Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 47 (2006), No. 2, 435-446.

On the Geometry of Symplectic Involutions

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Abstract. Let V be a $2n$-dimensional vector space over a field F and Ω be a non-degenerate symplectic form on V. Denote by $\mathfrak{H}_k(\Omega)$ the set of all $2k$-dimensional subspaces $U \subset V$ such that the restriction $\Omega|_U$ is non-degenerate. Our main result (Theorem 1) says that if $n \neq 2k$ and $\max(k, n-k) \geq 5$ then any bijective transformation of $\mathfrak{H}_k(\Omega)$ preserving the class of base subsets is induced by a semi-symplectic automorphism of V. For the case when $n \neq 2k$ this fails, but we have a weak version of this result (Theorem 2). If the characteristic of F is not equal to 2 then there is a one-to-one correspondence between elements of $\mathfrak{H}_k(\Omega)$ and symplectic $(2k, 2n-2k)$-involutions and Theorem 1 can be formulated as follows: for the case when $n \neq 2k$ and $\max(k, n-k) \geq 5$ any commutativity preserving bijective transformation of the set of symplectic $(2k, 2n-2k)$-involutions can be extended to an automorphism of the symplectic group.

MSC 2000: 51N30, 51A50
Keywords: hyperbolic symplectic geometry, symplectic group, Grassmannian

1. Introduction

Let W be an n-dimensional vector space over a division ring R and $n \geq 3$. We put $\mathcal{G}_k(W)$ for the Grassmannian of k-dimensional subspaces of W. The projective space associated with W will be denoted by $\mathcal{P}(W)$.

Let us consider the set $\mathfrak{G}_k(W)$ of all pairs

$$(S,U) \in \mathcal{G}_k(W) \times \mathcal{G}_{n-k}(W),$$

where $S+U=W$. If B is a base for $\mathcal{P}(W)$ then the *base subset* of $\mathfrak{G}_k(W)$ associated with the base B consists of all (S,U) such that S and U are spanned by elements of B. If $n \neq 2k$ then any bijective transformation of $\mathfrak{G}_k(W)$ preserving the class of base subsets is induced by a semi-linear isomorphism of W to itself or to the dual space W^* (for $n=2k$ this fails, but some weak version of this result holds true). Using Mackey's ideas [7] J. Dieudonné [2] and C. E. Rickart [9] have proved this statement for $k=1,n-1$. For the case when $1<k<n-1$ it was established by author [8]. Note that adjacency preserving transformations of $\mathfrak{G}_k(W)$ were studied in [6].

Now suppose that the characteristic of R is not equal to 2 and consider an involution $u \in \mathrm{GL}(W)$. There exist two subspaces $S_+(u)$ and $S_-(u)$ such that

$$u(x)=x \ \text{ if } \ x \in S_+(u)\,, \quad u(x)=-x \ \text{ if } \ x \in S_-(u)$$

and

$$W = S_+(u) + S_-(u).$$

We say that u is a $(k,n-k)$-*involution* if the dimensions of $S_+(u)$ and $S_-(u)$ are equal to k and $n-k$, respectively. The set of $(k,n-k)$-involutions will be denoted by $\mathfrak{I}_k(W)$. There is the natural one-to-one correspondence between elements of $\mathfrak{I}_k(W)$ and $\mathfrak{G}_k(W)$ such that each base subset of $\mathfrak{G}_k(W)$ corresponds to a maximal set of mutually permutable $(k,n-k)$-involutions. Thus any commutativity preserving transformation of $\mathfrak{I}_k(W)$ can be considered as a transformation of $\mathfrak{G}_k(W)$ preserving the class of base subsets, and our statement shows that if $n \neq 2k$ then any commutativity preserving bijective transformation of $\mathfrak{I}_k(W)$ can be extended to an automorphism of $\mathrm{GL}(W)$.

In the present paper we give symplectic analogues of these results.

2. Results

2.1.

Let V be a $2n$-dimensional vector space over a field F and $\Omega : V \times V \to F$ be a non-degenerate symplectic form. The form Ω defines on the set of subspaces of V the orthogonal relation which will be denoted by $\perp$. For any subspace $S \subset V$ we put $S^\perp$ for the orthogonal complement to S. A subspace $S \subset V$ is said to be *non-degenerate* if the restriction $\Omega|_S$ is non-degenerate; for this case S is even-dimensional and $S + S^\perp = V$. We put $\mathfrak{H}_k(\Omega)$ for the set of non-degenerate $2k$-dimensional subspaces. Any element of $\mathfrak{H}_k(\Omega)$ can be presented as the sum of k mutually orthogonal elements of $\mathfrak{H}_1(\Omega)$.

Let us consider the projective space $\mathcal{P}(V)$ associated with V. The points of this space are 1-dimensional subspaces of V, and each line consists of all 1-dimensional subspaces contained in a certain 2-dimensional subspace.

A line of $\mathcal{P}(V)$ is called *hyperbolic* if the corresponding 2-dimensional subspace belongs to $\mathfrak{H}_1(\Omega)$; otherwise, the line is said to be *isotropic*.

Points of $\mathcal{P}(V)$ together with the family of isotropic lines form the well-known *polar space*.

Some results related with the hyperbolic symplectic geometry (spanned by points of $\mathcal{P}(V)$ and hyperbolic lines) can be found in [1], [4], [5].

A base $B = \{P_1, \dots, P_{2n}\}$ of $\mathcal{P}(V)$ is called *symplectic* if for any $i \in \{1, \dots, 2n\}$ there is unique $\sigma(i) \in \{1, \dots, 2n\}$ such that $P_i \not\perp P_{\sigma(i)}$. Then the set $\mathfrak{S}_1$ consisting of all

$$S_i := P_i + P_{\sigma(i)}$$

is said to be the *base subset* of $\mathfrak{H}_1(\Omega)$ associated with the base B. For any $k \in \{2, \dots, n-1\}$ the set $\mathfrak{S}_k$ consisting of all $S_{i_1} + \dots + S_{i_k}$ ($S_{i_1}, \dots, S_{i_k}$ are different) will be called the *base subset* of $\mathfrak{H}_k(\Omega)$ associated with $\mathfrak{S}_1$ (or defined by $\mathfrak{S}_1$).

Now suppose that the characteristic of F is not equal to 2. An involution $u \in \mathrm{GL}(V)$ is symplectic (belongs to the group $\mathrm{Sp}(\Omega)$) if and only if $S_+(u)$ and $S_-(u)$ are non-degenerate and $S_-(u) = (S_+(u))^\perp$. We denote by $\mathfrak{I}_k(\Omega)$ the set of symplectic $(2k, 2n-2k)$-involutions. There is the natural bijection

$$i_k : \mathfrak{I}_k(\Omega) \to \mathfrak{H}_k(\Omega), \qquad u \to S_+(u).$$

We say that $\mathfrak{X} \subset \mathfrak{I}_k(\Omega)$ is an *MC-subset* if any two elements of $\mathfrak{X}$ are commutative and for any $u \in \mathfrak{I}_k(\Omega) \setminus \mathfrak{X}$ there exists $s \in \mathfrak{X}$ such that $su \neq us$ (in other words, $\mathfrak{X}$ is a maximal set of mutually permutable elements of $\mathfrak{I}_k(\Omega)$).

Fact 1. [2], [3] *$\mathfrak{X}$ is a MC-subset of $\mathfrak{I}_k(\Omega)$ if and only if $i_k(\mathfrak{X})$ is a base subset of $\mathfrak{H}_k(\Omega)$. For any two commutative elements of $\mathfrak{I}_k(\Omega)$ there is a MC-subset containing them.*

Fact 1 shows that a bijective transformation f of $\mathfrak{H}_k(\Omega)$ preserves the class of base subsets if and only if $i_k^{-1} f i_k$ is commutativity preserving.

2.2.

If l is an element of $\Gamma\mathrm{Sp}(\Omega)$ (the group of semi-linear automorphisms which preserved Ω to within a non-zero scalar and an automorphism of F) then for each number $k \in \{1, \dots, n-1\}$ we have the bijective transformation

$$(l)_k : \mathfrak{H}_k(\Omega) \to \mathfrak{H}_k(\Omega), \qquad U \to l(U)$$

which preserves the class of base subsets. The bijection

$$p_k : \mathfrak{H}_k(\Omega) \to \mathfrak{H}_{n-k}(\Omega), \qquad U \to U^\perp$$

sends base subsets to base subsets. We will need the following trivial fact.

Fact 2. *Let f be a bijective transformation of $\mathfrak{H}_k(\Omega)$ preserving the class of base subsets. Then the same holds for the transformation $p_k f p_{n-k}$. Moreover, if $f = (l)_k$ for certain $l \in \Gamma\mathrm{Sp}(\Omega)$ then $p_k f p_{n-k} = (l)_{n-k}$.*

Two distinct elements of $\mathfrak{H}_1(\Omega)$ are orthogonal if and only if there exists a base subset containing them, thus for any bijective transformation f of $\mathfrak{H}_1(\Omega)$ the following condition are equivalent:

- f preserves the relation $\perp$,
- f preserves the class of base subsets.

It is not difficult to prove (see [2], p. 26–27 or [9], p. 711–712) that if one of these conditions holds then f is induced by an element of $\Gamma\mathrm{Sp}(\Omega)$. Fact 2 guarantees that the same is fulfilled for bijective transformations of $\mathfrak{H}_{n-1}(\Omega)$ preserving the class of base subsets. This result was exploited by J. Dieudonné [2] and C. E. Rickart [9] to determine automorphisms of the group $\mathrm{Sp}(\Omega)$.

Theorem 1. *If $n \neq 2k$ and $\max(k, n-k) \geq 5$ then any bijective transformation of $\mathfrak{H}_k(\Omega)$ preserving the class of base subsets is induced by an element of $\Gamma\mathrm{Sp}(\Omega)$.*

Corollary 1. *Suppose that the characteristic of F is not equal to 2. If $n \neq 2k$ and $\max(k, n-k) \geq 5$ then any commutativity preserving bijective transformation f of $\mathfrak{I}_k(\Omega)$ can be extended to an automorphism of $\mathrm{Sp}(\Omega)$.*

Proof of Corollary. By Fact 1, $i_k f i_k^{-1}$ preserves the class of base subsets. Theorem 1 implies that $i_k f i_k^{-1}$ is induced by $l \in \Gamma\mathrm{Sp}(\Omega)$. The automorphism $u \to lul^{-1}$ is as required. □

2.3.

For the case when $n = 2k$ Theorem 1 fails.

Example 1. Suppose that $n = 2k$ and $\mathfrak{X}$ is a subset of $\mathfrak{H}_k(\Omega)$ such that for any $U \in \mathfrak{X}$ we have $U^\perp \in \mathfrak{X}$. Consider the transformation of $\mathfrak{H}_k(\Omega)$ which sends each $U \in \mathfrak{X}$ to $U^\perp$ and leaves fixed all other elements. This transformation preserves the class of base subsets (any base subset of $\mathfrak{H}_k(\Omega)$ contains U together with $U^\perp$), but it is not induced by a semilinear automorphism if $\mathfrak{X} \neq \emptyset, \mathfrak{H}_k(\Omega)$.

If $n = 2k$ then we denote by $\overline{\mathfrak{H}}_k(\Omega)$ the set of all subsets $\{U, U^\perp\} \subset \mathfrak{H}_k(\Omega)$. Then every $l \in \Gamma\mathrm{Sp}(\Omega)$ induces the bijection

$$(l)'_k : \overline{\mathfrak{H}}_k(\Omega) \to \overline{\mathfrak{H}}_k(\Omega), \qquad \{U, U^\perp\} \to \{l(U), l(U^\perp) = l(U)^\perp\}.$$

The transformation from Example 1 gives the identical transformation of $\overline{\mathfrak{H}}_k(\Omega)$.

Theorem 2. *Let $n = 2k \geq 14$ and f be a bijective transformation of $\mathfrak{H}_k(\Omega)$ preserving the class of base subsets. Then f preserves the relation $\perp$ and induces a bijective transformation of $\overline{\mathfrak{H}}_k(\Omega)$. The latter mapping is induced by an element of $\Gamma\mathrm{Sp}(\Omega)$.*

Corollary 2. *Let $n = 2k \geq 14$ and f be a commutativity preserving bijective transformation of $\mathfrak{I}_k(\Omega)$. Suppose also that the characteristic of F is not equal to 2. Then there exists an automorphism g of the group $\mathrm{Sp}(\Omega)$ such that $f(u) = \pm g(u)$ for any $u \in \mathfrak{I}_k(\Omega)$.*

3. Inexact subsets

In this section we suppose that $n \geq 4$ and $1 < k < n-1$.

3.1. Inexact subsets of $\mathfrak{G}_k(W)$

Let $B = \{P_1, \dots, P_n\}$ be a base of $\mathcal{P}(W)$. For any $m \in \{1, \dots, n-1\}$ we denote by $\mathfrak{B}_m$ the base subset of $\mathfrak{G}_m(W)$ associated with B (the definition was given in Section 1).

If $\alpha = (M, N) \in \mathfrak{B}_m$ then we put $\mathfrak{B}_k(\alpha)$ for the set of all $(S, U) \in \mathfrak{B}_k$ where S is incident to M or N (then U is incident to N or M, respectively), the set of all $(S, U) \in \mathfrak{B}_k$ such that S is incident to M will be denoted by $\mathfrak{B}^+_k(\alpha)$.

A subset $\mathfrak{X} \subset \mathfrak{B}_k$ is called *exact* if there is only one base subset of $\mathfrak{G}_k(W)$ containing $\mathfrak{X}$; otherwise, $\mathfrak{X}$ is said to be *inexact*.

If $\alpha \in \mathfrak{B}_2$ then $\mathfrak{B}_k(\alpha)$ is a maximal inexact subset of $\mathfrak{B}_k$ (Example 1 in [8]). Conversely, we have the following:

Lemma 1. (Lemma 2 of [8]) *If $\mathfrak{X}$ is a maximal inexact subset of $\mathfrak{B}_k$ then there exists $\alpha \in \mathfrak{B}_2$ such that $\mathfrak{X} = \mathfrak{B}_k(\alpha)$.*

Lemma 2. (Lemmas 5 and 8 of [8]) *Let g be a bijective transformation of $\mathfrak{B}_k$ preserving the class of maximal inexact subsets. Then for any $\alpha \in \mathfrak{B}_{k-1}$ there exists $\beta \in \mathfrak{B}_{k-1}$ such that*

$$g(\mathfrak{B}_k(\alpha)) = \mathfrak{B}_k(\beta);$$

moreover, we have

$$g(\mathfrak{B}^+_k(\alpha)) = \mathfrak{B}^+_k(\beta)$$

if $n \neq 2k$.

3.2. Inexact subsets of $\mathfrak{H}_k(\Omega)$

Let $\mathfrak{S}_1 = \{S_1, \dots, S_n\}$ be a base subset of $\mathfrak{H}_1(\Omega)$. For each number $m \in \{2, \dots, n-1\}$ we denote by $\mathfrak{S}_m$ the base subset of $\mathfrak{H}_m(\Omega)$ associated with $\mathfrak{S}_1$.

Let $M \in \mathfrak{S}_m$. Then $M^\perp \in \mathfrak{S}_{n-m}$. We put $\mathfrak{S}_k(M)$ for the set of all elements of $\mathfrak{S}_k$ incident to M or $M^\perp$. The set of all elements of $\mathfrak{S}_k$ incident to M will be denoted by $\mathfrak{S}^+_k(M)$.

Let $\mathfrak{X}$ be a subset of $\mathfrak{S}_k$. We say that $\mathfrak{X}$ is *exact* if it is contained only in one base subset of $\mathfrak{H}_k(\Omega)$; otherwise, $\mathfrak{X}$ will be called *inexact*. For any $i \in \{1, \dots, n\}$ we denote by $\mathfrak{X}_i$ the set of all elements of $\mathfrak{X}$ containing S_i. If $\mathfrak{X}_i$ is not empty then we define

$$U_i(\mathfrak{X}) := \bigcap_{U \in \mathfrak{X}_i} U,$$

and $U_i(\mathfrak{X}) := \emptyset$ if $\mathfrak{X}_i$ is empty. It is trivial that our subset is exact if $U_i(\mathfrak{X}) = S_i$ for each i.

Lemma 3. *$\mathfrak{X}$ is exact if $U_i(\mathfrak{X}) \neq S_i$ only for one i.*

Proof. Let $\mathfrak{S}'_1$ be a base subset of $\mathfrak{H}_1(\Omega)$ which defines a base subset of $\mathfrak{H}_k(\Omega)$ containing $\mathfrak{X}$. If $j \neq i$ then $U_j(\mathfrak{X}) = S_j$ implies that S_j belongs to $\mathfrak{S}'_1$. Let us take $S' \in \mathfrak{S}'_1$ which does not coincide with any S_j, $j \neq i$. Since S' is orthogonal to all such S_j, we have $S' = S_i$ and $\mathfrak{S}'_1 = \mathfrak{S}_1$. □

Example 2. Let $M \in \mathfrak{S}_2$. Then $M = S_i + S_j$ for some i, j. We choose orthogonal $S'_i, S'_j \in \mathfrak{H}_1(\Omega)$ such that $S'_i + S'_j = M$ and $\{S_i, S_j\} \neq \{S'_i, S'_j\}$. Then

$$(\mathfrak{S}_1 \setminus \{S_i, S_j\}) \cup \{S'_i, S'_j\}$$

is a base subset of $\mathfrak{H}_1(\Omega)$ which defines another base subset of $\mathfrak{H}_k(\Omega)$ containing $\mathfrak{S}_k(M)$. Therefore, $\mathfrak{S}_k(M)$ is inexact. Any $U \in \mathfrak{S}_k \setminus \mathfrak{S}_k(M)$ intersects M by S_i or S_j and

$$U_p(\mathfrak{S}_k(M) \cup \{U\}) = S_p$$

if $p = i$ or j; the same holds for all $p \neq i, j$. By Lemma 3, $\mathfrak{S}_k(M) \cup \{U\}$ is exact for any $U \in \mathfrak{S}_k \setminus \mathfrak{S}_k(M)$. Thus the inexact subset $\mathfrak{S}_k(M)$ is maximal.

Lemma 4. *Let $\mathfrak{X}$ be a maximal inexact subset of $\mathfrak{S}_k$. Then $\mathfrak{X} = \mathfrak{S}_k(M)$ for certain $M \in \mathfrak{S}_2$.*

Proof. By the definition, there exists another base subset of $\mathfrak{H}_k(\Omega)$ containing $\mathfrak{X}$; the associated base subset of $\mathfrak{H}_1(\Omega)$ will be denoted by $\mathfrak{S}'_1$. Since our inexact subset is maximal, we need to prove the existence of $M \in \mathfrak{S}_2$ such that $\mathfrak{X} \subset \mathfrak{S}_k(M)$.

Let us consider $i \in \{1, \dots, n\}$ such that U_i is not empty (from this moment we write U_i in place of $U_i(\mathfrak{X})$). We say that the number i is of *first type* if the inclusion $U_j \subset U_i$, $j \neq i$ implies that $U_j = \emptyset$ or $U_j = U_i$. If i is not of first type and the inclusion $U_j \subset U_i$, $j \neq i$ holds only for the case when $U_j = \emptyset$ or j is of first type then i is said to be of *second type*. Similarly, other types of numbers can be defined.

Suppose that there exists a number j of first type such that $\dim U_j \geq 4$. Then U_j contains certain $M \in \mathfrak{S}_2$. Since j is of first type, for any $U \in \mathfrak{X}$ one of the following possibilities is realized:

– $U \in \mathfrak{X}_j$ then $M \subset U_j \subset U$,

– $U \in \mathfrak{X} \setminus \mathfrak{X}_j$ then $U \subset U_j^{\perp} \subset M^{\perp}$.

This means that M is as required.

Now suppose that $U_j = S_j$ for all j of first type, so $S_j \in \mathfrak{S}'_1$ if j is of first type. Consider any number i of second type. If $U_i \in \mathfrak{S}_m$ then $m \geq 2$ and there are exactly $m - 1$ distinct j of first type such that $S_j = U_j$ is contained in U_i; since all such S_j belong to $\mathfrak{S}'_1$ and U_i is spanned by elements of $\mathfrak{S}'_1$, we have $S_i \in \mathfrak{S}'_1$. Step by step we establish the same for other types. Thus $S_i \in \mathfrak{S}'_1$ if U_i is not empty. Since $\mathfrak{X}$ is inexact, Lemma 3 implies the existence of two distinct numbers i and j such that $U_i = U_j = \emptyset$. We define $M := S_i + S_j$. Then any element of $\mathfrak{X}$ is contained in $M^{\perp}$ and we get the claim. □

Let $\mathfrak{S}'_1$ be another base subset of $\mathfrak{H}_1(\Omega)$ and $\mathfrak{S}'_m$, $m \in \{2, \dots, n-1\}$, be the base subset of $\mathfrak{H}_m(\Omega)$ defined by $\mathfrak{S}'_1$.

Lemma 5. *Let h be a bijection of $\mathfrak{S}_k$ to $\mathfrak{S}'_k$ such that h and h^{-1} send maximal inexact subsets to maximal inexact subsets. Then for any $M \in \mathfrak{S}_{k-1}$ there exists $M' \in \mathfrak{S}'_{k-1}$ such that*

$$h(\mathfrak{S}_k(M)) = \mathfrak{S}'_k(M');$$

moreover, we have

$$h(\mathfrak{S}^+_k(M)) = \mathfrak{S}'^+_k(M')$$

if $n \neq 2k$.

Proof. Let $\mathfrak{B}_m$, $m \in \{1, \dots, n-1\}$, be as in subsection 3.1.. For each m there is the natural bijection $b_m : \mathfrak{B}_m \to \mathfrak{S}_m$ sending $(S, U) \in \mathfrak{B}_m$, $S = P_{i_1} + \dots + P_{i_m}$ to $S_{i_1} + \dots + S_{i_m}$. For any $M \in \mathfrak{S}_m$ we have

$$\mathfrak{S}_k(M) = b_k(\mathfrak{B}_k(b_m^{-1}(M))) \quad \text{and} \quad \mathfrak{S}^+_k(M) = b_k(\mathfrak{B}^+_k(b_m^{-1}(M))).$$

Let b'_m be the similar bijection of $\mathfrak{B}_m$ to $\mathfrak{S}'_m$. Then $(b'_k)^{-1} h b_k$ is a bijective transformation of $\mathfrak{B}_k$ preserving the class of base subsets and our statement follows from Lemma 2. □

4. Proof of Theorems 1 and 2

By Fact 2, we need to prove Theorem 1 only for $k < n - k$. Throughout the section we suppose that $1 < k \leq n - k$ and $n - k \geq 5$; for the case when $n = 2k$ we require that $n \geq 14$.

4.1.

Let f be a bijective transformation of $\mathfrak{H}_k(\Omega)$ preserving the class of base subsets. The restriction of f to any base subset satisfies the condition of Lemma 5.

For any subspace $T \subset V$ we denote by $\mathfrak{H}_k(T)$ the set of all elements of $\mathfrak{H}_k(\Omega)$ incident to T or $T^\perp$, the set of all elements of $\mathfrak{H}_k(\Omega)$ incident to T will be denoted by $\mathfrak{H}^+_k(T)$.

In this subsection we show that Theorems 1 and 2 are simple consequences of the following lemma.

Lemma 6. *There exists a bijective transformation g of $\mathfrak{H}_{k-1}(\Omega)$ such that*

$$g(\mathfrak{H}^+_k(T)) = \mathfrak{H}^+_k(g(T)) \quad \forall\, T \in \mathfrak{H}_{k-1}(\Omega)$$

if $n \neq 2k$, and

$$g(\mathfrak{H}_k(T)) = \mathfrak{H}_k(g(T))) \quad \forall\, T \in \mathfrak{H}_{k-1}(\Omega)$$

for the case when $n = 2k$.

The proof of Lemma 6 will be given later.

Let $\mathfrak{S}_{k-1}$ be a base subset of $\mathfrak{H}_{k-1}(\Omega)$ and $\mathfrak{S}_k$ be the associated base subset of $\mathfrak{H}_k(\Omega)$ (these base subsets are defined by the same base subset of $\mathfrak{H}_1(\Omega)$). By our hypothesis, $f(\mathfrak{S}_k)$ is a base subset of $\mathfrak{H}_k(\Omega)$; we denote by $\mathfrak{S}'_{k-1}$ the associated base subset of $\mathfrak{H}_{k-1}(\Omega)$. It is easy to see that $g(\mathfrak{S}_{k-1}) = \mathfrak{S}'_{k-1}$, so g maps base subsets to base subsets. Since f^{-1} preserves the class of base subset, the same holds for g^{-1}. Thus g preserves the class of base subsets.

Now suppose that $g = (l)_{k-1}$ for certain $l \in \Gamma\mathrm{Sp}(\Omega)$. Let U be an element of $\mathfrak{H}_k(\Omega)$. We take $M, N \in \mathfrak{H}_{k-1}(\Omega)$ such that $U = M + N$. If $n \neq 2k$ then

$$\{U\} = \mathfrak{H}^+_k(M) \cap \mathfrak{H}^+_k(N) \ \text{ and } \ \{f(U)\} = \mathfrak{H}^+_k(l(M)) \cap \mathfrak{H}^+_k(l(N)),$$

so $f(U) = l(M) + l(N) = l(U)$, and we get $f = (l)_k$. For the case when $n = 2k$ we have

$$\{U, U^\perp\} = \mathfrak{H}_k(M) \cap \mathfrak{H}_k(N) \ \text{ and } \ \{f(U), f(U)^\perp\} = \mathfrak{H}_k(l(M)) \cap \mathfrak{H}_k(l(N));$$

since $l(M) + l(N) = l(U)$ and $l(M)^\perp \cap l(N)^\perp = (l(M) + l(N))^\perp = l(U)^\perp$,

$$\{f(U), f(U)^\perp\} = \{l(U), l(U)^\perp\};$$

the latter means that $f = (l)'_k$. Therefore, Theorem 1 can be proved by induction and Theorem 2 follows from Theorem 1.

To prove Lemma 6 we use the following:

Lemma 7. *Let $M \in \mathfrak{H}_m(\Omega)$ and N be a subspace contained in M. Then the following assertions are fulfilled:*

(1) *If $\dim N > m$ then N contains an element of $\mathfrak{H}_1(\Omega)$.*

(2) *If $\dim N > m + 2$ then N contains two orthogonal elements of $\mathfrak{H}_1(\Omega)$.*

(3) *If $\dim N > m + 4$ then N contains three distinct mutually orthogonal elements of $\mathfrak{H}_1(\Omega)$.*

Proof. The form $\Omega|_M$ is non-degenerate. If $\dim N > m$ then the restriction of $\Omega|_M$ to N is non-zero. This implies the existence of $S \in \mathfrak{H}_1(\Omega)$ contained in N. We have

$$\dim N \cap S^\perp \geq \dim N - 2,$$

and for the case when $\dim N > m + 2$ there is an element of $\mathfrak{H}_1(\Omega)$ contained in $N \cap S^\perp$. Similarly, (3) follows from (2). $\square$

4.2. Proof of Lemma 6 for $k < n - k$

Let $T \in \mathfrak{H}_{k-1}(\Omega)$ and $\mathfrak{S}_1 = \{S_1, \dots, S_n\}$ be a base subset of $\mathfrak{H}_1(\Omega)$ such that

$$T^\perp = S_1 + \dots + S_{n-k+1} \ \text{ and } \ T = S_{n-k+2} + \dots + S_n.$$

We put $\mathfrak{S}_k$ for the base subset of $\mathfrak{H}_k(\Omega)$ associated with $\mathfrak{S}_1$. Then $\mathfrak{S}^+_k(T)$ consists of all

$$U_i := T + S_i,$$

where $i \in \{1, \dots, n-k+1\}$. By Lemma 5, there exists $T' \in \mathfrak{H}_{k-1}(\Omega)$ such that

$$f(\mathfrak{S}^+_k(T)) \subset \mathfrak{H}^+_k(T').$$

We need to show that $f(\mathfrak{H}^+_k(T))$ coincides with $\mathfrak{H}^+_k(T')$.

Lemma 8. *Let $U \in \mathfrak{H}^+_k(T)$. Suppose that there exist two distinct $M, N \in \mathfrak{H}^+_k(T)$ such that $f(M), f(N)$ belong to $\mathfrak{H}^+_k(T')$ and there is a base subset of $\mathfrak{H}_k(\Omega)$ containing M, N and U. Then $f(U)$ is an element of $\mathfrak{H}^+_k(T')$.*

Proof. If there exists a base subset of $\mathfrak{H}_k(\Omega)$ containing M, N and U then T belongs to the associated base subset of $\mathfrak{H}_{k-1}(\Omega)$ and Lemma 5 implies the existence of $T'' \in \mathfrak{H}_{k-1}(\Omega)$ such that $f(M), f(N)$ and $f(U)$ belong to $\mathfrak{H}^+_k(T'')$. On the other hand, $f(M)$ and $f(N)$ are different elements of $\mathfrak{H}^+_k(T')$ and $f(M) \cap f(N)$ coincides with T'. Hence $T' = T''$. $\square$

For any $U \in \mathfrak{H}^+_k(T)$ we denote by $S(U)$ the intersection of U and $T^\perp$, it is clear that $S(U)$ is an element of $\mathfrak{H}_1(\Omega)$.

If $S(U)$ is contained in $S_1 + \dots + S_{n-k-1}$ then $S(U), S_{n-k}, S_{n-k+1}$ are mutually orthogonal and there exists a base subset of $\mathfrak{H}_k(\Omega)$ containing U, U_{n-k}, U_{n-k+1}. All $f(U_i)$ belong to $\mathfrak{H}^+_k(T')$ and Lemma 8 shows that $f(U) \in \mathfrak{H}^+_k(T')$.

Let U be an element of $\mathfrak{H}^+_k(T)$ such that $S(U)$ is contained in $S_1 + \dots + S_{n-k}$. We have

$$\dim(S_1 + \dots + S_{n-k-1}) \cap S(U)^\perp \geq 2(n-k-2) > n-k-1$$

(the latter inequality follows from the condition $n - k \geq 5$) and Lemma 7 implies the existence of $S' \in \mathfrak{H}_1(\Omega)$ contained in

$$(S_1 + \dots + S_{n-k-1}) \cap S(U)^\perp.$$

Then $S(U), S', S_{n-k+1}$ are mutually orthogonal and there exists a base subset of $\mathfrak{H}_k(\Omega)$ containing $U, T + S', U_{n-k+1}$. It was proved above that $f(T + S')$ belongs to $\mathfrak{H}^+_k(T')$. Since $f(U_i) \in \mathfrak{H}^+_k(T')$ for each i, Lemma 8 guarantees that $f(U)$ is an element of $\mathfrak{H}^+_k(T')$.

Now suppose that $S(U)$ is not contained in $S_1 + \dots + S_{n-k}$. Since $n - k \geq 5$,

$$\dim(S_1 + \dots + S_{n-k}) \cap S(U)^\perp \geq 2(n-k-1) > n-k+2.$$

By Lemma 7, there exist two orthogonal $S', S'' \in \mathfrak{H}_1(\Omega)$ contained in

$$(S_1 + \dots + S_{n-k}) \cap S(U)^\perp.$$

Then $S', S'', S(U)$ are mutually orthogonal and there exists a base subset of $\mathfrak{H}_k(\Omega)$ containing $S' + T, S'' + T$ and U. We have shown above that $f(S' + T)$, $f(S'' + T)$ belong to $\mathfrak{H}^+_k(T')$ and Lemma 8 shows that the same holds for $f(U)$.

So $f(\mathfrak{H}^+_k(T)) \subset \mathfrak{H}^+_k(T')$. Since f^{-1} preserves the class of base subsets, the inverse inclusion holds true. We define $g : \mathfrak{H}_{k-1}(\Omega) \to \mathfrak{H}_{k-1}(\Omega)$ by $g(T) := T'$. This transformation is bijective (otherwise, f is not bijective).

4.3. Proof of Lemma 6 for $n = 2k$

We start with the following:

Lemma 9. *If $n = 2k$ then $f(U^{\perp}) = f(U)^{\perp}$ for any $U \in \mathfrak{H}_k(\Omega)$.*

Proof. We take a base subset $\mathfrak{S}_k$ containing U. Then $U^{\perp} \in \mathfrak{S}_k$. Denote by $\mathfrak{S}_{k-1}$ the base subset of $\mathfrak{H}_{k-1}(\Omega)$ associated with $\mathfrak{S}_k$. Let $\mathfrak{S}'_{k-1}$ be the base subset of $\mathfrak{H}_{k-1}(\Omega)$ associated with $\mathfrak{S}'_k := f(\mathfrak{S}_k)$. We choose $M, N \in \mathfrak{S}_{k-1}$ such that $U = M + N$. Then

$$\{U, U^{\perp}\} = \mathfrak{S}_k(M) \cap \mathfrak{S}_k(N)$$

and Lemma 5 guarantees that

$$\{f(U), f(U^{\perp})\} = \mathfrak{S}'_k(M') \cap \mathfrak{S}'_k(N')$$

for some $M', N' \in \mathfrak{S}'_{k-1}$. The set $\mathfrak{S}'_k(M') \cap \mathfrak{S}'_k(N')$ is not empty if one of the following possibilities is realized:

- $M' + N'$ and $M'^{\perp} \cap N'^{\perp}$ are elements of $\mathfrak{H}_{k-1}(\Omega)$ and $\mathfrak{S}'_k(M') \cap \mathfrak{S}'_k(N')$ consists of these two elements.
- $M' \subset N'^{\perp}$ and $N' \subset M'^{\perp}$, then $\mathfrak{S}'_k(M') \cap \mathfrak{S}'_k(N')$ consists of 4 elements.

Thus

$$\{f(U), f(U^{\perp})\} = \{M' + N', M'^{\perp} \cap N'^{\perp}\}.$$

Since $M' + N'$ and $M'^{\perp} \cap N'^{\perp}$ are orthogonal, we get the claim. □

Let $T \in \mathfrak{H}_{k-1}(\Omega)$. As in the previous subsection we consider a base subset $\mathfrak{S}_1 = \{S_1, \dots, S_n\}$ of $\mathfrak{H}_1(\Omega)$ such that

$$T^{\perp} = S_1 + \cdots + S_{n-k+1} \text{ and } T = S_{n-k+2} + \cdots + S_n.$$

We denote by $\mathfrak{S}_k$ the base subset of $\mathfrak{H}_k(\Omega)$ associated with $\mathfrak{S}_1$. Then $\mathfrak{S}_k(T)$ consists of

$$U_i := T + S_i, \quad i \in \{1, \dots, n-k+1\}$$

and their orthogonal complements. Lemma 5 implies the existence of $T' \in \mathfrak{H}_{k-1}(\Omega)$ such that

$$f(\mathfrak{S}_k(T)) \subset \mathfrak{H}_k(T').$$

We show that $f(U)$ belongs to $\mathfrak{H}_k(T')$ for any $U \in \mathfrak{H}_k(T)$.

We need to establish this fact only for the case when U is an element of $\mathfrak{H}^+_k(T)$. Indeed, if $U \in \mathfrak{H}^+_k(T^{\perp})$ then $U^{\perp}$ is an element of $\mathfrak{H}^+_k(T)$ and $f(U^{\perp}) \in \mathfrak{H}_k(T')$ implies that $f(U) = f(U^{\perp})^{\perp}$ belongs to $\mathfrak{H}_k(T')$.

Lemma 10. *Let $U \in \mathfrak{H}^+_k(T)$. Suppose that there exist distinct $M_i \in \mathfrak{H}^+_k(T)$, $i = 1, 2, 3$ such that each $f(M_i)$ belongs to $\mathfrak{H}_k(T')$ and there is a base subset of $\mathfrak{H}_k(\Omega)$ containing M_1, M_2, M_3 and U. Then $f(U) \in \mathfrak{H}_k(T')$.*

Proof. By Lemma 5, there exists $T'' \in \mathfrak{H}_{k-1}(\Omega)$ such that $f(U)$, all $f(M_i)$, and their orthogonal complements belong to $\mathfrak{H}_k(T'')$. For any $i = 1, 2, 3$ one of the subspaces $f(M_i)$ or $f(M_i)^{\perp}$ is an element of $\mathfrak{H}^+_k(T'')$; we denote this subspace by M'_i. Then

$$T'' = \bigcap_{i=1}^{3} M'_i \quad \text{and} \quad T''^{\perp} = M'^{\perp}_i + M'^{\perp}_j,\ i \neq j;$$

note also that the intersection of any M'_i and $M'^{\perp}_j$ does not belong to $\mathfrak{H}_{k-1}(\Omega)$. Since all M'_i and $M'^{\perp}_i$ belong to $\mathfrak{H}_k(T')$, we have $T' = T''$. □

As in the previous subsection for any $U \in \mathfrak{H}^+_k(T)$ we denote by $S(U)$ the intersection of U and $T^{\perp}$, it is an element of $\mathfrak{H}_1(\Omega)$.

If $S(U)$ is contained in $S_1 + \dots + S_{n-k-2}$ then $S(U), S_{n-k-1}, S_{n-k}, S_{n-k+1}$ are mutually orthogonal and there exists a base subset of $\mathfrak{H}_k(\Omega)$ containing $U, U_{n-k-1}, U_{n-k}, U_{n-k+1}$. Since $f(U_i) \in \mathfrak{H}_k(T')$ for each i, Lemma 10 shows that $f(U)$ belongs to $\mathfrak{H}_k(T')$.

Suppose that $S(U)$ is contained in $S_1 + \dots + S_{n-k-1}$. We have

$$\dim(S_1 + \dots + S_{n-k-2}) \cap S(U)^{\perp} \geq 2(n-k-3) > n-k-2$$

(since $k = n-k \geq 7$) and Lemma 7 implies the existence of $S' \in \mathfrak{H}_1(\Omega)$ contained in

$$(S_1 + \dots + S_{n-k-2}) \cap S(U)^{\perp}.$$

Then $S(U), S', S_{n-k}, S_{n-k+1}$ are mutually orthogonal, so $U, T+S', U_{n-k}, U_{n-k+1}$ are contained in a certain base subset of $\mathfrak{H}_k(\Omega)$. It was shown above that $f(T+S')$ is an element of $\mathfrak{H}_k(T')$ and Lemma 10 guarantees that $f(U) \in \mathfrak{H}_k(T')$ (recall that all $f(U_i)$ belong to $\mathfrak{H}_k(T')$).

Consider the case when $S(U)$ is contained in $S_1 + \dots + S_{n-k}$. We have

$$\dim(S_1 + \dots + S_{n-k-1}) \cap S(U)^{\perp} \geq 2(n-k-2) > (n-k-1) + 2$$

(recall that $k = n-k \geq 7$) and there exist two orthogonal $S', S'' \in \mathfrak{H}_1(\Omega)$ contained in

$$S_1 + \dots + S_{n-k-1}) \cap S(U)^{\perp}$$

(Lemma 7). Then $S(U), S', S'', S_{n-k+1}$ are mutually orthogonal and there exists a base subset of $\mathfrak{H}_k(\Omega)$ containing $U, T+S', T+S'', U_{n-k+1}$. It follows from Lemma 10 that $f(U) \in \mathfrak{H}_k(T')$ (since $f(T+S')$, $f(T+S'')$ and any $f(U_i)$ belong to $\mathfrak{H}_k(T')$).

Let U be an element of $\mathfrak{H}_k(T')$ such that $S(U)$ is not contained in $S_1 + \dots + S_{n-k}$. Since $n = 2k \geq 14$,

$$\dim(S_1 + \dots + S_{n-k}) \cap S(U)^{\perp} \geq 2(n-k-1) > n-k+4.$$

By Lemma 7, there exist mutually orthogonal $S', S'', S''' \in \mathfrak{H}_1(\Omega)$ contained in

$$(S_1 + \dots + S_{n-k}) \cap S(U)^{\perp}.$$

A base subset of $\mathfrak{H}_k(\Omega)$ containing $U, T+S', T+S'', T+S'''$ exists. It was shown above that $f(T+S')$, $f(T+S'')$ and $f(T+S''')$ belong to $\mathfrak{H}_k(T')$ and Lemma 10 implies that the same holds for $f(U)$.

Thus $f(\mathfrak{H}_k(T)) \subset \mathfrak{H}_k(T')$. As in the previous subsection we have the inverse inclusion and define $g : \mathfrak{H}_{k-1}(\Omega) \to \mathfrak{H}_{k-1}(\Omega)$ by $g(T) := T'$.

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Received May 1, 2005