Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 47 (2006), No. 2, 419-434.

Four Dimensional Symplectic Lie Algebras

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Abstract. Invariant symplectic structures are determined in dimension four and the corresponding Lie algebras are classified up to equivalence. Symplectic four dimensional Lie algebras are described either as solutions of the cotangent extension problem or as symplectic double extension of $\mathbb{R}^2$ by $\mathbb{R}$. For this all extensions of a two dimensional Lie algebra are determined. We also find Lie algebras which do not admit a symplectic form in higher dimensions.

MSC 2000: 53D05 (primary), 22E25, 17B56 (secondary)
Keywords: symplectic structures, solvable Lie algebra, cotangent extension, symplectic double extensions

1. Introduction

Symplectic structures have proved to be an important tool in the description and geometrization of several phenomena. Special cases of symplectic manifolds are the Kähler ones, for which the methods of rational homotopy theory have been applied successfully (see [22] [12] [13] [17] [19] [26] for example), and algebraic tools were used to attack geometric aspects in [3] [10] [11] [18]. Another class of examples is provided by Lie groups, for which left invariant translations by elements of the Lie group are symplectomorphisms. In addition to these examples one of the most frequently encountered types of symplectic manifolds is the cotangent bundle of a differentiable manifold. To this type belongs the cotangent

*Partially supported by Conicet, SECyt - UNC and FONCYT

bundle T^*H of a Lie group H. If $\mathfrak{h}$ denotes the Lie algebra of H and let $\mathfrak{h}^*$ be the dual vector space of $\mathfrak{h}$, then H is identified with the zero section in T^*H and $\mathfrak{h}^*$ with the fiber over a neutral element of H. Let ω_0 be the skew-symmetric bilinear form defined in $\mathfrak{h}^* \oplus \mathfrak{h}$ by setting

$$\omega_0((\varphi, x), (\varphi', x')) = \varphi(x') - \varphi'(x)$$

It is easy to see that ω_0 spans a left invariant closed 2-form in the canonical Lie group structure of T^*H if and only if H is abelian (see Remark (3.2)). That leads to ask the following: *is there a Lie group structure on T^*H in such way that the left invariant two form induced by ω_0 is closed?* That is known as the *cotangent extension problem* [4]. More precisely, let $\mathfrak{h}$ be a Lie algebra and let $\mathfrak{h}^*$ be the dual vector space of $\mathfrak{h}$ and ω_0 be as defined above. The problem is to find a Lie algebra structure on $\mathfrak{h}^* \oplus \mathfrak{h}$ satisfying the following conditions:

(c1) $0 \longrightarrow \mathfrak{h}^* \longrightarrow \mathfrak{h}^* \oplus \mathfrak{h} \longrightarrow \mathfrak{h} \longrightarrow 0$ is an exact sequence of Lie algebras, $\mathfrak{h}^*$ endowed with the abelian Lie algebra structure;

(c2) the left invariant 2-form spanned by ω_0 is closed.

Note that $\mathfrak{h}^*$ is a lagrangian ideal on $\mathfrak{h}^* \oplus \mathfrak{h}$. The resulting Lie algebra is called a solution of the cotangent extension problem. This construction of symplectic manifolds was used by Boyom [4] [5] to give models for symplectic Lie algebras.

Another construction arises by the symplectic double extension found by Medina and Revoy [21] and generalized by Dardié and Medina [9]. This construction realizes a symplectic group as the reduction of another symplectic group. In this situation one has a symplectic Lie algebra $\mathfrak{g}$ and an isotropic ideal $\mathfrak{h}$, such that $\mathfrak{h}^\perp$ is an ideal and the symplectic structure on $\mathfrak{h}^\perp/\mathfrak{h}$ is induced from that of $\mathfrak{g}$.

Whenever we deal with symplectic structures we can study the existence and classification of such structures on any given Lie algebra such as in [6] [1] or the algebraic structure of a Lie algebra (group) admitting a symplectic structure, as in [4] [5] [9] [21]. In this paper we compute all symplectic structures on any real solvable four dimensional Lie algebras. Then we consider the action of the authomorphism group of the Lie algebra on the space of symplectic structures and we get the classification of symplectic Lie algebras up to this equivalence relation, completing the table in [21]. In the four dimensional case we study the action of the adjoint representation and we make use of this to find Lie algebras do not admitting symplectic structures in higher dimensions. A next goal in this paper is to reconstruct these symplectic Lie algebras in terms of the above described models. In this sense the principal result we prove is that any symplectic Lie algebra which is either completely solvable or $\mathfrak{aff}(\mathbb{C})$ is a solution of the cotangent extension problem. Furthermore we classify all solutions of the cotangent extension problem up to equivalence for $\mathfrak{h}$ of dimension two. For the other symplectic four dimensional Lie algebras we apply results of [21], [9] to prove that $\mathfrak{g}$ is obtained as a double extension of the two dimensional abelian Lie algebra by $\mathbb{R}$. In both models lagrangian and isotropic ideals play an important role.

The author thanks I. Dotti for general supervision, L. Cagliero for very useful discussions and the referee for very helpful comments to get an improvement of a previous version.

2. Four dimensional symplectic Lie algebras

2.1. Four dimensional solvable Lie algebras

Since we are interested on left invariant structures, our work is reduced to the Lie algebras of the corresponding Lie groups. As a first step we exhibit in the following proposition the different classes of four dimensional solvable Lie algebras (see for instance [2] for a proof).

Proposition 2.1. *Let $\mathfrak{g}$ be a solvable four dimensional real Lie algebra. Then if $\mathfrak{g}$ is not abelian, it is equivalent to one and only one of the Lie algebras listed below:*

$$
\begin{array}{ll}
\mathfrak{rh}_3: & [e_1,e_2]=e_3 \\
\mathfrak{rr}_3: & [e_1,e_2]=e_2,\ [e_1,e_3]=e_2+e_3 \\
\mathfrak{rr}_{3,\lambda}: & [e_1,e_2]=e_2, [e_1,e_3]=\lambda e_3 \qquad \lambda\in[-1,1] \\
\mathfrak{rr}'_{3,\gamma}: & [e_1,e_2]=\gamma e_2-e_3, [e_1,e_3]=e_2+\gamma e_3 \qquad \gamma\geq 0 \\
\mathfrak{r}_2\mathfrak{r}_2: & [e_1,e_2]=e_2,\ [e_3,e_4]=e_4 \\
\mathfrak{r}'_2: & [e_1,e_3]=e_3,\ [e_1,e_4]=e_4,\ [e_2,e_3]=e_4,\ [e_2,e_4]=-e_3 \\
\mathfrak{n}_4: & [e_4,e_1]=e_2,\ [e_4,e_2]=e_3 \\
\mathfrak{r}_4: & [e_4,e_1]=e_1,\ [e_4,e_2]=e_1+e_2, [e_4,e_3]=e_2+e_3 \\
\mathfrak{r}_{4,\mu}: & [e_4,e_1]=e_1,\ [e_4,e_2]=\mu e_2,\ [e_4,e_3]=e_2+\mu e_3 \qquad \mu\in\mathbb{R} \\
\mathfrak{r}_{4,\alpha,\beta}: & [e_4,e_1]=e_1,\ [e_4,e_2]=\alpha e_2,\ [e_4,e_3]=\beta e_3, \\
 & \text{with } -1<\alpha\leq\beta\leq 1,\ \alpha\beta\neq 0, \text{ or } -1=\alpha\leq\beta\leq 0 \\
\mathfrak{r}'_{4,\gamma,\delta}: & [e_4,e_1]=e_1,\ [e_4,e_2]=\gamma e_2-\delta e_3,\ [e_4,e_3]=\delta e_2+\gamma e_3 \quad \gamma\in\mathbb{R},\delta>0 \\
\mathfrak{d}_4: & [e_1,e_2]=e_3,\ [e_4,e_1]=e_1,\ [e_4,e_2]=-e_2 \\
\mathfrak{d}_{4,\lambda}: & [e_1,e_2]=e_3,\ [e_4,e_3]=e_3,\ [e_4,e_1]=\lambda e_1,\ [e_4,e_2]=(1-\lambda)e_2 \quad \lambda\geq\frac{1}{2}
\end{array}
$$

$$
\begin{array}{ll}
\mathfrak{d}'_{4,\delta}: & [e_1,e_2]=e_3,\ [e_4,e_1]=\frac{\delta}{2}e_1-e_2, [e_4,e_3]=\delta e_3,\ [e_4,e_2]=e_1+\frac{\delta}{2}e_2 \quad \delta\geq 0 \\
\mathfrak{h}_4 & [e_1,e_2]=e_3,\ [e_4,e_3]=e_3,\ [e_4,e_1]=\frac{1}{2}e_1,\ [e_4,e_2]=e_1+\frac{1}{2}e_2
\end{array}
$$

A Lie algebra is called *unimodular* if tr(ad$_x$)=0 for all $x\in\mathfrak{g}$, where tr denotes the trace of the map. The unimodular four dimensional solvable Lie algebras are: $\mathbb{R}^4$, $\mathfrak{rh}_3$, $\mathfrak{rr}_{3,-1}$, $\mathfrak{rr}'_{3,0}$, $\mathfrak{n}_4$, $\mathfrak{r}_{4,-1/2}$, $\mathfrak{r}_{4,\mu,-1-\mu}$, $-1<\mu\leq -1/2$, $\mathfrak{r}'_{4,\mu,-\mu/2}$, $\mathfrak{d}_4$, $\mathfrak{d}'_{4,0}$.

Recall that a solvable Lie algebra is *completely solvable* when ad$_x$ has real eigenvalues for all $x\in\mathfrak{g}$.

2.2. Classification of symplectic Lie algebras

A *symplectic structure* on a $2n$-dimensional Lie algebra $\mathfrak{g}$ is a closed 2-form $\omega\in\Lambda^2(\mathfrak{g}^*)$ such that ω has maximal rank, that is, ω^n is a volume form on the corresponding Lie group. Lie algebras (groups) admitting symplectic structures are called *symplectic* Lie algebras (resp. Lie groups).

It is known that if $\mathfrak{g}$ is four dimensional and symplectic then it must be solvable [8]. However not every four dimensional solvable Lie group admits a symplectic structure. In this section we determine all left invariant symplectic structures on simply connected four dimensional Lie groups and we classify the corresponding Lie algebras, up to equivalence.

Denoting by $\{e^i\}$ the dual basis on $\mathfrak{g}^*$ of the basis $\{e_i\}$ on $\mathfrak{g}$ (see (2.1)), the next Proposition 2.2 describes symplectic structures in the four dimensional case and the proof follows by working on each Lie algebra. We first compute the closed two forms, that is, $\omega = \sum_{i\geq 1, j>i} e^i \wedge e^j$ such that $d\omega = 0$, where d denotes the antiderivation operator. The next step is to compute the rank of ω. If ω has maximal rank, then $\mathfrak{g}$ will be endowed with a symplectic structure.

Proposition 2.2. *Let $\mathfrak{g}$ be a symplectic real Lie algebra of dimension four. Then $\mathfrak{g}$ is isomorphic to one of the following Lie algebras equipped with a symplectic form as follows:*

$\mathfrak{rh}_3$: $\omega = a_{12}e^1 \wedge e^2 + a_{13}e^1 \wedge e^3 + a_{14}e^1 \wedge e^4 + a_{23}e^2 \wedge e^3 + a_{24}e^2 \wedge e^4$
$a_{14}a_{23} - a_{13}a_{24} \neq 0$

$\mathfrak{rr}_{3,0}$: $\omega = a_{12}e^1 \wedge e^2 + a_{13}e^1 \wedge e^3 + a_{14}e^1 \wedge e^4 + a_{34}e^3 \wedge e^4, \quad a_{12}a_{34} \neq 0$

$\mathfrak{rr}_{3,-1}$: $\omega = a_{12}e^1 \wedge e^2 + a_{13}e^1 \wedge e^3 + a_{14}e^1 \wedge e^4 + a_{23}e^2 \wedge e^3, \quad a_{14}a_{23} \neq 0$

$\mathfrak{rr}'_{3,0}$: $\omega = a_{12}e^1 \wedge e^2 + a_{13}e^1 \wedge e^3 + a_{14}e^1 \wedge e^4 + a_{23}e^2 \wedge e^3, \quad a_{14}a_{23} \neq 0$

$\mathfrak{r}_2\mathfrak{r}_2$: $\omega = a_{12}e^1 \wedge e^2 + a_{13}e^1 \wedge e^3 + a_{34}e^3 \wedge e^4, \quad a_{12}a_{34} \neq 0$

$\mathfrak{r}'_2$: $\omega = a_{12}e^1 \wedge e^2 + a_{13-24}(e^1 \wedge e^3 - e^2 \wedge e^4) + a_{14+23}(e^1 \wedge e^4 + e^2 \wedge e^3)$
$a^2_{14+23} + a^2_{13-24} \neq 0$

$\mathfrak{n}_4$: $\omega = a_{12}e^1 \wedge e^2 + a_{14}e^1 \wedge e^4 + a_{24}e^2 \wedge e^4 + a_{34}e^3 \wedge e^4, \quad a_{12}a_{34} \neq 0$

$\mathfrak{r}_{4,0}$: $\omega = a_{14}e^1 \wedge e^4 + a_{23}e^2 \wedge e^3 + a_{24}e^2 \wedge e^4 + a_{34}e^3 \wedge e^4, \quad a_{14}a_{23} \neq 0$

$\mathfrak{r}_{4,-1}$: $\omega = a_{13}e^1 \wedge e^3 + a_{14}e^1 \wedge e^4 + a_{24}e^2 \wedge e^4 + a_{34}e^3 \wedge e^4, \quad a_{13}a_{24} \neq 0$

$\mathfrak{r}_{4,-1,\beta}$: $\omega = a_{12}e^1 \wedge e^2 + a_{14}e^1 \wedge e^3 + a_{24}e^2 \wedge e^4 + a_{34}e^3 \wedge e^4,$
$a_{14}a_{23} \neq 0, \ \beta \neq -1, 0, 1$

$\mathfrak{r}_{4,-1,-1}$: $\omega = a_{12}e^1 \wedge e^2 + a_{13}e^1 \wedge e^3 + a_{14}e^1 \wedge e^4 + a_{24}e^2 \wedge e^4 + a_{34}e^3 \wedge e^4,$
$a_{12}a_{34} - a_{13}a_{24} \neq 0$

$\mathfrak{r}_{4,\alpha,-\alpha}$: $\omega = a_{14}e^1 \wedge e^4 + a_{23}e^2 \wedge e^3 + a_{24}e^2 \wedge e^4 + a_{34}e^3 \wedge e^4,$
$a_{14}a_{23} \neq 0, \ \alpha \neq -1, 0$

$\mathfrak{r}'_{4,0,\delta}$: $\omega = a_{14}e^1 \wedge e^4 + a_{23}e^2 \wedge e^3 + a_{24}e^2 \wedge e^4 + a_{34}e^3 \wedge e^4,$
$a_{14}a_{23} \neq 0, \ \delta \neq 0$

$\mathfrak{d}_{4,1}$: $\omega = a_{12-34}(e^1 \wedge e^2 - e^3 \wedge e^4) + a_{14}e^1 \wedge e^4 + a_{24}e^2 \wedge e^4, \quad a_{12-34} \neq 0$

$\mathfrak{d}_{4,2}$: $\omega = a_{12-34}(e^1 \wedge e^2 - e^3 \wedge e^4) + a_{14}e^1 \wedge e^4 + a_{23}e^2 \wedge e^3 + a_{24}e^2 \wedge e^4,$
$-a^2_{12-34} + a_{14}a_{23} \neq 0$

$\mathfrak{d}_{4,\lambda}$: $\omega = a_{12-34}(e^1 \wedge e^2 - e^3 \wedge e^4) + a_{14}e^1 \wedge e^4 + a_{24}e^2 \wedge e^4,$
$a_{12-34} \neq 0, \ \lambda \neq 1, 2$

$\mathfrak{d}'_{4,\delta}$: $\omega = a_{12-\delta 34}(e^1 \wedge e^2 - \delta e^3 \wedge e^4) + a_{14}e^1 \wedge e^4 + a_{24}e^2 \wedge e^4$
$a_{-12+\delta 34} \neq 0, \ \delta \neq 0$

$\mathfrak{h}_4$: $\omega = a_{12-34}(e^1 \wedge e^2 - e^3 \wedge e^4) + a_{14}e^1 \wedge e^4 + a_{24}e^2 \wedge e^4, \quad a_{12-34} \neq 0$

Recall that an element $\omega \in \Lambda^p(\mathfrak{g}^*)$ is called *exact* if $\omega = d\eta$ for some $\eta \in \Lambda^{p-1}(\mathfrak{g}^*)$. Thus the computations of the previous propositions give also the exact symplectic Lie algebras, which were obtained by Campoamor in [7].

Corollary 2.3. *A four dimensional solvable Lie algebra admits an exact symplectic structure if and only if* $\mathfrak{g}$ *is one of the Lie algebras of Table* (2.3) *attached with the respective symplectic structure.*

Case	ω	Condition
$\mathfrak{r}_2\mathfrak{r}_2$	$a_{12}e^1\wedge e^2 + a_{34}e^3\wedge e^4$	$a_{12}a_{34}\neq 0$
$\mathfrak{r}_2'$	$a_{13-24}(e^1\wedge e^3 - e^2\wedge e^4) + a_{14+23}(e^1\wedge e^4 + e^2\wedge e^3)$	$a_{14+23}^2 + a_{13-24}^2 \neq 0$
$\mathfrak{d}_{4,1}$	$a_{12-34}(e^1\wedge e^2 - e^3\wedge e^4) + a_{14}e^1\wedge e^4$	$a_{12-34}\neq 0$
$\mathfrak{d}_{4,\lambda}\lambda\neq 1$	$a_{12-34}(e^1\wedge e^2 - e^3\wedge e^4) + a_{14}e^1\wedge e^4 + a_{24}e^2\wedge e^4$	$a_{12-34}\neq 0$
$\mathfrak{d}_{4,\delta}'\delta\neq 0$	$a_{-12+\delta 34}(-e^1\wedge e^2 + \delta e^3\wedge e^4) + a_{14}e^1\wedge e^4 + a_{24}e^2\wedge e^4$	$a_{-12+\delta 34}\neq 0$
$\mathfrak{h}_4$	$a_{12-34}(e^1\wedge e^2 - e^3\wedge e^4) + a_{14}e^1\wedge e^4 + a_{24}e^2\wedge e^4$	$a_{12-34}\neq 0$

Table 2.3

Two symplectic Lie algebras $(\mathfrak{g}_1,\omega_1)$ and $(\mathfrak{g}_2,\omega_2)$ are said to be (symplectomorphically) *equivalent* if there exists an isomorphism of Lie algebras $\varphi:\mathfrak{g}_1\to\mathfrak{g}_2$, which preserves the symplectic forms, that is $\varphi^*\omega_2=\omega_1$.

Proposition 2.4. *Let* $\mathfrak{g}$ *be a symplectic real Lie algebra of dimension four. Then* $\mathfrak{g}$ *is symplectomorphically equivalent to one of the following Lie algebras equipped with a symplectic form as follows:*

$\mathfrak{rh}_3$: $\omega = e^1\wedge e^4 + e^2\wedge e^3$ $\quad\mathfrak{rr}_{3,0}$: $\omega = e^1\wedge e^2 + e^3\wedge e^4$
$\mathfrak{rr}_{3,-1}$: $\omega = e^1\wedge e^4 + e^2\wedge e^3$ $\quad\mathfrak{rr}_{3,0}'$: $\omega = e^1\wedge e^4 + e^2\wedge e^3$
$\mathfrak{r}_2\mathfrak{r}_2$: $\omega_\lambda = e^1\wedge e^2 + \lambda e^1\wedge e^3 + e^3\wedge e^4,\quad \lambda\geq 0$
$\mathfrak{r}_2'$: $\omega = e^1\wedge e^4 + e^2\wedge e^3$ $\quad\mathfrak{n}_4$: $\omega = e^1\wedge e^2 + e^3\wedge e^4$
$\mathfrak{r}_{4,0}$: $\omega_+ = e^1\wedge e^4 + e^2\wedge e^3,\quad \omega_- = e^1\wedge e^4 - e^2\wedge e^3$
$\mathfrak{r}_{4,-1}$: $\omega = e^1\wedge e^3 + e^2\wedge e^4$
$\mathfrak{r}_{4,-1,\beta}$: $\omega = e^1\wedge e^2 + e^3\wedge e^4,\quad -1\leq\beta<1$
$\mathfrak{r}_{4,\alpha,-\alpha}$: $\omega = e^1\wedge e^4 + e^2\wedge e^3,\quad -1<\alpha<0$
$\mathfrak{r}_{4,0,\delta}'$: $\omega_+ = e^1\wedge e^4 + e^2\wedge e^3,\quad \omega_- = e^1\wedge e^4 - e^2\wedge e^3,\quad \delta>0$
$\mathfrak{d}_{4,1}$: $\omega_1 = e^1\wedge e^2 - e^3\wedge e^4,\quad \omega_2 = e^1\wedge e^2 - e^3\wedge e^4 + e^2\wedge e^4$
$\mathfrak{d}_{4,2}$: $\omega_1 = e^1\wedge e^2 - e^3\wedge e^4$, $\omega_2 = e^1\wedge e^4 + e^2\wedge e^3$, $\omega_3 = e^1\wedge e^4 - e^2\wedge e^3$
$\mathfrak{d}_{4,\lambda}$: $\omega = e^1\wedge e^2 - e^3\wedge e^4,\quad \lambda\geq\frac{1}{2},\quad \lambda\neq 1,2$
$\mathfrak{d}_{4,\delta}'$: $\omega_+ = e^1\wedge e^2 - \delta e^3\wedge e^4,\quad \omega_- = -e^1\wedge e^2 + \delta e^3\wedge e^4,\quad \delta>0$
$\mathfrak{h}_4$: $\omega_+ = e^1\wedge e^2 - e^3\wedge e^4,\quad \omega_- = -e^1\wedge e^2 + e^3\wedge e^4$

Proof. For the proof one applies the definition making use of the authomorphisms of symplectic Lie algebras (see [23] for instance). As an example, for the case $\mathfrak{r}_2'$ the authomorphism given by $\sigma e_1 = e_1$, $\sigma e_2 = e_2+\alpha e_4$, $\sigma e_3 = \gamma e_3 - \beta e_4$, $\sigma e_4 = \beta e_3 + \gamma e_4$ does $\sigma^*(e^1\wedge e^4 + e^2\wedge e^3) = \alpha e^1\wedge e^2 + \beta(e^1\wedge e^3 - e^2\wedge e^4) + \gamma(e^1\wedge e^4 + e^2\wedge e^3)$ where $\beta^2+\gamma^2\neq 0$. Similar computations on each symplectic Lie algebra complete the proof. □

The following corollaries follow from the result above and they should be compared with the results in Section 4.

Corollary 2.5. *Let* $\mathfrak{g}$ *be a unimodular four dimensional solvable non abelian Lie algebra. Then* $\mathfrak{g}$ *is symplectic if and only if* $\mathfrak{g}$ *is isomorphic to* $\mathfrak{n}_4$ *or* $\mathfrak{g}$ *is isomorphic to a direct product of* $\mathbb{R}$ *and a three dimensional unimodular solvable Lie algebra.*

Remark 2.6. In [18] the authors proved that unimodular symplectic Lie algebras must be solvable.

Corollary 2.7. *Let $\mathfrak{g}$ be a non unimodular four dimensional solvable Lie algebra. If $\mathfrak{g}$ is symplectic then either:*

i) $\mathfrak{g}' \simeq \mathbb{R}$ *or*

ii) *if* $\mathfrak{g}' \simeq \mathbb{R}^2$ *then* $\mathfrak{g}$ *is isomorphic either to* $\mathfrak{r}_2\mathfrak{r}_2$ *or* $\mathfrak{r}_2'$*, or*

iii) $\mathfrak{g}' \simeq \mathfrak{h}_3$ *or*

iv) $\mathfrak{g}' \simeq \mathbb{R}^3$ *and the adjoint action of an element* $e_0 \notin \mathfrak{g}'$ *is equivalent to one of the following ones:*

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \quad \underset{-1 \leq \beta < 0}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \beta \end{pmatrix}} \quad \underset{-1 < \alpha < 0}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & -\alpha \end{pmatrix}} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \delta \\ 0 & -\delta & 0 \end{pmatrix}$$

3. Models for symplectic Lie algebras

In this section we describe symplectic four dimensional Lie algebras in terms of two basic constructions: either as solutions of the cotangent extension problem or as symplectic double extensions. In both models isotropic ideals, in particular lagrangian ones, play an important role.

Let $(\mathfrak{g}, \Omega)$ be a Lie algebra endowed with a non-degenerate skew-symmetric bilinear form. If $W \subset \mathfrak{g}$ is a subspace of $\mathfrak{g}$ then the orthogonal subspace $W^\perp$ is

$$W^\perp = \{x \in \mathfrak{g} \,/\, \Omega(x, y) = 0 \text{ for all } y \in W\}.$$

In particular it always holds that $\dim \mathfrak{g} = \dim W + \dim W^\perp$. The subspace W is called *isotropic* if $\Omega(W, W) = 0$ (that is $W \subset W^\perp$) and is called *lagrangian* if $W^\perp = W$. From now on the space generated by $e_1, \ldots, e_k$ will be denoted $\langle e_1, \ldots, e_k \rangle$.

It is easy to see that a subspace W is lagrangian if and only if W is isotropic and $\dim \mathfrak{g} = 2 \dim W$. Moreover, since Ω is closed, an isotropic ideal W must be abelian and $W^\perp$ is a subalgebra.

Lemma 3.1. *Let $\mathfrak{g}$ be a symplectic four dimensional Lie algebra, then $\mathfrak{g}$ always admits an isotropic ideal $\mathfrak{j}$. Moreover except for the Lie algebras $\mathfrak{rr}'_{3,0}$, $\mathfrak{r}'_{4,0,\delta}$, $\mathfrak{d}'_{4,\lambda}$ all other Lie algebras admit lagrangian ideals.*

Proof. For each symplectic four dimensional Lie algebra we will exhibit an isotropic ideal with respect to every symplectic form - see Proposition 2.2.

$$\begin{array}{llllllll}
\mathfrak{rh}_3 : & \langle e_3, e_4 \rangle, & \mathfrak{rr}_{3,-1} : & \langle e_2, e_4 \rangle & \mathfrak{rr}_{3,0} : & \langle e_2, e_3 \rangle, & \mathfrak{rr}'_{3,0} : & \langle e_4 \rangle, \\
\mathfrak{r}_2\mathfrak{r}_2 : & \mathfrak{g}', & \mathfrak{r}'_2 : & \mathfrak{g}' & \mathfrak{n}_4 : & \mathfrak{g}', & \mathfrak{r}_{4,0} : & \mathfrak{g}' \\
\mathfrak{r}_{4,-1} : & \langle e_1, e_2 \rangle, & \mathfrak{r}_{4,-1,-1} : & \langle e_2, e_3 \rangle & \mathfrak{r}_{4,-1,\beta} : & \langle e_1, e_2 \rangle, & \mathfrak{r}_{4,\alpha,-\alpha} : & \langle e_1, e_3 \rangle, \\
\mathfrak{r}'_{4,0,\delta} : & \langle e_1 \rangle, & \mathfrak{d}_{4,\lambda} : & \langle e_1, e_3 \rangle, & \mathfrak{d}'_{4,\delta} : & \langle e_3 \rangle, & \mathfrak{h}_4 : & \langle e_1, e_3 \rangle
\end{array}$$

Moreover $\mathfrak{j} \subset \mathfrak{g}' + \mathfrak{z}(\mathfrak{g})$ and is abelian. □

3.1. Cotangent extension problem

Let $\mathfrak{h}$ be a Lie algebra and let $\mathfrak{h}^*$ be the dual vector space of $\mathfrak{h}$, consider $\mathfrak{h}^* \oplus \mathfrak{h}$ as a vector space and let ω_0 be the skew-symmetric two form defined as:

$$\omega_0((\varphi, x), (\varphi', x')) = \varphi(x') - \varphi'(x) \tag{1}$$

The *cotangent extension problem* consists in finding a Lie algebra structure on $\mathfrak{h}^* \oplus \mathfrak{h}$ which satisfies the following conditions:

(c1) $0 \longrightarrow \mathfrak{h}^* \longrightarrow \mathfrak{h}^* \oplus \mathfrak{h} \longrightarrow \mathfrak{h} \longrightarrow 0$ is an exact sequence of Lie algebras, $\mathfrak{h}^*$ endowed with its abelian Lie algebra structure;

(c2) the left invariant 2-form spanned by ω_0 is closed.

A symplectic Lie algebra $(\mathfrak{g}, \omega)$ is said to be a solution of the cotangent extension problem if $\mathfrak{g}$ is symplectomorphically equivalent to a Lie algebra of the form $(\mathfrak{h}^* \oplus \mathfrak{h}, \omega_0)$ satisfying conditions (c1) and (c2). Thus $\mathfrak{g}$ is an extension of the Lie algebra $\mathfrak{h}$.

In what follows we describe conditions to get solutions of the cotangent extension problem. Let $(\mathfrak{h}, [\,,\,]_{\mathfrak{h}})$ be a Lie algebra and $\rho : \mathfrak{h} \to \mathrm{End}(\mathfrak{h}^*)$ be a representation of $\mathfrak{h}$ on the dual space of left invariant 1-forms of $\mathfrak{h}$, denoted $\mathfrak{h}^*$. Thus $\mathfrak{h}^*$ inherits a structure of $\mathfrak{h}$-module, denoted $x.\varphi = \rho(x)\varphi$.

Let $\mathfrak{g}$ be the direct sum of the vector spaces $\mathfrak{h}^* \oplus \mathfrak{h}$ and define a skew-symmetric map on $\mathfrak{g}$, $[\,,\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ by:

$$[\varphi, \eta] = \varphi,\ \eta \in \mathfrak{h}^* \quad [\varphi, x] = -x.\varphi\, x \in \mathfrak{h},\ \varphi \in \mathfrak{h}^* \quad [x, y] = [x, y]_{\mathfrak{h}} + \alpha(x, y)\, x,\ y \in \mathfrak{h}$$

where α is a 2-cochain. Then $[,]$ defines a structure of Lie algebra on $\mathfrak{g}$ if and only if α is a 2-cocycle, $\alpha \in Z^2(\mathfrak{h}, \mathfrak{h}^*)$, that is

$$\begin{aligned}&\alpha([x_1, x_2]_{\mathfrak{h}}, x_3) + \alpha([x_2, x_3]_{\mathfrak{h}}, x_1) + \alpha([x_3, x_1]_{\mathfrak{h}}, x_2)\\ &\quad = x_3.\alpha(x_1, x_2) + x_1.\alpha(x_2, x_3) + x_2.\alpha(x_3, x_1).\end{aligned} \tag{2}$$

In this situation $\mathfrak{g}$ is an extension of $\mathfrak{h}$ and we have the following exact sequence of Lie algebras

$$0 \longrightarrow \mathfrak{h}^* \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{h} \longrightarrow 0.$$

The two-form ω_0 on $\mathfrak{g}$ defined by

$$\omega_0((\varphi_1, x_1), (\varphi_2, x_2)) = \varphi_1(x_2) - \varphi_2(x_1)$$

is closed if and only if

$$\alpha(x_1, x_2)(x_3) + \alpha(x_2, x_3)(x_1) + \alpha(x_3, x_1)(x_2) = 0 \tag{3}$$

for all $x_1, x_2, x_3 \in \mathfrak{h}$ and

$$x.\varphi(y) - y.\varphi(x) = \varphi([x, y]_{\mathfrak{h}}). \tag{4}$$

The condition (3), known as the "Bianchi identity", is equivalent to say that α belongs to the kernel of the canonical map $(\Lambda^2\mathfrak{h} \otimes \mathfrak{h} \to \Lambda^3\mathfrak{h})$, which is $\mathbb{S}_{(2,1)}(\mathfrak{h})$ the Weyl space corresponding to the partition 3=2+1 (see [14]).

Then the resulting Lie algebra $\mathfrak{g}$ (attached to the triple $(\mathfrak{h}, \rho, [\alpha])$) satisfying (2), (3) and (4) is a solution of the cotangent extension problem.

Remark 3.2. The coadjoint representation satisfies (4) if and only if $\mathfrak{h}$ is abelian. Thus ω_0 spans a symplectic structure on the cotangent bundle T^*H of a Lie group H, endowed with its canonical Lie group structure, if and only if H is abelian.

Remark 3.3. If one defines a connection ∇ on $\mathfrak{h}$ by

$$\varphi(\nabla_x y) = -x.\varphi(y) \qquad (\text{so } \varphi \circ \nabla_x = -x.\varphi) \quad x, y \in \mathfrak{h},\ \varphi \in \mathfrak{h}^*$$

then (4) is equivalent to the fact that the connection is torsion free (see [4]).

Remark 3.4. Observe that $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$ if and only if $\alpha = 0$ and in this case $\mathfrak{g}$ is the semidirect product of $\mathfrak{h}$ and $\mathfrak{h}^*$. Moreover ω_0 is closed if and only if (4) is verified. This case was studied by Boyom in [4].

Remark 3.5. If the representation ρ is trivial, then (2) becomes

$$\alpha([x_1, x_2]_{\mathfrak{h}}, x_3) + \alpha([x_2, x_3]_{\mathfrak{h}}, x_1) + \alpha([x_3, x_1]_{\mathfrak{h}}, x_2) = 0$$

and ω_0 is closed if and only if (3) holds and (4) becomes $\varphi([x, y]_{\mathfrak{h}}) = 0$. So $\mathfrak{h}$ must be abelian (direct sum of vector spaces) and $\mathfrak{g}$ is a two-step nilpotent Lie algebra with Lie bracket $[,]$ defined by $[x, y] = \alpha(x, y) \quad x,\ y \in \mathfrak{h}$, therefore $\mathfrak{h}' = \operatorname{Im}\alpha$. The 2-form ω_0 is closed if and only if (3) holds.

The previous explanation proofs the first assertion of the following theorem. The second assertion relates Lie algebras having a lagrangian ideal with solutions of the cotangent extension problem. We include the proof, which is useful for our purposes. However similar results are known in a more general context (see (3.7)).

Theorem 3.6. i) *Let $\mathfrak{g}$ be a $2n$-dimensional Lie algebra with an abelian ideal $\mathfrak{h}^*$ of dimension n (so we have* (c1)*). Then $\mathfrak{g}$ is a solution of the cotangent extension problem, if and only if conditions* (3) *and* (4) *are satisfied.*

ii) *Let $(\mathfrak{g}, \omega)$ be a symplectic Lie algebra with a lagrangian ideal. Then $\mathfrak{g}$ is a solution of the cotangent extension problem.*

Proof. ii) Let $\mathfrak{j}$ be a lagrangian ideal, then one has the following exact sequences of Lie algebras:

$$0 \longrightarrow \mathfrak{j} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{j} \longrightarrow 0.$$

Let $\mathfrak{h} := \mathfrak{g}/\mathfrak{j}$ be the quotient Lie algebra. Since $\mathfrak{j}$ is abelian it can be identified with $\mathfrak{h}^*$, the last one endowed with the abelian Lie algebra structure. Let $\mathfrak{g} = \mathfrak{j} \oplus \mathfrak{v}$ be a splitting into lagrangian subspaces (this always exists, see [24] Lect. 2), then the map $\beta : \mathfrak{j} \to (\mathfrak{h})^*$ given by $\beta(x)(y + \mathfrak{j}) = \omega(x, y)$ for $x \in \mathfrak{j}$, $y + \mathfrak{j} \in h = \mathfrak{g}/\mathfrak{j}$, induces an isomorphism $\tilde{\beta} : \mathfrak{j} \to \mathfrak{v}^*$. Giving $\mathfrak{h}^* \oplus \mathfrak{h}$ the Lie algebra structure via the isomorphism $1 \oplus \tilde{\beta} : \mathfrak{j} \oplus \mathfrak{v} \to \mathfrak{h}^* \oplus \mathfrak{h}$, one may check that $1 \oplus \tilde{\beta}$ is a symplectomorphism from $(\mathfrak{g}, \omega)$ to $(\mathfrak{h}^* \oplus \mathfrak{h}, \omega_0)$, completing the proof of the second assertion of the theorem. □

Remark 3.7. It is known that every lagrangian foliation is locally symplectomorphic to the foliation of $\mathbb{R}^{2n}$ by the manifolds x_i =constant, and that the leaves of a lagrangian foliation carry a natural flat torsion free affine connection (see [25]).

Definition 3.8. *Two solutions of the cotangent extension problem $\mathfrak{g}_1$ and $\mathfrak{g}_2$ resulting as extensions of $\mathfrak{h}_1$ and $\mathfrak{h}_2$ respectively are said to be equivalent if there exists a isomorphism $\psi : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that $\psi\mathfrak{h}_1 = \mathfrak{h}_2$.*

Theorem 3.9. *The following tables show all solutions $\mathfrak{g}$ up to equivalence of the cotangent extension problem for $\mathfrak{h}$ of dimension two.*

The elements of $H^2_\rho(\mathfrak{h}, \mathbb{R}^2)$ are denoted by α. The basis of $\mathfrak{h}$ is $\{x, y\}$ and for $\mathfrak{h}^*$ we choose the corresponding dual basis. The symbol (*) indicates that the Lie algebra obtained as an extension of $\mathfrak{h}$, via ρ and α, is a solution of the cotangent extension problem.

Representation ρ	$H^2_\rho(\mathbb{R}^2, \mathbb{R}^2)$	$\mathfrak{g}$
$\rho \equiv 0$	2	$\begin{cases} \mathbb{R}^4 & \alpha = 0(*) \\ \mathfrak{rh}_3 & \alpha \neq 0(*) \end{cases}$
$\rho(x) = 0 \quad \rho(y) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$	1	$\begin{cases} \mathfrak{rr}_{3,0} & \alpha = 0(*) \\ \mathfrak{r}_{4,0} & \alpha \neq 0(*) \end{cases}$
$\rho(x) = 0 \quad \rho(y) = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$ $\lambda \neq 0$	0	$\mathfrak{rr}_{3,\lambda}$
$\rho(x) = 0 \quad \rho(y) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$	1	$\begin{cases} \mathfrak{rh}_3 & \alpha = 0(*) \\ \mathfrak{n}_4 & \alpha \neq 0(*) \end{cases}$
$\rho(x) = 0 \quad \rho(y) = \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix}$ $\lambda \neq 0$	0	$\mathfrak{rr}_3$
$\rho(x) = 0 \quad \rho(y) = \begin{pmatrix} \gamma & 1 \\ -1 & \gamma \end{pmatrix}$	0	$\mathfrak{rr}'_{3,\gamma}$
$\rho(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \rho(y) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$	0	$\mathfrak{r}_2\mathfrak{r}_2 \quad (*)$
$\rho(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \rho(y) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	0	$\mathfrak{d}_{4,1} \quad (*)$
$\rho(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \rho(y) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	0	$\mathfrak{r}'_2 \quad (*)$

Table 3.1 for $\mathfrak{h} = \mathbb{R}^2$

Representation ρ	$H^2_\rho(\mathfrak{aff}(\mathbb{R}),\mathbb{R}^2)$	$\mathfrak{g}$
$\rho \equiv 0$	0	$\mathfrak{rr}_{3,0}(*)$
$\rho(y)=0 \quad \rho(x)=\begin{pmatrix}0&0\\0&\lambda\end{pmatrix}$ $\lambda\in[-1,1]$	0	$\mathfrak{rr}_{3,\lambda}(**)$
$\rho(y)=0 \quad \rho(x)=\begin{pmatrix}\mu&0\\0&\lambda\end{pmatrix}$ $-1<\mu\le\lambda, \mu\lambda\neq 0 \quad \text{or} \quad -1=\mu\le\lambda\le 0$	0	$\mathfrak{r}_{4,\mu,\lambda}(***)$
$\rho(y)=0 \quad \rho(x)=\begin{pmatrix}\mu&1\\0&\mu\end{pmatrix}$	0	$\mathfrak{r}_{4,\mu}$
$\rho(y)=0 \quad \rho(x)=\begin{pmatrix}\gamma&\delta\\-\delta&\gamma\end{pmatrix}$ $\gamma\in\mathbb{R},\ \delta>0$	0	$\mathfrak{r}'_{4,\gamma,\delta}$
$\rho(y)=\begin{pmatrix}0&0\\1&0\end{pmatrix} \quad \rho(x)=\begin{pmatrix}1&0\\0&2\end{pmatrix}$	1	$\begin{cases}\mathfrak{d}_{4,1/2} & \alpha=0(*)\\ \mathfrak{h}_4 & \alpha\neq 0(*)\end{cases}$
$\rho(y)=\begin{pmatrix}0&0\\1&0\end{pmatrix} \quad \rho(x)=\begin{pmatrix}-1&0\\0&0\end{pmatrix}$	0	$\mathfrak{d}_4$
$\rho(y)=\begin{pmatrix}0&0\\1&0\end{pmatrix} \quad \rho(x)=\begin{pmatrix}a-1/2&0\\0&a+1/2\end{pmatrix}$ $a\neq 3/2,-1/2$	0	$\mathfrak{d}_{4,\frac{1}{a+1/2}}(*)$

Table 3.1 for $\mathfrak{h}=\mathfrak{aff}(\mathbb{R})$

$(**)$ solution for $\lambda=-1$.

$(***)$ solution for (μ,λ) of the form $(-1,-1)$ $(-1,\lambda),(\mu,-\mu)$ with restrictions of Proposition 2.2.

Remark 3.10. In [2] it was considered a special case of the sequence of Lie algebras (c1), that is when $\mathfrak{h}$ is a two dimensional Lie algebra and the exact sequence (c1) splits, so that $\mathfrak{g}$ is a semidirect product of $\mathfrak{h}$ and $\mathbb{R}^2$.

Proof. To construct the tables, we make use of the information given in [2] to get all semidirect extensions of $\mathfrak{h}$. The idea of this proof is to study the image of ρ in $\mathfrak{gl}(2,\mathbb{R})$. Thus if $\mathfrak{h}\simeq\mathbb{R}^2$, then the image of ρ can be described in terms of the following subalgebras of $\mathfrak{gl}(2,\mathbb{R})$:

$$\left\{\begin{pmatrix}a&0\\0&b\end{pmatrix}:a,b\in\mathbb{R}\right\},\quad \left\{\begin{pmatrix}a&1\\0&a\end{pmatrix}:a\in\mathbb{R}\right\},\quad \left\{\begin{pmatrix}a&b\\b&-a\end{pmatrix}:a,b\in\mathbb{R}\right\}.$$

If the image of ρ is one dimensional, then $\mathfrak{g}$ is an extension of a three dimensional Lie algebra. If the image of ρ is two dimensional, then $\mathfrak{g}$ is either isomorphic to $\mathfrak{r}_2\mathfrak{r}_2$ or to $\mathfrak{r}'_2$.

If $\mathfrak{h}$ is isomorphic to $\mathfrak{aff}(\mathbb{R})$, then we may assume that $\mathfrak{h}=\langle x,y\rangle$, with $[x,y]=y$. Thus if ρ is trivial then $\mathfrak{g}\simeq\mathfrak{rr}_{3,0}$. If the image of ρ is one dimensional then $\rho(y)=0$ and $\rho(x)$ acts on $\mathfrak{h}^*$ as follows:

$$\left\{\begin{pmatrix}\mu&0\\0&\lambda\end{pmatrix}:\mu,\lambda\in\mathbb{R},\lambda\neq 0\right\},\ \left\{\begin{pmatrix}\mu&0\\1&\mu\end{pmatrix}:\mu\in\mathbb{R}\right\},\ \left\{\begin{pmatrix}\gamma&\delta\\\delta&-\gamma\end{pmatrix}:\gamma,\delta\in\mathbb{R},\delta\neq 0\right\}.$$

In these cases we get the first part of the second table. If the image of ρ is two dimensional, then one may assume that

$$\rho(y) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and $\rho(x)$ takes the following form

$$\rho(x) = \begin{pmatrix} \alpha + 1/2 & 0 \\ 0 & \alpha - 1/2 \end{pmatrix}$$

and so we get all semidirect extensions. To complete the proof we need to compute the second cohomology group and to determine the resulting Lie algebra. □

Theorem 3.11. *Let $\mathfrak{g}$ be four dimensional Lie algebra; if $\mathfrak{g}$ is a solution of the cotangent extension problem then $\mathfrak{g}$ is either symplectic completely solvable or isomorphic to $\mathfrak{aff}(\mathbb{C})$.*

Proof. A first proof follows by reading the results of the previous tables. A second proof is obtained as follows: by Lemma 3.1 there always exists a abelian lagrangian ideal $\mathfrak{j}$ on any symplectic completely solvable four dimensional Lie algebra. The proof will be completed be applying Theorem 3.6. □

Remark 3.12. The symplectic four dimensional Lie algebras $\mathfrak{rr}'_{3,0}, \mathfrak{r}'_{4,0,\delta}$ do not admit lagrangian ideals however they are semidirect products of two symplectic 2-dimensional Lie algebras. But $\mathfrak{d}'_{4,\delta}$ cannot be written as a semidirect product of 2-dimensional Lie algebras. It was proved in [2] that this Lie algebra does not admit a decomposition as a direct sum of two 2-dimensional subalgebras (sum as vector spaces).

3.2. Symplectic double extensions

[21],[9] In this section we modelize some four dimensional symplectic Lie algebras as symplectic double extensions in a classical or in a generalized sense. To this end we recall the main ideas of these constructions and we remit to the papers of Medina A. and Revoy P. [21] or Dardié J. and Medina A. [9] for the details.

Let (B, ω') be a symplectic Lie algebra, let δ be a derivation of B and let $z \in$ B. Let $I = \mathbb{R}e \oplus B$ be the central extension of B by $\mathbb{R}e$ defined by

$$[a,b]_I = [a,b]_B + \omega'(\delta a, b)e \qquad a, b \in B$$

where $[\,,\,]_B$ denotes the Lie bracket on B. Let A be the semidirect product of I by $\mathbb{R}d$ given by

$$[d,e] = 0 \qquad [d,a] = -\omega'(z,a)e - \delta(a) \qquad a \in B.$$

Extending the symplectic structure of B to A via ω, defined as $\omega(e,d) = 1$, and $\omega(B,e) = 0 = \omega(B,d)$, then A is said to be a symplectic double extension of (B, ω') by $\mathbb{R}$.

Theorem 3.13. [21] *Let (A, ω) be a symplectic $2n$-dimensional Lie algebra with non trivial center. Then A is a symplectic double extension by $\mathbb{R}$ of a $2n-2$-dimensional symplectic Lie algebra (B,ω').*

This result motivates the following definition in [9].

Definition 3.14. *A symplectic Lie algebra $(\mathfrak{g}, \omega)$ is called a symplectic double extension of a symplectic Lie algebra (W, ω') if there exists a central one dimensional subalgebra $\mathfrak{j} \subset \mathfrak{g}$ such that the reduced symplectic Lie algebra $\mathfrak{j}^\perp/\mathfrak{j}$ is isomorphic to (W, ω').*

Note that in this case $\mathfrak{j}^\perp$ is an ideal on $\mathfrak{g}$, but this is not true in general for every isotropic ideal on $\mathfrak{g}$.

Corollary 3.15. *If $\mathfrak{g}$ is a symplectic Lie algebra isomorphic to either $\mathfrak{rh}_3$, or $\mathfrak{rr}_{3,0}$ or $\mathfrak{rr}_{3,-1}$ or $\mathfrak{rr}'_{3,0}$ or $\mathfrak{n}_4$ or $\mathfrak{r}_{4,0}$, then $\mathfrak{g}$ is a symplectic double extension of $\mathbb{R}^2$.*

Proof. Since these symplectic Lie algebras have non trivial center, the assertion follows from the previous theorem. □

Let $\mathfrak{h}$ be a central one dimensional subalgebra of a symplectic Lie algebra $\mathfrak{g}$, then the following exact sequences of Lie algebras describe central extensions of Lie algebras.

$$0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{h}^\perp \longrightarrow \mathfrak{h}^\perp/\mathfrak{h} \longrightarrow 0 \tag{5}$$

$$0 \longrightarrow \mathfrak{h}^\perp \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{h}^\perp \longrightarrow 0 \tag{6}$$

$$0 \longrightarrow \mathfrak{h}^\perp/\mathfrak{h} \longrightarrow \mathfrak{g}/\mathfrak{h} \longrightarrow \mathfrak{g}/\mathfrak{h}^\perp \longrightarrow 0 \tag{7}$$

$$0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{h} \longrightarrow 0 \tag{8}$$

Since $\dim \mathfrak{h} = 1$ then (7) and (8) describe semidirect products of Lie algebras.

For the symplectic Lie algebra $\mathfrak{r}'_{3,0}$, let $\mathfrak{h}$ be the central ideal spanned by e_4, then $\mathfrak{h}^\perp =< e_2, e_3, e_4 >$. The first exact sequence (5) describe a trivial central extension, that is, $\mathfrak{h}^\perp$ is the extension of $\mathbb{R}^2 =< e_2, e_3 >$ by $\mathbb{R}$ defined by the zero class in $Z^2(\mathfrak{h}^\perp/\mathfrak{h}, \mathbb{R})$. The second exact sequence (6) describe a semidirect product of Lie algebras, by the action of $\mathbb{R}$ on $\mathfrak{h}^\perp$ via $\mathrm{ad} e_1$.

In order to give a model for the symplectic Lie algebras $\mathfrak{r}'_{4,0}$ and $\mathfrak{d}'_{4,\delta}$ for $\delta > 0$, we make use of generalized symplectic Lie algebras [9]. A generalized symplectic Lie algebra $\mathfrak{a}$ admits a decomposition $V^* \oplus B \oplus V$ as vector spaces, where (B, ω') is a symplectic Lie algebra, V^* is the dual vector space of V, which is a the underlying Lie algebra of a left symmetric algebra and such that there exists a representation $\Gamma : V \to Der(B)$, a cocycle $\varphi \in Z^2_{S.G.}(V, V^*)$ and a symmetric bilinear form $f : V \times V \to B$, satisfying some extra conditions. These conditions assert that $\mathfrak{a}$ is a symplectic Lie algebra endowed with the symplectic structure $\omega' + \omega_0$ where ω_0 is the canonical symplectic form on $V^* \oplus V$ (see Theorem 2.3 in [9]). We remit to this paper for more explanation. We will apply the following result of Dardié and Medina [9] to characterize some four dimensional symplectic Lie algebras.

Theorem 3.16. *Let $(\mathfrak{g}, \omega)$ be a symplectic Lie algebra and let $\mathfrak{h}$ be a isotropic ideal of $\mathfrak{g}$. If $\mathfrak{h}^\perp$ is a ideal of $\mathfrak{g}$ and if the following exact sequence of Lie algebras*

$$0 \longrightarrow \mathfrak{h}^\perp/\mathfrak{h} \longrightarrow \mathfrak{g}/\mathfrak{h} \longrightarrow \mathfrak{g}/\mathfrak{h}^\perp \longrightarrow 0 \tag{9}$$

splits, then $\mathfrak{g}$ is a generalized symplectic double extension of the reduced symplectic Lie algebra $\mathfrak{h}^\perp/\mathfrak{h}$ by the Lie algebra $\mathfrak{g}/\mathfrak{h}^\perp$.

The exact sequences (5), (6), (7), (8) characterize the generalized symplectic double extension and Lie algebras having a lagrangian ideal are examples of generalized symplectic double extensions.

Proposition 3.17. *Let $\mathfrak{g}$ be a symplectic Lie algebra isomorphic either to $\mathfrak{r}'_{4,0,\delta}$ or $\mathfrak{d}'_{4,\delta}$ $\delta > 0$, then $\mathfrak{g}$ is a generalized symplectic double extension of $\mathbb{R}^2$ by $\mathbb{R}$.*

Proof. In the cases $\mathfrak{r}'_{4,0,\delta}$, $\mathfrak{d}'_{4,\lambda}$, let $\mathfrak{h}$ be the isotropic ideal generated by e_1 in the first case and by e_3 in the second one. Then the orthogonal subspace $\mathfrak{h}^\perp$ is the ideal $\mathfrak{h}^\perp =< e_1, e_2, e_3 >$ in both cases. It is easy to see that $\mathfrak{h}^\perp/\mathfrak{h}$ is abelian and two dimensional. Thus one has the exact sequences of Lie algebras:

$$0 \to \mathfrak{h} \to \mathfrak{h}^\perp \to \mathfrak{h}^\perp/\mathfrak{h} \to 0 \tag{10}$$

$$0 \to \mathfrak{h}^\perp \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}^\perp \to 0 \tag{11}$$

where $\mathfrak{b} := \mathfrak{h}^\perp/\mathfrak{h}$ is an abelian two dimensional Lie algebra endowed with a symplectic structure induced from $\mathfrak{g}$. The first sequence (10) is defined by the cohomology class of a 2-cocycle $\varphi \in Z^2(\mathfrak{b}, \mathbb{R})$. If the commutator of $\mathfrak{g}$ is abelian then φ is trivial and thus $\mathfrak{h}^\perp$ is a direct product of $\mathfrak{h}$ and $\mathfrak{b}$. If $\mathfrak{g}$ is isomorphic to $\mathfrak{d}'_{4,\lambda}$, then φ is no trivial. Thus for $\mathfrak{b}$ the abelian two dimensional Lie algebra generated by e_1 and e_2, the Lie bracket on $\mathfrak{h}^\perp = \mathfrak{b} \oplus \mathbb{R}e_3$ is given by $[e_1, e_2] = \varphi(e_1, e_2)e_3$. The second sequence (11) splits (see [2]) and so we prove that $\mathfrak{g}$ is a generalized symplectic double extension. $\square$

4. On Lie algebras do not admitting symplectic structures

Motivated by the four dimensional case, we search for Lie algebras do not admitting symplectic structures. As before the antiderivation operator in $\Lambda(\mathfrak{g}^*)$ will be denoted d throughout this section.

The Heisenberg Lie algebra of dimension $2n+1$, denoted $\mathfrak{h}_{2n+1}$, is generated by elements e_i $i = 1, \ldots, 2n+1$, with the relations $[e_i, e_{i+1}] = e_{2n+1}$, $i = 2k+1$, $k = 0, \ldots, n-1$. Using this Lie bracket relations the following lemma follows.

Lemma 4.1. *Let D be a derivation of the Heisenberg Lie algebra h_{2n+1}. Then, in the basis $e_1, \ldots, e_{2n+1}$, the matrix of D is*

$$\begin{pmatrix} A & * \\ 0 & \lambda \end{pmatrix} \qquad A = (a_{ij}) \in gl_n(\mathbb{R}),\ a_{i+1,i+1} = \lambda - a_{i,i},\ i = 2k+1,\ k = 0, \ldots, n-1.$$

It follows that $\lambda = \mathrm{tr}D/n$. In particular the unimodular extensions of $\mathfrak{h}_{2n+1}$ are those such that $\lambda = 0$.

Observe that changing D by $\tilde{D} := D - \sum_{i=1}^{2n} a_{i,2n+1}e_i$ we get a new derivation of $\mathfrak{h}_{2n+1}$ of the form,

$$\begin{pmatrix} A & 0 \\ 0 & \lambda \end{pmatrix} \quad A=(a_{ij})\in gl_n(\mathbb{R}),\ a_{i+1,i+1}=\lambda - a_{i,i},\ i=2k+1,\ k=0,\dots,n-1. \quad (12)$$

The extended Lie algebras resulting as semidirect products of the Heisenberg Lie algebra by $\mathbb{R}$, using D or $\tilde{D}$, are isomorphic.

Proposition 4.2. *Let $\mathfrak{g} = \mathbb{R}e_0 \ltimes \mathfrak{h}_{2n+1}$ be a unimodular extension of the Heisenberg Lie algebra $\mathfrak{h}_{2n+1}$ such that, A as in* (12)*, belongs to* $\mathrm{GL}_n(\mathbb{R})$*. Then $\mathfrak{g}$ does never admit a symplectic structure.*

Proof. It holds $de^0 = 0$ and if $A \in \mathrm{GL}_n$ then $\{z_j = Ae_j\}$, $j = 1, \dots, 2n$, is a basis of the subspace spanned by $e_1, \dots, e_{2n}$. One can see that $d(e^0 \wedge e^{2n+1}) \neq 0$ and $d(z^j \wedge e^{2n+1}) \neq 0$. Therefore if ω is a closed two-form then $\omega \in \Lambda^2(W^*)$ where W is the subspace of $\mathfrak{g}$ generated by e_i, $i = 1, \dots, 2n$ and this implies $\omega^{n+1} = 0$. $\square$

Remark 4.3. Let $\mathfrak{g}$ be a Lie algebra as in the previous proposition. Then by computing one can see that $H_{2n+2}(\mathfrak{g}) \neq 0$ and however $H^2(\mathfrak{g})$ does not necessarily vanishes. In fact, for instance the derivation D of $\mathfrak{h}_5$ given by $De_1 = e_1$, $De_e = -e_2$ $De_3 = -e_3$ and $De_4 = e_4$ proves the last assertion.

The following result is known and the proof can be achieved by canonical computations.

Proposition 4.4. *Let $\mathfrak{g}$ be a trivial extension of the Heisenberg Lie algebra $\mathfrak{h}_{2n+1}$. Then $\mathfrak{g}$ is symplectic if and only if $n = 1$.*

Proposition 4.5. *Let $\mathfrak{g}$ be a semidirect product of $\mathbb{R}e_0$ and the $2n-1$-dimensional abelian ideal. Thus*

i) *if ad_{e_0} diagonalizes and the eigenvalues of ad_{e_0} satisfy $\lambda_i + \lambda_j \neq 0$ for all i,j. Then $\mathfrak{g}$ cannot be equipped with a symplectic structure.*

ii) *If $\mathrm{ad}_{e_0} e_i = e_{i-1}$, $2 \leq i \leq 2n-1$. Then $\mathfrak{g}$ is symplectic if and only if $n = 2$.*

Proof. i) follows from direct computations.

To prove ii) notice that if $n = 2$ then (2.2) proves that $\mathfrak{g} \simeq \mathfrak{n}_4$ is symplectic and if $n \geq 3$ then one verifies that any closed two-form ω satisfies $\omega^3 = 0$, hence $\mathfrak{g}$ cannot be symplectic. $\square$

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Received December 30, 2004