or $O^-_a(2^b)$, $a = 2^{c+1}$, $c \geq 2$, and $b = 2^d$, $c + d = s$.

If $S \cong L_2(2^m)$, then $\mu(\mathrm{Aut}(S)) = \{m, 2^m - 1, 2^m + 1\} = \{m, p - 2, p\}$. First, assume that n is odd. In this case we have $(p - 2, p^n - 1) = 1$, in fact if $(p - 2, p^n - 1) = d$ then d divides $2^n - 1$, and so $d \mid (p - 2, 2^n - 1) = (2^m - 1, 2^n - 1) = 2^{(m,n)} - 1 = 1$. Now since $\pi(S) \subseteq \pi(G)$, it follows that $\pi(p - 2) \subset \pi(p^n + 1)$. Moreover, it is easy to see that $2^{m-1} + 1$ divides $p^n + 1$ and $(p - 2, 2^{m-1} + 1) = 3$. Now we consider the primitive prime divisors

$$r := p_n \in \pi(p^n - 1) \quad \text{and} \quad s := 2_{2(m-1)} \in \pi(2^{m-1} + 1).$$

Evidently $r, s \notin \pi(\mathrm{Aut}(S))$, and so $r, s \in \pi(N)$. From the nilpotency of N it follows that $r \sim s$, which is a contradiction. Next, we suppose that n is even. In this case we have $2^{m-1} + 1$ divides $p^n - 1$ and $(2^{m-1} + 1, p - 2) = 1$. Now, if $\pi(p - 2) \subset \pi(p^n - 1)$ then $(p - 2, p^n + 1) = 1$ and again we consider the following primitive prime divisors

$$r := p_{2n} \in \pi(p^n + 1) \quad \text{and} \quad s := 2_{m-1} \in \pi(2^{m-1} - 1),$$

and we get $r \sim s$, as before. But this is a contradiction. Therefore we must have $\pi(p - 2) \subseteq \pi(p^n + 1)$. Let $r \in \pi(2^{m-1} + 1) \subseteq \pi(p^n - 1)$. Clearly $r \notin \pi(\mathrm{Aut}(S))$, hence $r \in \pi(N)$. Now since $2^m : 2^m - 1 \leq L_2(2^m)$, by Lemma 5 we deduce that $r(2^m - 1) \in \omega(G)$, which is a contradiction.

If $S \cong F_4(2^e)$, then the maximal odd factors set $\psi(\mathrm{Aut}(S))$ of $\mu(\mathrm{Aut}(S))$ is equal to the same set of $\mu(S)$ since $|\mathrm{Aut}(S)| = 2^{e+1}$. From Lemma 9 we have

$$\psi(\mathrm{Aut}(S)) = \{q'^4 - 1, q'^4 + 1, q'^4 - q'^2 + 1, (q' - 1)(q'^3 + 1), (q' + 1)(q'^3 - 1)\},$$

where $q' = 2^e, e \geq 1$.

In this case $q'^4+1=p, q'^4-1=p-2$. Since G is an extension of a $\pi_1(G)$-group N by a group G_1, where $S \leq G_1 \leq \mathrm{Aut}(S)$, and $\mu(G)=\{p^n-1, p, p^n+1\}$, we may get a contradiction dividing the two cases. If $n \geq 4$, then the odd number $\frac{1}{2}(p^n+1)$ and the odd factor of p^n-1 are all greater than any number in $\psi(\mathrm{Aut}(S))$. Hence we have r, s such that

$$r \in \pi(p^n+1) \quad \text{and} \quad s \in \pi(p^n-1),$$

and $r, s \notin \pi(\mathrm{Aut}(S))$, so $r, s \in \pi(N)$. From the nilpotency of N it follows that $r \sim s$, which is a contradiction. If $n=2$, then we may infer that $(p-2, p^2+1)=5$ and $(p-2, p^2-1)=3$. It is impossible. Also we may get a similar contradiction if $n=3$.

If $S \cong S_a(2^b)$, then the maximal odd factors set $\psi(\mathrm{Aut}(S))$ of $\mu(\mathrm{Aut}(S))$ is equal to the same set of $\mu(S)$ since $|\mathrm{Aut}(S)|=b=2^d$. From [7], §3(3) we have

$$\{q'^{\frac{1}{2}(a)}-1, q'^{\frac{1}{2}(a)}+1\} \subseteq \psi(\mathrm{Aut}(S)),$$

where $q'=2^b, b \geq 1$. In this case $q'^{\frac{1}{2}(a)}+1=p$, and $q'^{\frac{1}{2}(a)}-1=p-2$, since the other numbers are not primes in $\psi(\mathrm{Aut}(S))$. The rest of proof is similar to the case of $S \cong F_4(2^e)$ by comparing the two sets of $\psi(\mathrm{Aut}(S))$ and $\mu(G)$.

If $S \cong O^-_{2(m+1)}(2), m=2^s, s>1$, then the maximal odd factors set $\psi(Aut(S))$ of $\mu(\mathrm{Aut}(S))$ is equal to the same set of $\mu(S)$ since $|\mathrm{Aut}(S)|=2$. From [7], §3(5) we have

$$\{q'^{m+1}+1, q'^m+1, q'^m-1\} \subseteq \psi(\mathrm{Aut}(S)),$$

where $q'=2$. In this case $q'^m+1=p$, and $q'^m-1=p-2$. The rest of proof is similar to the above cases.

If $S \cong O^-_a(2^b)$, $a=2^{c+1}$, $c \geq 2$, and $b=2^d$, $c+d=s$, the proof is similar.

Case 11. *$q=97^n$ or $q=p^n$, where $p=2^\alpha 3^\beta+1>109$ is a prime, $\beta \neq 0$ and $n \geq 2$.*

In this case S is a simple C_{pp}-group, and from Table 1 and Lemma 7, we obtain that $S \cong L_2(q)$.

Step 2. *N is a 2-group.*

Let P/N be a Sylow p-subgroup of S and X/N be the normalizer in S of P/N. Then X/N is a Frobenius group of order $q(q-1)/2$, with cyclic complement of order $(q-1)/2$. Now, by Lemma 5, we deduce that N is a 2-group.

Step 3. $h(G) \in \{1, \infty\}$.

First suppose that $N=1$. In this case, we have $S=L_2(q)$, $q=p^n$, $S \leq G \leq \mathrm{Aut}(S)$. Denote the factor group G/S by M. Obviously, $M \leq \mathrm{Aut}(S)$. Therefore, every element of M is a product of a field automorphism f, whose order is a divisor of n, and diagonal automorphism d of order dividing 2. Let $f \neq 1$ and r be a prime dividing the order of f. Without loss of generality, we may assume that $o(f)=r$. Evidently, r divides n, and we put $\bar{q}=p^{n/r}$. Denote by φ an automorphism of the field $\mathbb{F}_q$ inducing f. Since φ fixes a subfield $\mathbb{F}_{\bar{q}}$ of $\mathbb{F}_q$, f centralizes a subgroup $\overline{S}$ of S isomorphic to $L_2(\bar{q})$. But then G can not be a C_{pp}-group, which is a contradiction. Thus $f = 1$. Hence, we have $M \leq \langle d \rangle$ and so $|G/S| \leq 2$. Therefore $G \cong S$ or $G \cong \mathrm{PGL}(2, q)$. From $q+1 \in \omega(\mathrm{PGL}(2, q)) \backslash \omega(S)$, we have $G \cong \mathrm{PGL}(2, q)$. Thus, in this case $h(G) = 1$. Next, suppose that $N \neq 1$. Now, by Lemma 4, we get $h(G) = \infty$. The proof of Theorem 1 is complete. $\square$

Proof of Theorem 2. The proof follows immediately from Lemma 7(2) and Lemma 4. $\square$

Proof of Theorem 3. Let H be an extension of a group of order 2 by S_4 such that a Sylow 2-subgroup of H is a quaternion group. Then $\mu(H) = \{6, 8\}$. By Lemma 8 in [9], there exists an extension G of an elementary Abelian 7-group by H, which is a Frobenius group. It follows that $\mu(G) = \{6, 7, 8\}$, and then Theorem 1 follows from Lemma 4. $\square$

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