Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 47 (2006), No. 2, 527-541.

The Betten-Walker Spread and Cayley's Ruled Cubic Surface

Unserem lieben Freund Dieter Betten zum 65. Geburtstag gewidmet

Hans Havlicek Rolf Riesinger

Institut für Diskrete Mathematik und Geometrie, Technische Universität Wien
Wiedner Hauptstraße 8–10/104, A-1040 Wien, Austria.
e-mail: havlicek@geometrie.tuwien.ac.at

Patrizigasse 7/14, A-1210 Wien, Austria.
e-mail: rolf.riesinger@chello.at

Abstract. We establish that, over certain ground fields, the set of osculating tangents of Cayley's ruled cubic surface gives rise to a (maximal partial) spread which is also a dual (maximal partial) spread. It is precisely the Betten-Walker spreads that allow for this construction. Every infinite Betten-Walker spread is not an algebraic set of lines, but it turns into such a set by adding just one pencil of lines.

MSC 2000: 51A40, 51M30, 14J26
Keywords: Cayley's ruled cubic surface, osculating tangents, maximal partial spread, maximal partial dual spread, algebraic set of lines

1. Introduction

1.1 In this paper we deal with a spread which was discovered independently, and approximately at the same time, by D. Betten [2] and M. Walker [28].

Betten used the concept of a transversal homeomorphism in order to describe and classify topological translation planes in terms of partitions of the vector space $\mathbb{R}^4$ into 2-dimensional subspaces. What we call the Betten-Walker spread (BW-spread) is described in [2, Satz 3]. Betten's paper contains also a short remark that the construction of this spread works also for finite fields of characteristic $\neq 3$

without a primitive third root of unity [2, pp. 338–339]. We refer to [15] or [21] for the connection between spreads and translation planes; it is due to J. André and was found independently by R. H. Bruck and R. C. Bose.

Walker adopted the projective point of view, which leads to spreads of lines in a projective 3-space. He focussed on the case of a finite ground field $\mathrm{GF}(q)$, $q \equiv -1 \pmod 6$, and on the reguli contained in the spread. Thereby he laid the cornerstone for a concept which is now called the *Thas-Walker construction*. It links spreads with flocks of quadrics via the Klein mapping.

The BW-spread corresponds to a flock of a quadratic cone. In the finite case this flock is due to C. Fisher and J. A. Thas, who weakened Walker's condition $q \equiv -1 \pmod 6$. We refer to [25, pp. 334–338] for further details. Some authors use the term "FTW-spread" for a finite BW-spread.

The BW-spread and its corresponding translation plane were revisited by A. G. Spera [23]. The comprehensive paper by V. Jha and N. L. Johnson [13] (which should be read together with its second part [14]) contains more information about the BW-spread and its associated flock. In both papers the existence of the BW-spread is established for an arbitrary ground field K with characteristic $\neq 3$ subject to the condition that each element of K has precisely one third root in K. We add in passing that the BW-spread is among the "likeable structures" of W. M. Kantor; see [7].

Finally, there is a neat connection, found by J. A. Thas, between flocks of quadratic cones over finite fields and certain generalized quadrangles; see [25, p. 334], [26], and the references given there. The infinite case was treated by F. De Clerck and H. Van Maldeghem [5]. However, this connection with generalized quadrangles is beyond the scope of the present paper. Let us just add the following remark: In the finite case, the BW-spread corresponds to a generalized quadrangle discovered by W. M. Kantor; cf. [24, p. 398]. Thus some authors speak of the "FTWKB generalized quadrangle" in order to bring together the names of all the involved mathematicians; see, for example, [18, p. 222].

1.2 One aim of the present note is to present a short, direct, and self-contained approach to the BW-spread, thereby establishing a connection with an algebraic surface which was discovered already in the 19th century, namely *Cayley's ruled cubic surface*. According to [17, p. 181] this name is not completely appropriate, since M. Chasles published his discovery of that surface in 1861, three years before A. Cayley.

Our starting point is a Cayley surface F, say, in the projective 3-space over an arbitrary field K. At each simple point of F there is a unique osculating tangent other than a generator. The set of all those osculating tangents, together with one particular line on F, gives then a set of lines, say $\mathcal{O}$, which easily turns out to be a spread if K satisfies the conditions mentioned above (characteristic $\neq 3$, each element of K has precisely one third root in K). Moreover, when "precisely" is replaced with "at most" then $\mathcal{O}$ is a maximal partial spread; see Theorem 3.3 and cf. [13, Theorem 6.3]. By our approach the maximality of such a partial spread follows from the observation that all points of a distinguished plane are incident with a line of the partial spread.

By a classical result, there exists a duality which maps the Cayley surface (as a set of points) onto the set of its tangent planes. Any mapping of this kind fixes $\mathcal{O}$, as a set of lines. Therefore, all our results hold together with their dual counterparts. So, depending on the ground field, $\mathcal{O}$ will be a (maximal partial) spread and at the same time a (maximal partial) dual spread.

In case of characteristic three there is a line of nuclei for the Cayley surface. The existence of that line was noted by M. de Finis and M. J. de Resmini [6] without giving a geometric interpretation. We show in Theorem 4.7 that the line of nuclei is the axis of a parabolic congruence which contains all lines of $\mathcal{O}$.

1.3 The transversal homeomorphism used in [2] to describe the BW-spread is given in terms of polynomial functions. This raises the question whether or not the BW-spread is *algebraic*, i.e., its image under the Klein mapping is an algebraic variety. In the finite case every set of points is an algebraic variety (by [11, Lemma 3.5 (a)], even a hypersurface), whence we exclude that case from our investigation. On the other hand, infinite algebraic spreads seem to be rare. The only examples known to the authors are the regular spreads (or, in other words, the elliptic linear congruences) and some spreads found by the second author; see [19, Table 1]. Unfortunately, our hope to find another example of an algebraic spread did not come true. However, the BW-spread is very close to being algebraic. We establish in Theorem 4.3 that the union of the BW-spread and one pencil of lines is algebraic. More precisely, the Klein image of that set is the smallest algebraic variety containing the Klein image of the BW-spread (Theorem 4.5). When looking for equations describing that variety (in terms of Plücker coordinates) the thesis of R. Koch [16] turned out extremely useful, even though we could not directly implement his results in our work. It is worth mentioning that the BW-spread (over $\mathbb{R}$) is ubiquitous in Koch's thesis under the German name "Schmiegtangentenkongruenz" (congruence of osculating tangents), but the property of being a spread never seems to be mentioned in the text. Likewise, the authors were unable to find a remark on this property in the older literature on the Cayley surface.

2. The Cayley surface

2.1 We consider the three-dimensional projective space $\mathbb{P}_3(K)$ over a commutative field K. As we shall use column vectors, a point has the form $K\boldsymbol{p}$ with $(0,0,0,0)^{\mathrm{T}} \neq \boldsymbol{p} = (p_0,p_1,p_2,p_3)^{\mathrm{T}} \in K^{4\times 1}$. The set of lines of $\mathbb{P}_3(K)$ is written as $\boldsymbol{\mathcal{L}}$.

Let $\boldsymbol{X} := (X_0,X_1,X_2,X_3)$ be a family of independent indeterminates over K. We refer to [10, pp. 48–51] for those basic notions of algebraic geometry which will be used in this paper. However, in contrast to [10], we write

$$\mathcal{V}\big(g_1(\boldsymbol{X}), g_2(\boldsymbol{X}), \ldots, g_r(\boldsymbol{X})\big) := \big\{K\boldsymbol{p} \in \mathbb{P}_3(K) \mid g_1(\boldsymbol{p}) = g_2(\boldsymbol{p}) = \cdots = g_r(\boldsymbol{p}) = 0\big\}$$

for the set of K-rational points of the variety given by homogeneous polynomials $g_1(\boldsymbol{X}),\ g_2(\boldsymbol{X}),\ \ldots,\ g_r(\boldsymbol{X}) \in K[\boldsymbol{X}]$.

Each matrix $M = (m_{ij})_{0\leq i,j\leq 3} \in \mathrm{GL}_4(K)$ acts on the column space $K^{4\times 1}$ by multiplication from the left hand side and therefore as a projective collineation on $\mathbb{P}_3(K)$. Moreover, M acts as a K-algebra isomorphism on $K[\boldsymbol{X}]$ via $X_i \mapsto \sum_{j=0}^{3} m_{ij}X_j$ for $i \in \{0,1,2,3\}$. Given a form $g(\boldsymbol{X}) \in K[\boldsymbol{X}]$ and its image under M, say $h(\boldsymbol{X})$, the collineation induced by M takes $\mathcal{V}\big(h(\boldsymbol{X})\big)$ to $\mathcal{V}\big(g(\boldsymbol{X})\big)$.

In what follows the plane $\omega := \mathcal{V}(X_0)$ will be considered as plane at infinity; thus we turn $\mathbb{P}_3(K)$ into a projectively closed affine space.

2.2 We refer to [1], [3], [4], [8], [20], and [22] for the definition and basic properties of Cayley's ruled cubic surface or, for short, the Cayley surface. It is, to within projective collineations, the point set $F := \mathcal{V}\big(f(\boldsymbol{X})\big)$, where

$$f(\boldsymbol{X}) := X_0X_1X_2 - X_1^3 - X_0^2X_3 \in K[\boldsymbol{X}].$$

Let $\partial_i := \frac{\partial}{\partial X_i}$. Hence we obtain

$$\begin{array}{rclrcl} \partial_0 f(\boldsymbol{X}) &=& X_1X_2 - 2X_0X_3, & \partial_1 f(\boldsymbol{X}) &=& X_0X_2 - 3X_1^2,\\ \partial_2 f(\boldsymbol{X}) &=& X_0X_1, & \partial_3 f(\boldsymbol{X}) &=& -X_0^2. \end{array} \tag{1}$$

These partial derivatives vanish simultaneously at $(p_0,p_1,p_2,p_3)^{\mathrm{T}} \in K^{4\times 1}$ if, and only if, at least one of the following conditions holds:

$$p_0 = p_1 = 0; \tag{2}$$

$$p_0 = p_2 = 0 \text{ and } \operatorname{Char} K = 3. \tag{3}$$

The parametrization

$$K^2 \to \mathbb{P}_3(K) : (u_1,u_2) \mapsto K(1,u_1,u_2,u_1u_2-u_1^3)^{\mathrm{T}} =: P(u_1,u_2)$$

is injective, and its image coincides with $F\setminus\omega$ (the affine part of F). According to (1), all points of $F\setminus\omega$ are simple. The tangent plane at $P(u_1,u_2)$ equals

$$\mathcal{V}\left((2u_1^3 - u_1u_2)X_0 + (-3u_1^2+u_2)X_1 + u_1X_2 - X_3\right). \tag{4}$$

The points subject to (2) comprise the line $\mathcal{V}(X_0,X_1) = F\cap\omega =: g_\infty$. They are easily seen to be double points of F. The *tangent cone* (or *tangent space* [10, p. 49]) at a point $U := K(0,0,s_2,s_3)^{\mathrm{T}}$, $(s_2,s_3)\neq(0,0)$, is

$$\mathcal{V}\big(X_0(s_2X_1 - s_3X_0)\big); \tag{5}$$

this is either a pair of distinct planes (if $U \neq Z := K(0,0,0,1)^{\mathrm{T}}$) or a repeated plane (if $U = Z$). We call each of these planes a *tangent plane* at U. The point Z is a so-called *pinch point* [17, p. 76], and its tangent plane is ω. See Figure 1 which displays the Cayley surface in $\mathbb{P}_3(\mathbb{R})$ in an affine neighbourhood of Z. (The plane $\mathcal{V}(X_3)$ is at infinity in this illustration.)

The tangent plane of F at $P(0,0)$ is $\mathcal{V}(X_3)$; this plane meets F along the line $\mathcal{V}(X_1,X_3)$ and the parabola

$$l := \mathcal{V}(X_0X_2 - X_1^2, X_3). \tag{6}$$

For each $(s_0, s_1) \in K^2 \setminus \{(0,0)\}$ the line

$$g(s_0, s_1) := K(s_0^2, s_0 s_1, s_1^2, 0)^{\mathrm{T}} + K(0, 0, s_0, s_1)^{\mathrm{T}}$$

is a generator of F. There are no other lines on F. The line $g(0,1) = g_\infty$ is not only a generator of F, but also a directrix, as it has non-empty intersection with every generator. Each point of g_∞, except the point Z, is on precisely two generators of F; each affine point of F is incident with precisely one generator (Figure 2).

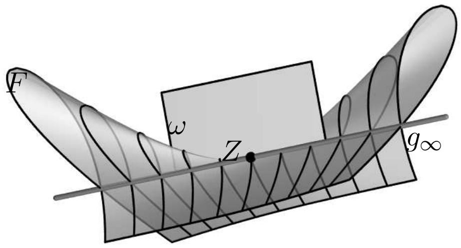

Figure 1

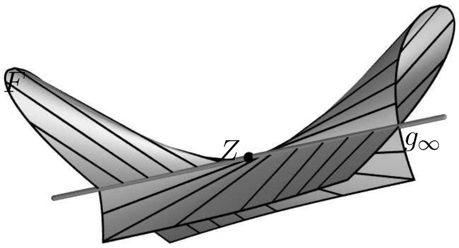

Figure 2

Next we describe the automorphic projective collineations of F: The set of all matrices

$$M_{a,b,c} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & c & 0 & 0 \\ b & 3\,ac & c^2 & 0 \\ ab - a^3 & bc & ac^2 & c^3 \end{pmatrix}$$

where $a, b \in K$ and $c \in K \setminus \{0\}$ is a group, say G, under multiplication. Each matrix in G leaves invariant the cubic form $f(\boldsymbol{X}) = X_0 X_1 X_2 - X_1^3 - X_0^2 X_3$ to within the factor c^3. Consequently, the group G acts on F as a group of projective collineations. Under the action of G, the points of F fall into three orbits: $F \setminus \omega$, $g_\infty \setminus \{Z\}$, and $\{Z\}$. Except for the case when $|K| \leq 3$, the group G yields all projective collineations of F; see [8, Section 3].

Observe that the following holds irrespective of the characteristic of K.

Lemma 2.3. *There exists a duality which maps the set of points of the Cayley surface F onto the set of its tangent planes. Thus the set of all tangent planes of F is a Cayley surface in the dual projective space.*

Proof. By (4) and (5), a plane $\mathcal{V}\left(\sum_{i=0}^{3} a_i X_i\right)$, where $a_i \in K$, is a tangent plane of F if, and only if, $a_1 a_2 a_3 - a_2^3 - a_0 a_3^2 = 0$. Consequently, the linear bijection

$$K^{4\times 1} \to K^{1\times 4} : (x_0, x_1, x_2, x_3)^{\mathrm{T}} \mapsto (x_3, x_2, x_1, x_0)$$

gives a duality of $\mathbb{P}_3(K)$ with the required properties. □

We note that the duality from the above takes, for all $(u_1, u_2) \in K^2$, the point $P(u_1, u_2)$ to the tangent plane at the point $P(-u_1, 3u_1^2 - u_2)$.

3. Osculating tangents and the Betten-Walker spread

3.1 If a line t meets F at a simple point P with multiplicity ≥ 3 then it is called an *osculating tangent* at P. Such a tangent line is either a generator or it meets F at P only. In the latter case it will be called a *proper osculating tangent* of F. Observe that we are not dealing with those lines which meet F with multiplicity ≥ 3 at a double point. In fact, g_∞ is the set of double points, and at any point $U \in g_\infty$, the lines meeting F at U with multiplicity ≥ 3 comprise two pencils (one pencil if $U = Z$) lying in the two tangent planes at U (only tangent plane at $U = Z$); see formula (5). The following is part of the folklore:

Lemma 3.2. *At each point $P(u_1, u_2) \in F \setminus g_\infty$ there is a unique proper osculating tangent, namely the line which joins $P(u_1, u_2)$ with the point $K(0, 1, 3u_1, u_2)^{\mathrm{T}}$.*

Proof. Let $(u_1, u_2) = (0, 0)$. The tangent plane at $P(0, 0)$ is $\mathcal{V}(X_3)$. Any proper osculating tangent through $P(0, 0)$ is necessarily incident with this plane, and it meets F at $P(0, 0)$ only. By (6), only the tangent t of the parabola l at $P(0, 0)$ can be a proper osculating tangent, since every other tangent of F at $P(0, 0)$ meets l residually at a point $\neq P(0, 0)$. The point at infinity of t is $K(0, 1, 0, 0)$. It is straightforward to verify that t meets F at $P(0, 0)$ with multiplicity three. By the action of the matrix $M_{u_1,u_2,1} \in G$ the assertion follows for any point $P(u_1, u_2) \in F \setminus g_\infty$. $\square$

Theorem 3.3. *Let*

$$\mathcal{O} := \{t \in \mathcal{L} \mid t \text{ is a proper osculating tangent of } F\} \cup \{g_\infty\}.$$

This set of lines has the following properties:

(a) *$\mathcal{O}$ is a partial spread of $\mathbb{P}_3(K)$ if, and only if,* $\operatorname{Char} K \neq 3$ *and K does not contain a third root of unity other than* 1.

(b) *If $\mathcal{O}$ is a partial spread then it is maximal, i.e., it is not a proper subset of any partial spread of $\mathbb{P}_3(K)$.*

(c) *$\mathcal{O}$ is a covering of $\mathbb{P}_3(K)$ if, and only if,* $\operatorname{Char} K \neq 3$ *and each element of K has a third root in K.*

Proof. (a) It is immediate from Lemma 3.2 that all proper osculating tangents of F are skew to g_∞. So it suffices to discuss whether or not two distinct proper osculating tangents of F have a point in common. As the group G acts transitively on $F \setminus g_\infty$, all proper osculating tangents of F are in one orbit of G. So it is enough consider the osculating tangents at distinct points $P(0, 0)$ and $P(u_1, u_2)$. By Lemma 3.2, these lines are skew if, and only if,

$$\det \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & u_1 & 1 \\ 0 & 0 & u_2 & 3u_1 \\ 0 & 0 & u_1u_2 - u_1^3 & u_2 \end{pmatrix} = u_2^2 - 3u_1^2u_2 + 3u_1^4 \neq 0. \tag{7}$$

If $u_1 = 0$ then $u_2 \neq 0$, whence (7) holds irrespective of the ground field. Otherwise we substitute $u_2 = (2+y)u_1^2$ with $y \in K$. Hence (7) turns into $u_1^4(y^2+y+1) \neq 0$. Observing $(X^2+X+1)(X-1) = X^3-1 \in K[X]$, we see that X^2+X+1 has a zero in K precisely when the following holds: Either $\operatorname{Char} K = 3$, since in this case $X^2+X+1 = (X-1)^2$, or $\operatorname{Char} K \neq 3$ and there exists a third root of unity $w \neq 1$ in K, since then $1^2+1+1 \neq 0$.

(b) We infer from (a) that $\operatorname{Char} K \neq 3$. Thus Lemma 3.2 implies that each point at infinity is incident with a proper osculating tangent or just with the line g_∞. As every line of $\mathbb{P}_3(K)$ has a point in common with the plane at infinity and each point of the plane at infinity is on a line of the partial spread, the partial spread $\mathcal{O}$ is maximal.

(c) First, let $\operatorname{Char} K = 3$. It suffices to show that $\mathcal{O}$ cannot be a covering of $\mathbb{P}_3(K)$. By Lemma 3.2, all proper osculating tangents meet the line

$$n := \mathcal{V}(X_0, X_2). \tag{8}$$

Clearly, there exists a point in $\omega \setminus (n \cup g_\infty)$. This point is not incident with any line of $\mathcal{O}$.

Next, assume $\operatorname{Char} K \neq 3$. By the proof of (b), we may restrict ourselves to affine points. A point $K(1, p_1, p_2, p_3)$ is on a line of $\mathcal{O}$ if, and only if, there is a pair $(u_1, u_2) \in K^2$ and an $s \in K$ such that

$$(1, p_1, p_2, p_3)^{\mathrm{T}} = (1, u_1, u_2, u_1u_2 - u_1^3)^{\mathrm{T}} + s(0, 1, 3u_1, u_2)^{\mathrm{T}}.$$

So we obtain the following system of equations in the unknowns u_1, u_2, s:

$$u_1 = p_1 - s, \quad u_2 = p_2 - 3s(p_1 - s), \quad s^3 = p_3 - (p_1p_2 - p_1^3).$$

This system has a solution precisely when $p_3 - (p_1p_2 - p_1^3)$ has a third root in K. As $p_3 - (p_1p_2 - p_1^3)$ can assume any value in K, the assertion follows. $\square$

3.4 By the above, an affine point lies on a line of $\mathcal{O}$ if, and only if, it can be written in the form $K(1, p_1, p_2, p_1p_2 - p_1^3 + s^3)$ with $p_1, p_2, s \in K$.

The results of Theorem 3.3 were established in [13, Theorem 6.1] and [23, Teorema 1] in a completely different way. In those papers the reader will also find conditions for a field K to meet one of the algebraic conditions given in (a), (b), or (c).

We noted in Lemma 2.3 that F admits a duality which is easily seen to fix $\mathcal{O}$, as a set of lines. Hence the dual counterparts of the characterizations given in Theorem 3.3 hold as well. Thus $\mathcal{O}$ is a (maximal partial) spread if, and only if, it is a (maximal partial) dual spread. Observe that the point $Z = K(0,0,0,1)^{\mathrm{T}}$ takes over the role of the plane ω in the dual setting.

3.5 Let $\operatorname{Char} K = 3$. Recall from (8) that $n = \mathcal{V}(X_0, X_2)$. By (3), every point of $n \setminus \{Z\}$ is a *nucleus* of F, i.e. a point off F, where all partial derivatives (1) vanish; see [10, p. 50]. We refer also to [6, Proposition 3.17], where nuclei are defined in a slightly different way (including double points of F). Even though Z is not a nucleus according to our definition, we shall refer to n as being the *line of nuclei*. We established in the proof of Theorem 3.3 (c) that all proper osculating tangents meet the line of nuclei. This result will be improved in Theorem 4.7.

3.6 Let $K = \mathbb{R}$ so that $\mathcal{O}$ is a spread. In order to show that this is in fact the BW-spread, as described in [2, Satz 3], we apply the collineation

$$\alpha : \mathbb{P}_3(K) \to \mathbb{P}_3(K) : K(x_0, x_1, x_2, x_3)^{\mathrm{T}} \mapsto K\left(x_0, x_1, \frac{x_2}{3}, \frac{x_3}{3}\right)^{\mathrm{T}}$$

which fixes the line g_∞. By Lemma 3.2, any line of $\alpha(\mathcal{O} \setminus \{g_\infty\})$ has the form

$$K\left(1, u_1, \frac{u_2}{3}, \frac{u_1 u_2}{3} - \frac{u_1^3}{3}\right)^{\mathrm{T}} + K\left(0, 1, u_1, \frac{u_2}{3}\right)^{\mathrm{T}}$$

with $(u_1, u_2) \in \mathbb{R}^2$. By joining this line with $Z = K(0,0,0,1)^{\mathrm{T}}$ and $K(0,0,1,0)^{\mathrm{T}}$, we obtain two distinct planes with equations

$$x_2 = \left(\frac{u_2}{3} - u_1^2\right) x_0 + u_1 x_1 \text{ and } x_3 = -\frac{u_1^3}{3} x_0 + \frac{u_2}{3} x_1,$$

respectively. The substitutions $x_0 =: x$, $x_1 =: y$, $x_2 =: u$, $x_3 =: v$, $u_2/3 - u_1^2 =: t$, and $u_1 =: s$ turn these equations into

$$u = tx + sy \text{ and } v = -\frac{s^3}{3}x + (s^2 + t)y.$$

These are the formulas from [2, Satz 3]. In particular, we have the transversal homeomorphism of $\mathbb{R}^2$ with $(t, s) \mapsto (-s^3/3, s^2 + t)$.

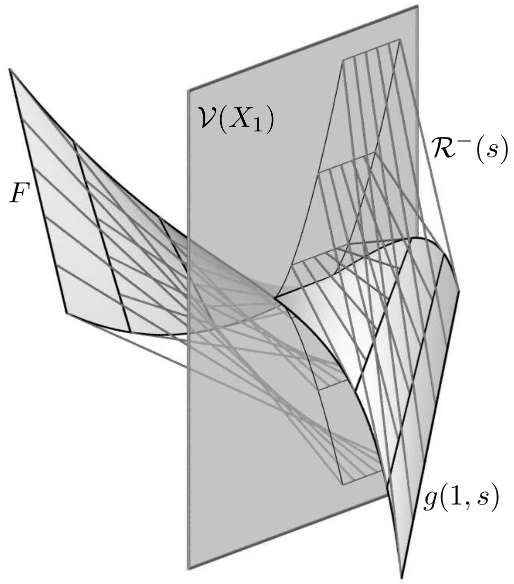

Figure 3

It is also easy to see that our results coincide with [28], where homogeneous coordinates with indices running from 1 to 4, say x'_1, x'_2, x'_3, x'_4, were used. The appropriate transformation from our coordinates x_0, x_1, x_2, x_3 is given by $x'_1 = x_3/3$, $x'_2 = x_2/3$, $x'_3 = x_1$, and $x'_4 = x_0$. The group S_1 used in [28] is a subgroup of our group G, whereas the reguli $\mathfrak{R}_i$ from [28] arise in our setting as follows: Take the set of all proper osculating tangents at the points of a generator $g(1,s)$, $s \in K$, together with g_∞. This is easily seen to be a regulus, say $\mathcal{R}^-(s)$, which clearly is contained in $\mathcal{O}$. In affine terms each such regulus is one family of generators on a hyperbolic paraboloid. These hyperbolic paraboloids have g_∞ as a common generator and they share a common tangent plane at each point of g_∞. Thus, for example, each such paraboloid meets the plane $\mathcal{V}(X_1) \supset g_\infty$ residually in a line; all these lines are parallel, as they pass through $K(0,0,1,0)^{\mathrm{T}}$. This is depicted in Figure 3.

3.7 Assume Char $K \neq 3$. Then the lines of $\mathcal{O}$ other than g_∞ define (by intersection) an injective mapping $\omega \setminus g_\infty \to \mathcal{V}(X_1) \setminus g_\infty$; compare with the construction of a spread via a transversal mapping due to N. Knarr [15, pp. 26–29]. An illustration is given in Figure 4, where temporarily $\mathcal{V}(X_0 + X_1)$ takes over the role of the plane at infinity. (The curves in $\mathcal{V}(X_1)$ are semicubical parabolas.)

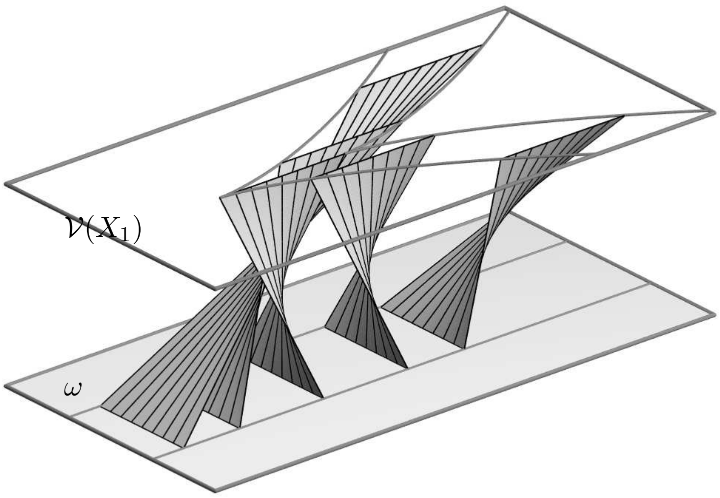

Figure 4

4. The Klein image of the Betten-Walker spread

4.1 In terms of coordinates, the exterior square $K^{4\times 1} \wedge K^{4\times 1}$ coincides with $K^{6\times 1}$ by setting

$$\boldsymbol{p} \wedge \boldsymbol{q} = (p_0, p_1, p_2, p_3)^{\mathrm{T}} \wedge (q_0, q_1, q_2, q_3)^{\mathrm{T}} = (y_{01}, y_{02}, y_{03}, y_{12}, y_{13}, y_{23})^{\mathrm{T}}$$

where $y_{ij} = p_i q_j - p_j q_i$. Given that $\boldsymbol{p}, \boldsymbol{q}$ are linearly independent the entries of the column vector $(y_{01}, y_{02}, \ldots, y_{23})^{\mathrm{T}}$ are the well known Plücker coordinates of the line $K\boldsymbol{p} + K\boldsymbol{q}$. The Klein mapping $\kappa : \mathcal{L} \to Q : K\boldsymbol{p} + K\boldsymbol{q} \mapsto K(\boldsymbol{p} \wedge \boldsymbol{q})$ is a bijection from the line set $\mathcal{L}$ of $\mathbb{P}_3(K)$ onto Klein quadric $Q := \mathcal{V}(k(\boldsymbol{Y})) \subset \mathbb{P}_5(K)$, where $\boldsymbol{Y} = (Y_{01}, Y_{02}, \dots, Y_{23})$ denotes a family of six independent indeterminates over K and

$$k(\boldsymbol{Y}) := Y_{01}Y_{23} - Y_{02}Y_{13} + Y_{03}Y_{12}.$$

The polarity of the Klein quadric will be denoted by $\perp$. Observe that $\perp$ is symplectic if, and only if, $\operatorname{Char} K = 2$. Table 15.10 in [9, pp. 29–31] contains all the information on the Klein mapping which we shall use in this section without further reference.

A set of lines in $\mathbb{P}_3(K)$ is said to be *algebraic* if its Klein image is the set of K-rational points of an algebraic variety in $\mathbb{P}_5(K)$.

4.2 Let us first calculate the Klein image of the set of generators of F: We obtain, for all $(s_0, s_1) \in K^2 \setminus \{(0,0)\}$, that

$$\kappa(g(s_0, s_1)) = K(0, s_0^3, s_0^2 s_1, s_0^2 s_1, s_0 s_1^2, s_1^3)^{\mathrm{T}} \in \mathbb{P}_5(K). \tag{9}$$

So we get a twisted cubic [9, Chapter 21] lying in the three-dimensional subspace

$$C := \mathcal{V}(X_{01}, X_{03} - X_{12}) \subset \mathbb{P}_5(K).$$

The intersection $C \cap Q$ is a quadratic cone with vertex $W_\infty := \kappa(g_\infty) = K\boldsymbol{w}_\infty$, where

$$\boldsymbol{w}_\infty := (0,0,0,0,0,1)^{\mathrm{T}}. \tag{10}$$

This cone is the Klein image of a parabolic linear congruence with axis g_∞, which contains all generators of F. The subspace $C^\perp$ is the line spanned by W_∞ and $W := K\boldsymbol{w}$, where

$$\boldsymbol{w} := (0,0,1,-1,0,0)^{\mathrm{T}}.$$

This line meets the Klein quadric at W_∞ only. We have $C^\perp \cap C = \{W_\infty\}$ for $\operatorname{Char} K \neq 2$, but $C^\perp \subset C$ otherwise.

In the subsequent theorem we exhibit algebraic equations which are satisfied by the Klein image of $\mathcal{O}$; we shall explain in 4.4 how these equations were found.

Theorem 4.3. *Suppose* $\operatorname{Char} K \neq 3$. *Let* $\mathcal{O}$ *be given as in Theorem* 3.3, *and let* $\mathcal{L}[Z,\omega]$ *be the pencil of lines in the plane* $\omega = \mathcal{V}(X_0)$ *with centre* $Z = K(0,0,0,1)^{\mathrm{T}}$. *Consider the polynomials*

$$h_1(\boldsymbol{Y}) := 3Y_{01}(Y_{12} + Y_{03}) - Y_{02}^2, \tag{11}$$
$$h_2(\boldsymbol{Y}) := 3Y_{02}Y_{13} - (Y_{12} + Y_{03})^2, \tag{12}$$
$$h_3(\boldsymbol{Y}) := 9Y_{01}Y_{13} - Y_{02}(Y_{12} + Y_{03}). \tag{13}$$

Then $\kappa(\mathcal{O} \cup \mathcal{L}[Z,\omega])$ *equals the intersection of the variety*

$$J := \mathcal{V}(h_1(\boldsymbol{Y}), h_2(\boldsymbol{Y}), h_3(\boldsymbol{Y})) \subset \mathbb{P}_5(K)$$

with the Klein quadric $Q = \mathcal{V}(k(\boldsymbol{Y}))$.

Proof. (a) For all $(u_1, u_2) \in K^2$, the Klein image of the only proper osculating tangent at $P(u_1, u_2)$ is the point with coordinates

$$(1, 3u_1, u_2, 3u_1^2 - u_2, u_1^3, 3u_1^4 - 3u_1^2 u_2 + u_2^2)^{\mathrm{T}}. \tag{14}$$

The Klein image of the pencil $\mathcal{L}[Z, \omega]$ is the line spanned by $K(0,0,0,0,1,0)$ and $K\boldsymbol{w}_\infty = \kappa(g_\infty)$; see (10). Now a direct verification shows $\kappa(\mathcal{O} \cup \mathcal{L}[Z, \omega]) \subset (J \cap Q)$.
(b) In order to show $(J \cap Q) \subset \kappa(\mathcal{O} \cup \mathcal{L}[Z, \omega])$ we determine all vectors

$$\boldsymbol{y} := (y_{01}, y_{02}, y_{03}, y_{12}, y_{13}, y_{23})^{\mathrm{T}} \in K^{6\times 1}$$

subject to $h_1(\boldsymbol{y}) = h_2(\boldsymbol{y}) = h_3(\boldsymbol{y}) = k(\boldsymbol{y}) = 0$.

In a first step we determine all such vectors with $y_{01} \neq 0$. Without loss of generality we may assume $y_{01} = 1$; also we let $y_{02} =: u_1$ and $y_{03} =: u_2$. From $h_1(\boldsymbol{y}) = 3y_{12} + 3u_2 - 9u_1^2$ follows $y_{12} = 3u_1^2 - u_2$ which can be substituted in $h_3(\boldsymbol{y}) = 9y_{13} - 9u_1^3$. This gives $y_{13} = u_1^3$. We calculate $k(\boldsymbol{y}) = y_{23} - 3u_1^4 + u_2(u_1^2 - u_2)$ which yields $y_{23} = 3u_1^4 - 3u_1^2 u_2 + u_2^2$. Altogether, we obtain precisely the vectors given in (14), whence $h_2(\boldsymbol{y}) = 0$ holds too.

The second step is to look for all solutions with $y_{01} = 0$. We infer from $h_1(\boldsymbol{y}) = -y_{02}^2$ that $y_{02} = 0$, from which we obtain $h_2(\boldsymbol{y}) = -(y_{12} + y_{03})^2$. So $y_{12} = -y_{03}$. Now $k(\boldsymbol{y}) = -y_{03}^2$ gives $y_{03} = -y_{12} = 0$. Summing up we obtain

$$\boldsymbol{y} := (0, 0, 0, 0, y_{13}, y_{23})^{\mathrm{T}} \in K^{6\times 1}$$

which is either the zero vector or a representative of a point in $\kappa(\mathcal{L}[Z, \omega])$. Consequently, also $h_3(\boldsymbol{y}) = 0$ is satisfied. □

4.4 Let us shortly describe how the polynomials $h_i(\boldsymbol{Y})$ were found: We noted already in 3.7 that all proper osculating tangents at the points of a generator $g(1, s)$, $s \in K$, together with g_∞ form a regulus $\mathcal{R}^-(s) \subset \mathcal{O}$. The opposite regulus $\mathcal{R}^+(s)$, say, contains the generator $g(1, s)$. Both reguli lie on a hyperbolic paraboloid which is known in differential line geometry under the name *Lie quadric* of F along the generator $g(1, s)$; cf., among others, [12, pp. 33–37] or [22, pp. 67–68]. In Figure 5 some reguli $\mathcal{R}^-(s)$ are visualized in an affine neighbourhood of the point Z. In this picture the plane $\mathcal{V}(X_3)$ appears at infinity. See also Figures 3 and 4 for a different view of these reguli.

Given a point $\kappa(g(1, s))$ of the twisted cubic (9) let us denote by $\pi(s)$ the osculating plane at this point. The plane $\pi(s)$ meets the cone $C \cap Q$ in a conic which is the Klein image of the regulus $\mathcal{R}^+(s)$. The Klein image of its opposite regulus $\mathcal{R}^-(s)$ is a conic lying in the plane $\pi(s)^\perp \supset C^\perp$. We choose the three-dimensional subspace

$$B := \mathcal{V}(Y_{03}, Y_{23})$$

which is skew to $C^\perp$. Thus $\pi(s)^\perp$ meets B at a unique point. We obtain this point by letting $u_1 = s$ in (14) and by projecting through $C^\perp$ to B. This gives (for every $u_2 \in K$)

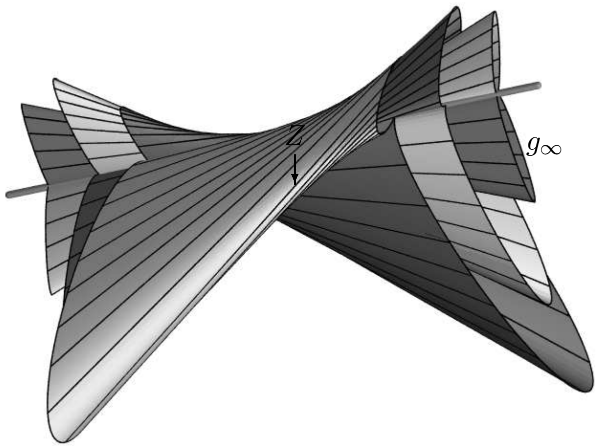

Figure 5

$$K(1, 3\,s, 0, 3\,s^2, s^3, 0)^{\mathrm{T}}. \tag{15}$$

Now we consider $s \in K$ to be variable. Up to the exceptional case when $\operatorname{Char} K = 3$, the planes $\pi(s)$ belong to a cubic developable, so that the points of intersection $\pi(s)^\perp \cap B$ will belong to a twisted cubic. This is also immediate from (15). It is well known, that a twisted cubic can be obtained as the intersection of three quadrics in B. (We used two quadratic cones projecting the cubic from two different points, and a hyperbolic quadric.) Each of these three quadrics in B gives rise to a quadratic cone in $\mathbb{P}_5(K)$ by joining it with the line $C^\perp$. The quadratic polynomials in Theorem 4.3 describe these three quadratic cones in $\mathbb{P}_5(K)$ so that J actually is a cone with vertex $C^\perp$ having a twisted cubic in B as its base.

We wish to emphasize that our approach is motivated by the Thas-Walker construction linking flocks of cones with spreads: The osculating planes $\pi(s)$ yield a flock of the quadratic cone $C \cap Q$ if, and only if, the set $\mathcal{O}$ is a spread of $\mathbb{P}_3(K)$. See [25, pp. 334–338] and [13, Theorem 6.2].

It should be noted here that we did not use results from the thesis of R. Koch [16, pp. 18–19]. He described, over the real numbers, the congruence of osculating tangents in terms of a cubic and a quadratic form (equations (2.27) and (2.28) loc. cit.). In fact, his equation (2.28) corresponds, up to a change of coordinates, to our polynomial (11). However, the two equations of Koch are also satisfied by the Plücker coordinates of all lines through the point Z, whereas our system of equations yields less lines through that point.

The next result says that, from an algebraic point of view, our system of equations (11) – (13) is the best possible:

Theorem 4.5. *Suppose that K is an infinite field with* $\operatorname{Char} K \neq 3$. *Let $\mathcal{O}$ and $\mathcal{L}[Z, \omega]$ be given as in Theorems* 3.3 *and* 4.3, *respectively. If $h(\boldsymbol{Y}) \in K[\boldsymbol{Y}]$ is a form such that $\kappa(\mathcal{O}) \subset \mathcal{V}(h(\boldsymbol{Y}))$ then $\kappa(\mathcal{L}[Z, \omega]) \subset \mathcal{V}(h(\boldsymbol{Y}))$.*

Proof. Given such a form $h(\boldsymbol{Y}) \in K[X]$ we obtain from (14) the identity

$$h((1, 3u_1, u_2, 3u_1^2 - u_2, u_1^3, 3u_1^4 - 3u_1^2 u_2 + u_2^2)^{\mathrm{T}}) = 0 \text{ for all } (u_1, u_2) \in K^2. \tag{16}$$

Due to $\operatorname{Char} K \neq 3$ there exists a field extension $\overline{K}/K$ of degree $[\overline{K} : K] \leq 2$ containing a third root of unity $w \neq 1$. We infer from a standard result on zeros of polynomials over an infinite domain (see, for example, [27, § 28]) that (16) holds also for all $(u_1, u_2) \in \overline{K}^2$. We allow $u_1 \in \overline{K}$ and replace u_2 with $(1-w)u_1^2 + mu_1$ in (14), where $m \in \overline{K}$ is fixed, but arbitrary. Thus (14) turns into

$$(1, 3u_1, (1-w)u_1^2 + mu_1, (2+w)u_1^2 - mu_1, u_1^3, -(2w+1)mu_1^3 + m^2u_1^2)^{\mathrm{T}}.$$

This is for $u_1 \in \overline{K}$ a rational parametrization of all but one points of a twisted cubic which is contained in the variety of $\mathbb{P}_3(\overline{K})$ determined by $h(\boldsymbol{Y})$ considered as an element of $\overline{K}[\boldsymbol{Y}]$. As $\overline{K}$ is infinite, also the remaining point of the twisted cubic, namely

$$\overline{K}(0, 0, 0, 0, 1, -(2w+1)m)^{\mathrm{T}},$$

is a point of that variety. We claim that $2w+1 \neq 0$: This is trivial when $\operatorname{Char} K = 2$. For $\operatorname{Char} K \neq 2$ the assertion holds, because our assumption $\operatorname{Char} K \neq 3$ guarantees that $(-1/2)^3 \neq 1$, whence $w \neq -1/2$.

Thus, for appropriate values of $m \in \overline{K}$, we see that all points of the line $\kappa(\mathcal{L}[Z, \omega])$ except $\kappa(g_\infty)$ belong to the variety $\mathcal{V}(h(\boldsymbol{Y}))$. Finally, since K is infinite, we obtain $\kappa(g_\infty) \in \mathcal{V}(h(\boldsymbol{Y}))$. □

Corollary 4.6. *Infinite Betten-Walker spreads are not algebraic sets of lines.*

We now turn to the remaining case of characteristic 3. Here the situation is completely different:

Theorem 4.7. *Let* $\operatorname{Char} K = 3$. *Then* $\mathcal{O} \cup \mathcal{L}[Z, \omega]$ *is a subset of a parabolic linear congruence* $\mathcal{N}$ *whose Klein image equals the quadratic cone* $Q \cap D$, *where*

$$D := \mathcal{V}(X_{02}, X_{03} + X_{12})$$

is a three-dimensional subspace of $\mathbb{P}_5(K)$. *The axis of the congruence* $\mathcal{N}$ *is the line* n *of nuclei. The congruence* $\mathcal{N}$ *coincides with* $\mathcal{O} \cup \mathcal{L}[Z, \omega]$ *if, and only if, each element of* K *has a third root in* K.

Proof. The polar subspace $D^\perp$ (with respect to the Klein quadric Q) is the line joining

$$\kappa(n) = K(0, 0, 0, 0, 1, 0)^{\mathrm{T}} \in Q \text{ and } K(0, 0, 1, 1, 0, 0)^{\mathrm{T}} \notin Q.$$

So $D^\perp \not\subset Q$ is a tangent of the Klein quadric and $D \cap Q$ is a quadratic cone with vertex $\kappa(n)$. For all $(u_1, u_2) \in K^2$, the Klein image of the only proper osculating tangent at $P(u_1, u_2)$ is the point with coordinates

$$(1, 0, u_2, -u_2, u_1^3, u_2^2)^{\mathrm{T}}. \tag{17}$$

The Klein image of the pencil $\mathcal{L}[Z, \omega]$ is the line spanned by $\kappa(n)$ and $K\boldsymbol{w}_\infty = \kappa(g_\infty)$; see (10). Now a direct verification shows $(\mathcal{O} \cup \mathcal{L}[Z, \omega]) \subset \mathcal{N}$.

We read off from the penultimate coordinate in (17) that $\mathcal{N} = \mathcal{O} \cup \mathcal{L}[Z, \omega]$ holds precisely when every element of K has a third root in K. □

Of course, when K is finite with characteristic 3 then each of its elements has a third root in K.

The line $D^{\perp}$ has yet another natural interpretation: Formula (9) can be rewritten in the form

$$\kappa(g(s_0, s_1)) = K(s_0^3\boldsymbol{v}_0 + s_0^2 s_1\boldsymbol{v}_1 + s_0 s_1^2\boldsymbol{v}_2 + s_1^3\boldsymbol{v}_3) \tag{18}$$

with linearly independent $\boldsymbol{v}_i \in K^{6\times 1}$. By the above, we obtain a twisted cubic as $(s_0, s_1) \neq (0, 0)$ varies in K^2. Due to Char $K = 3$ all osculating planes of this cubic belong to the pencil of planes (see [9, Theorem 21.1.2]) with axis $K\boldsymbol{v}_1 + K\boldsymbol{v}_2 = D^{\perp}$.

References

[1] Bertini, E.: *Introduzione alla Geometria Proiettiva degli Iperspazi.* G. Principato, Messina 1923. JFM 49.0484.08

[2] Betten, D.: *4-dimensionale Translationsebenen mit 8-dimensionaler Kollineationsgruppe.* Geom. Dedicata **2** (1973), 327–339. Zbl 0272.50028

[3] Brauner, H.: *Geometrie auf der Cayleyschen Fläche.* Österr. Akad. Wiss., Abt. II **173** (1964), 93–128. Zbl 0134.17402

[4] Brauner, H.: *Neuere Untersuchungen über windschiefe Flächen.* Jahresber. Dtsch. Math.-Ver. **70** (1967), 61–85. Zbl 0153.50803

[5] De Clerck, F.; Van Maldeghem, H.: *On flocks of infinite quadratic cones.* Bull. Belg. Math. Soc. Simon Stevin **1** (1994), 399–415. Zbl 0811.51004

[6] de Finis, M.; de Resmini, M. J.: *On cubic surfaces over a field of characteristic* 3. Ann. Discrete Math. **18** (1983), 241–277. Zbl 0504.05016

[7] Gevaert, H.; Johnson, N. L.: *Flocks of quadratic cones, generalized quadrangles and translation planes.* Geom. Dedicata **27** (1988), 301–317. Zbl 0653.51012

[8] Gmainer, J.; Havlicek, H.: *Isometries and collineations of the Cayley surface.* Innov. Incidence Geom. **2** (2005), 109–127. Zbl 1096.51007

[9] Hirschfeld, J. W. P.: *Finite Projective Spaces of Three Dimensions.* Oxford University Press, Oxford 1985. Zbl 0574.51001

[10] Hirschfeld, J. W. P.: *Projective Geometries over Finite Fields.* Clarendon Press, Oxford, second edition, 1998. Zbl 0899.51002

[11] Hirschfeld, J. W. P.; Shaw, R.: *Projective geometry codes over prime fields.* In G. Mullen and P. J.-S. Shiue, editors, Finite fields: theory, applications, and algorithms (Las Vegas, NV, 1993), Contemp. Math. **168**, 151–163. Amer. Math. Soc., Providence, RI, 1994. Zbl 0872.94053

[12] Hoschek, J.: *Liniengeometrie.* Bibliographisches Institut, Zürich 1971. B. I. Hochschulskripten, 733 a/b. Zbl 0227.53007

[13] Jha, V.; Johnson, N. L.: *Infinite flocks of a quadratic cone.* J. Geom. **57** (1996), 123–150. Zbl 0866.51002

[14] Jha, V.; Johnson, N. L.: *Infinite flocks of quadratic cones–II Generalized Fisher flocks.* J. Korean Math. Soc. **39** (2002), 653–664. Zbl 1011.51005

[15] Knarr, N.: *Translation Planes.* Lecture Notes in Mathematics **1611**. Springer, Berlin 1995. Zbl 0843.51004

[16] Koch, R.: *Geometrien mit einer CAYLEYschen Fläche 3.Grades als absolutem Gebilde.* PhD thesis, Universität Stuttgart, 1968.

[17] Müller, E.; Krames, J.: *Vorlesungen über Darstellende Geometrie, 3. Band: Konstruktive Behandlung der Regelflächen.* Deuticke, Wien 1931. Zbl 0001.34704

[18] O'Keefe, C. M.; Penttila, T.: *Subquadrangles of generalized quadrangles of order* (q^2, q)*, q even.* J. Comb. Theory Ser. A **94**(2001), 218–229. Zbl 0989.51003

[19] Riesinger, R.: *Constructing non-regular algebraic spreads with asymplecticly complemented regulization.* Beitr. Algebra Geom. **46** (2005), 179–206. Zbl 1073.51002

[20] Rosati, L. A.: *Sul numero dei punti di una superficie cubica in uno spazio lineare finito.* Boll. Unione Mat. Ital. **11**(3) (1956), 412–418. Zbl 0073.38104

[21] Salzmann, H.; Betten, D.; Grundhöfer, T.; Hähl, H.; Löwen, R.; Stroppel, M.: *Compact Projective Planes.* W. de Gruyter, Berlin 1996. Zbl 0851.51003

[22] Sauer, R.: *Projektive Liniengeometrie.* W. de Gruyter, Berlin, Leipzig 1937. Zbl 0016.21804

[23] Spera, A. G.: *Piani di Betten-Walker infiniti ed alcune applicazioni.* Boll. Unione Mat. Ital. B **5**(6) (1986), 623–638.

[24] Thas, J. A.: *Generalized polygons.* In F. Buekenhout, editor: Handbook of incidence geometry, 383–431. North-Holland, Amsterdam, 1995. Zbl 0823.51009

[25] Thas, J. A.: *Projective geometry over a finite field.* In F. Buekenhout, editor: Handbook of incidence geometry, 295–347. North-Holland, Amsterdam, 1995. Zbl 0822.51007

[26] Thas, J. A.: *Geometrical constructions of flock generalized quadrangles.* J. Comb. Theory Ser. A **94** (2001), 51–62. Zbl 1008.51010

[27] van der Waerden, B. L.: *Algebra I.* Springer, Berlin-Heidelberg-New York 1966. cf. 9. Auflage, Springer-Verlag, Berlin 1993 Zbl 0781.12002

[28] Walker, M.: *A class of translation planes.* Geom. Dedicata **5** (1976), 135–146. Zbl 0356.50022

Received June 17, 2005