Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 47 (2006), No. 2, 397-417.

On the Invariant Theory of the Bézoutiant

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Abstract. We study the classical invariant theory of the Bézoutiant $\mathcal{R}(A, B)$ of a pair of binary forms A, B. It is shown that $\mathcal{R}(A, B)$ admits a Taylor expansion whose coefficients are (essentially) the odd transvectants $(A, B)_{2r+1}$; moreover $\mathcal{R}(A, B)$ is entirely determined by the first two terms $M = (A, B)_1, N = (A, B)_3$. Using the Plücker relations, we give equivariant formulae which express the higher transvectants $(A, B)_5, (A, B)_7$ in terms of M, N. We also describe a 'generic reduction formula' which recovers B from $\mathcal{R}(A, B)$ and A.

MSC 2000: 13A50
Keywords: binary forms, Bézoutiant, transvectant, covariant, Grassmannian

1. Introduction

We begin by recalling the construction of the Bézoutiant of two binary forms. Let $\mathbf{x} = (x_0, x_1), \mathbf{y} = (y_0, y_1)$ be pairs of variables, and write $\omega = x_0\, y_1 - x_1\, y_0$. If A, B are (homogeneous) forms of order d in $\mathbf{x}$, then their Bézoutiant is defined to be

$$\mathcal{R}(A, B) = \frac{1}{\omega}\,[\,A(x_0, x_1)B(y_0, y_1) - B(x_0, x_1)A(y_0, y_1)].$$

Since $\mathcal{R}$ is symmetric in $\mathbf{x}$ and $\mathbf{y}$ of order $(d-1)$ in each, it can be seen as a quadratic form over the vector space of binary forms of order $(d-1)$.

If $V = \mathrm{Span}\{x_0, x_1\}$, then the construction of $\mathcal{R}$ corresponds to the isomorphism of $SL(V)$-representations

$$\wedge^2 \mathrm{Sym}^d V \xrightarrow{\sim} \mathrm{Sym}^2(\mathrm{Sym}^{d-1} V), \quad A \wedge B \longrightarrow \mathcal{R}(A, B).$$

It is easy to see that

$$\mathcal{R}(\alpha\, A + \beta\, B, \gamma\, A + \delta\, B) = (\alpha\, \delta - \beta\, \gamma)\, \mathcal{R}(A, B),$$

i.e., up to a scalar, $\mathcal{R}$ depends only on the pencil spanned by A, B (denoted $\Pi_{A,B}$). Conversely, $\mathcal{R}$ determines the pair (A, B) up to a unimodular transformation.

Bézoutiants have been principally studied for their use in elimination theory (e.g., see [8] or [11, vol. I, §136 ff]). In contradistinction, our interest lies in their invariant theoretic properties (understood in the sense of Grace and Young [5]).

1.1. A summary of results

In Section 2, we recall some fundamental facts about transvectants. We will show that $\mathcal{R}(A, B)$ admits a 'Taylor series' in ω as follows:

$$\mathcal{R}(A, B) = c_0\, T_1^{\mathfrak{p}} + c_1\, \omega^2\, T_3^{\mathfrak{p}} + c_2\, \omega^4\, T_5^{\mathfrak{p}} + \cdots,$$

where

- T_{2r+1} denotes the $(2r + 1)$-th transvectant of A, B,
- $\mathfrak{p}$ denotes the operation of symmetric polarization, and
- c_r are rational constants dependent on d and r.

Hence, from our viewpoint, a study of $\mathcal{R}(A, B)$ will be tantamount to a study of the odd transvectants $\{T_{2r+1} : r \geq 0\}$ of A and B.

In Section 3, we formulate a second order differential equation derived from T_1, T_3 whose solution space is $\Pi_{A,B}$. This shows that the terms of degree ≤ 2 in the Taylor series implicitly determine those of higher degree. The former cannot be chosen arbitrarily, and we give an algebraic characterization of terms which can so appear. Specifically, we construct a set of joint covariants $\Phi_0, \ldots, \Phi_d$ with arguments M, N, with the following property: *There exist* A, B *such that* $M = (A, B)_1, N = (A, B)_3$, *if and only if* $\Phi_0(M, N) = \cdots = \Phi_d(M, N) = 0$.

We have remarked earlier that $\mathcal{R}$ determines $\Pi_{A,B}$. Hence, given A and $\mathcal{R}$, the form B is determined up to an additive multiple of A. In Section 4, we give an equivariant formula for B in terms of A and $\mathcal{R}$. This is called a 'generic reduction formula', in analogy with a device introduced by D'Alembert in the theory of differential equations.

In Section 5, we use the classical Plücker relations to describe formulae which calculate T_5, T_7 from a knowledge of T_1 and T_3. The question of a formula in the general case of $T_{2r+1}, (r \geq 4)$ is left open. Three more open problems (with some supporting examples) are given in Section 6.

Acknowledgements. It is a pleasure to thank my colleague A. Abdesselam for several helpful conversations. I also thank Jim Carrell and Zinovy Reichstein for an invitation to the University of British Columbia, where this work was done.

2. Preliminaries

We will heavily use [5] as a standard reference for classical invariant theory. Glenn's treatise [2] covers substantially the same ground. In particular, we assume some familiarity with transvectants, covariants, and the symbolic calculus. A more recent exposition of this material is given in [12]. Basic facts about the representation theory of SL_2 can be found in [1, 15, 16].

The base field is throughout $\mathbf{C}$. A form will always mean a homogeneous polynomial in $\mathbf{x}$. By contrast, an $\mathbf{xy}$-form will involve both sets of variables, and will be homogeneous in each set. The $\mathbf{x}$-degree of a form will be called its order (to avoid conflict with [5]). The order of an $\mathbf{xy}$-form is a pair of integers.

Sometimes we will let $\mathbf{k}$ denote a nonzero constant which need not be precisely specified.

2.1. SL_2-modules

Let V be a $\mathbf{C}$-vector space of dimension two with the natural action of $SL(V)$. We write S_e for the symmetric power representation $\operatorname{Sym}^e V$, and $S_e(S_f)$ for $\operatorname{Sym}^e(\operatorname{Sym}^f V)$ etc.

The $\{S_e : e \geq 0\}$ are a complete set of finite dimensional irreducible $SL(V)$-modules. By complete reducibility, each finite dimensional $SL(V)$-module is isomorphic to a direct sum of the S_e. If $\{x_0, x_1\}$ is a basis of V, then an element of S_e is a form of order e in $\mathbf{x}$. We identify the projective space $\mathbb{P}^e$ with $\mathbb{P}S_e$, and write $A \in \mathbb{P}^e$ for the point represented by a (nonzero) form A. By convention, $S_e = 0$ if $e < 0$.

2.2. Transvectants

For integers $e, f \geq 0$, we have a decomposition of $SL(V)$-modules

$$S_e \otimes S_f \simeq \bigoplus_{r=0}^{\min\{e,f\}} S_{e+f-2r}. \tag{1}$$

If E, F are forms of orders e, f, the image of the projection of $E \otimes F$ into the r-th summand is called their r-th transvectant, denoted $(E, F)_r$. It is a form of order $e + f - 2r$, whose coefficients are linear in the coefficients of E and F. In coordinates, it is given by the formula

$$(E, F)_r = \frac{(e-r)!(f-r)!}{e!f!} \sum_{i=0}^{r} (-1)^i \binom{r}{i} \frac{\partial^r E}{\partial x_0^{r-i}\, \partial x_1^i} \frac{\partial^r F}{\partial x_0^i\, \partial x_1^{r-i}} \tag{2}$$

(The initial scaling factor is conventional, some authors choose it differently.) In particular $(E, F)_0 = E\,F$, and $(E, F)_1 = \mathbf{k} \times$ Jacobian(E, F). Note that

$$\begin{aligned} (E, F)_r &= (-1)^r (F, E)_r, && (3)\\ (E, F)_r &= 0 \qquad \text{for } r > \min\{e, f\}. && (4) \end{aligned}$$

If E, F have the same order, then

$$(\alpha\, E + \beta\, F, \gamma\, E + \delta\, F)_{2r+1} = (\alpha\,\delta - \beta\,\gamma)\,(E, F)_{2r+1}, \tag{5}$$

for arbitrary constants $\alpha, \beta, \gamma, \delta$. This shows that the odd transvectants $(E, F)_{2r+1}$ are combinants of E, F, i.e., up to a scalar, they depend only on the pencil spanned by E, F.

If E, F are given symbolically, then [5, §49] gives an algorithm for calculating their transvectants. See Proposition 3.2 for a typical instance of its use.

The following lemma is elementary (see [3, Lemma 2.2]).

Lemma 2.1. *If E, F are nonzero forms of order e such that $(E, F)_1 = 0$, then $E = \mathbf{k}\, F$.* □

2.3.

Each representation S_e is self-dual, i.e., we have an isomorphism

$$S_e \xrightarrow{\sim} S_e^* = \mathrm{Hom}_{SL(V)}(S_e, \mathbf{C}).$$

This map sends an order e form E to the functional

$$\delta_E : S_e \longrightarrow \mathbf{C}, \quad F \longrightarrow (E, F)_e.$$

2.4. The Gordan series

Given three forms E, F, G, this very useful series describes certain linear dependency relations between transvectants of the type $((E, F)_\star, G)_\star$ and $((E, G)_\star, F)_\star$.

Let E, F, G be of orders e, f, g respectively, and a_1, a_2, a_3 three integers satisfying the following conditions:

- $a_2 + a_3 \le e$, $a_1 + a_3 \le f$, $a_1 + a_2 \le g$, and
- either $a_1 = 0$ or $a_2 + a_3 = e$ (or both).

Then we have an identity

$$\begin{aligned} &\sum_{i \ge 0} \frac{\binom{f-a_1-a_3}{i}\binom{a_2}{i}}{\binom{e+f-2a_3-i+1}{i}}\, ((E, F)_{a_3+i}, G)_{a_1+a_2-i} \\ &= (-1)^{a_1} \sum_{i \ge 0} \frac{\binom{g-a_1-a_2}{i}\binom{a_3}{i}}{\binom{e+g-2a_2-i+1}{i}}\, ((E, G)_{a_2+i}, F)_{a_1+a_3-i}\,. \end{aligned} \tag{6}$$

This identity is usually denoted $\begin{pmatrix} E & F & G \\ e & f & g \\ a_1 & a_2 & a_3 \end{pmatrix}$.

2.5. The Clebsch-Gordan series

Let $\mathbf{y}\,\partial_{\mathbf{x}}$ denote the polarization operator

$$y_0 \frac{\partial}{\partial x_0} + y_1 \frac{\partial}{\partial x_1}.$$

If E is a form of order e, then define its m-th polar to be

$$E^{\langle m \rangle} = \frac{(e-m)!}{e!} (\mathbf{y}\,\partial_{\mathbf{x}})^m \, E,$$

which is an $\mathbf{xy}$-form of order $(e-m, m)$. By Euler's theorem, we can recover E from $E^{\langle m \rangle}$ by the substitution $\mathbf{y} := \mathbf{x}$. If e is even, we will denote $E^{\langle e/2 \rangle}$ by $E^{\mathfrak{p}}$. It is symmetric in $\mathbf{x}, \mathbf{y}$, and naturally thought of as an element of $S_2(S_{e/2})$.

The Clebsch-Gordan series is a more precise statement of decomposition (1). For forms E, F of orders e, f, it gives an identity

$$E(\mathbf{x})\,F(\mathbf{y}) = \sum_{r=0}^{\min\{e,f\}} \frac{\binom{e}{r}\binom{f}{r}}{\binom{e+f-r+1}{r}} \, \omega^r \, (E, F)_r^{\langle f-r \rangle}. \tag{7}$$

Remark 2.2. The notional distinction between the Gordan series and Clebsch-Gordan series is merely for convenience of reference, and has no historical basis. In fact (7) directly leads to (6) (see [5, §52]).

Now let $\mathcal{U} \in S_2(S_{d-1})$. We identify $\mathcal{U}$ with an $\mathbf{xy}$-form of order $(d-1, d-1)$ which is symmetric in both sets of variables. It can then be expressed as a 'Taylor series' in ω. Define constants

$$c_r = \frac{2\binom{d}{2r+1}^2}{\binom{2d-2r}{2r+1}} \quad \text{for } 0 \leq r \leq \lfloor \frac{d-1}{2} \rfloor. \tag{8}$$

Proposition 2.3. *There exists a unique sequence of forms*

$$U_\bullet = (U_1, U_3, \dots, U_{2r+1}, \dots),$$

where $\mathrm{ord}\ \mathrm{U}_{2r+1} = 2(\mathrm{d} - 2\mathrm{r} - 1)$*, such that*

$$\mathcal{U} = \sum_{r \geq 0} c_r \, \omega^{2r} \, (U_{2r+1})^{\mathfrak{p}}.$$

Proof. First we prove the existence. Since $\mathcal{U}$ is symmetric in $\mathbf{x}$ and $\mathbf{y}$, it is a linear combination of expressions of the form

$$\langle i\, j \rangle = x_0^{d-1-i} x_1^i \, y_0^{d-1-j} y_1^j + x_0^{d-1-j} x_1^j \, y_0^{d-1-i} y_1^i.$$

Let $A = x_0^{d-1-i} x_1^i, B = x_0^{d-1-j} x_1^j$, so

$$\langle i\, j \rangle = A(\mathbf{x}) B(\mathbf{y}) + B(\mathbf{x}) A(\mathbf{y}).$$

Rewrite the right-hand side as a sum of two Clebsch-Gordan series. By property (3), only the even powers of ω will survive. This shows the existence claim for $\langle ij\rangle$, and hence in general by linearity.

Conversely, let $U_\bullet, U'_\bullet$ be two such sequences for $\mathcal{U}$. By the substitution $\mathbf{y} := \mathbf{x}$, we deduce $U_1 = U'_1$. Now divide $\mathcal{U} - U_1$ by ω^2 and again let $\mathbf{y} := \mathbf{x}$ etc., then we successively see that $U_{2r+1} = U'_{2r+1}$ for all r. □

Henceforth, A, B will always denote linearly independent forms of order d. We will write

$$T_i := (A, B)_i, \quad \Pi_{A,B} := \mathrm{Span}\{A, B\} \subseteq S_d. \tag{9}$$

Proposition 2.4. *With notation as above,*

$$\mathcal{R}(A, B) = \sum_{r \geq 0} c_r \, \omega^{2r} \, (T_{2r+1})^{\mathfrak{p}}. \tag{10}$$

Proof. Express $A(\mathbf{x})B(\mathbf{y})$ and $B(\mathbf{x})A(\mathbf{y})$ as Gordan series and subtract. By property (3), only the odd powers of ω will survive. Now divide by ω, then they all become even powers. □

It follows that the collection $\{T_{2r+1} : r \geq 0\}$ determines $\mathcal{R}(A, B)$. It will be shown below that the terms $r = 0, 1$ are already sufficient.

3. The Wronskian o.d.e.

3.1. Generalities on Wronskians

Given integers p, q with $q \leq p+1$, there is an isomorphism of $SL(V)$-modules (see [1, §11])

$$\wedge^q S_p \xrightarrow{\sim} S_q(S_{p-q+1}). \tag{11}$$

Composing it with the natural surjection

$$S_q(S_{p-q+1}) \longrightarrow S_{q(p-q+1)}, \tag{12}$$

we get the Wronskian map

$$\Theta : \wedge^q S_p \longrightarrow S_{q(p-q+1)}.$$

If $F_1, \dots, F_q$ are order p forms, then their Wronskian $\Theta(F_1 \wedge \dots \wedge F_q)$ is given by the $q \times q$ determinant

$$(i, j) \longrightarrow \frac{\partial^{q-1} F_i}{\partial x_0^{q-j} \, \partial x_1^{j-1}} \quad \text{for } 1 \leq i, j \leq q.$$

The crucial property of the construction is that Θ is nonzero on decomposable tensors, i.e., $\Theta(F_1 \wedge \dots \wedge F_q) = 0 \iff F_1 \wedge \dots \wedge F_q = 0 \iff$ the F_i are linearly dependent.

3.2.

Now let A, B, F be of order d, with Wronskian

$$\mathbb{W} = \Theta(A \wedge B \wedge F) = \begin{vmatrix} A_{x_0^2} & A_{x_0 x_1} & A_{x_1^2} \\ B_{x_0^2} & B_{x_0 x_1} & B_{x_1^2} \\ F_{x_0^2} & F_{x_0 x_1} & F_{x_1^2} \end{vmatrix}.$$

We will evaluate $\mathbb{W}$ symbolically. Let us write

$$A = \alpha_x^d, \ B = \beta_x^d, \ F = f_x^d. \tag{13}$$

As usual, α_x stands for the symbolic linear form $\alpha_0 x_0 + \alpha_1 x_1$, and $(\alpha\,\beta)$ for $\alpha_0\,\beta_1 - \alpha_1\,\beta_0$ etc.

Lemma 3.1. *With notation as above,*

$$\frac{1}{(d^2-d)^3}\,\mathbb{W} = (\alpha\,\beta)(\alpha\,f)(\beta\,f)\,\alpha_x^{d-2}\,\beta_x^{d-2}\,f_x^{d-2}. \tag{14}$$

Proof. Differentiating (13), we get expressions such as

$$A_{x_0 x_1} = d(d-1)\,\alpha_x^{d-2}\,\alpha_0\,\alpha_1.$$

Substitute these into $\mathbb{W}$ and factor out $\alpha_x^{d-2}\,\beta_x^{d-2}\,f_x^{d-2}$. We are left with a Vandermonde determinant which evaluates to $(\alpha\,\beta)(\alpha\,f)(\beta\,f)$. ☐

Now we will rewrite $\mathbb{W}$ in terms of transvectants.

Proposition 3.2. *With notation as in* (9)*, we have an identity*

$$\frac{1}{(d^2-d)^3}\,\mathbb{W} = (T_1, F)_2 - \frac{d-2}{4d-6}\,T_3\,F.$$

Proof. Symbolically, the transvectants can be written as

$$T_1 = (\alpha\,\beta)\,\alpha_x^{d-1}\beta_x^{d-1}, \quad T_3 = (\alpha\,\beta)^3\,\alpha_x^{d-3}\beta_x^{d-3}.$$

First we calculate the transvectant $(T_1, F)_2$ using the algorithm given in (see [5, §49]):

- Calculate the second polar T_1. It is equal to

$$\frac{(2d-4)!}{(2d-2)!}\,(\mathbf{y}\,\partial_{\mathbf{x}})^2\,T_1 = \frac{1}{(2d-2)(2d-3)}(\alpha\,\beta)\,\alpha_x^{d-3}\beta_x^{d-3}\times \\ \{(d-1)(d-2)\,\alpha_x^2\,\beta_y^2 + 2(d-1)^2\,\alpha_x\,\beta_x\,\alpha_y\,\beta_y + (d-1)(d-2)\,\beta_x^2\,\alpha_y^2\}.$$

- Make substitutions $\alpha_y := (\alpha\,f), \beta_y := (\beta\,f)$, and multiply by f_x^{d-2}. The result is

$$\begin{aligned}(T_1, F)_2 = &\frac{d-2}{4d-6}(\alpha\,\beta)\,\alpha_x^{d-3}\beta_x^{d-3}f_x^{d-2} \times \{(\beta\,f)^2\,\alpha_x^2 \\ &+ \frac{2d-2}{d-2}(\alpha\,f)(\beta\,f)\,\alpha_x\beta_x + (\alpha\,f)^2\,\beta_x^2\}.\end{aligned} \tag{15}$$

We would like to compare (14) and (15), so we will rewrite both of them in terms of standard monomials (see [16, Ch. 3]). Order the variables as $\alpha < \beta < f < x$. The monomial $(\beta\, f)\, \alpha_x$ is nonstandard, so use the Plücker syzygy to rewrite it as

$$(\beta\, f)\, \alpha_x = (\alpha\, f)\, \beta_x - (\alpha\, \beta) f_x.$$

Substitute this into the right hand sides of (14) and (15). Subtracting the two expressions, we get

$$(T_1, F)_2 - \frac{1}{(d^2-d)^3}\mathbb{W} = \frac{d-2}{4d-6}\, (\alpha\, \beta)^3\, \alpha_x^{d-3} \beta_x^{d-3} f_x^d = \frac{d-2}{4d-6}\, T_3\, F.$$

This completes the proof. □

3.3.

If M, N are forms of orders $2d-2, 2d-6$ respectively, then we define

$$\psi_{M,N}(F) := (M, F)_2 - \frac{d-2}{4d-6}\, N\, F. \tag{16}$$

We are interested in the differential equation

$$\psi_{M,N}(F) = 0, \tag{17}$$

which we may call the Wronskian (second order) ordinary differential equation with parameters M, N. (It is always assumed that $M \neq 0$, otherwise the equation is of no interest.) The following corollary is immediate.

Corollary 3.3. *If F is of order d, then $F \in \Pi_{A,B}$ iff $\psi_{T_1,T_3}(F) = 0$.*

Proof. Indeed, $\psi_{T_1,T_3}(F) = 0$ iff A, B, F are linearly dependent. □

Hence, given T_1, T_3, the pair $\{A, B\}$ is determined up to a unimodular transformation (cf. (5)). It follows that T_1, T_3 together determine all the T_{2r+1}.

Proposition 3.4. *Let M, N be of orders $2d-2, 2d-6$. Assume that* (17) *has two linearly independent solutions A, B of order d. Then there exists a nonzero constant λ such that $M = \lambda\, T_1,\ N = \lambda\, T_3$.*

Proof. Multiply the identities $\psi_{M,N}(A) = 0, \psi_{M,N}(B) = 0$ by B, A respectively and subtract, this gives $B\,(M, A)_2 = A\,(M, B)_2$. Now the Gordan series

$$\begin{pmatrix} A & M & B \\ d & 2d-2 & d \\ 0 & 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} B & M & A \\ d & 2d-2 & d \\ 0 & 0 & 2 \end{pmatrix}$$

respectively give identities

$$(A, M)_2\, B = (AB, M)_2 + ((A, B)_1, M)_1 + \frac{d}{4d-2}(A, B)_2\, M$$

$$(B, M)_2\, A = (BA, M)_2 + ((B, A)_1, M)_1 + \frac{d}{4d-2}(B, A)_2\, M.$$

Subtracting and using property (3) for A, B, we get $((A, B)_1, M)_1 = 0$. Now $(A, B)_1 \neq 0$ since A, B are independent, but then Lemma 2.1 implies that $M = \lambda\,(A, B)_1$ for some λ. Finally

$$\frac{d-2}{4d-6}\,N A = (M, A)_2 = \lambda\,(T_1, A)_2 = \lambda\,\frac{d-2}{4d-6}\,T_3\,A,$$

hence $N = \lambda\,T_3$. □

3.4.

We have shown that the following conditions are equivalent for the pair (M, N).

(i) There exist A, B such that $M = (A, B)_1, N = (A, B)_3$.

(ii) There exist A, B such that

$$\mathcal{R}(A, B) = c_0\,M^{\mathfrak{p}} + c_1\,\omega^2\,N^{\mathfrak{p}} + O(\omega^4).$$

(iii) The dimension of the kernel of the map $\psi_{M,N} : S_d \longrightarrow S_{3d-6}$ is two. (Since we have a second order o.d.e., it can never exceed two in any case.)

We can now construct the covariants Φ_r as in the introduction. Clearly (iii) is equivalent to the condition that the map

$$\wedge^d \psi_{M,N} : \wedge^d S_d \longrightarrow \wedge^d (S_{3d-6})$$

be zero. Identify $\wedge^d S_d$ with S_d via (11). Let f_1 denote the image of $\wedge^d \psi_{M,N}$ via the isomorphism

$$\mathrm{Hom}_{SL(V)}(S_d, \wedge^d S_{3d-6}) \simeq \mathrm{Hom}_{SL(V)}(\mathbf{C}, \wedge^d S_{3d-6} \otimes S_d).$$

Consider the composite morphism

$$\mathbf{C} \xrightarrow{f_1} \wedge^d S_{3d-6} \otimes S_d \xrightarrow{f_2} S_d(S_{2d-5}) \otimes S_d \xrightarrow{f_3} S_{d(2d-5)} \otimes S_d,$$

where f_2 comes from the isomorphism (11), and f_3 from the natural surjection (12). For each $0 \leq r \leq d$, we have projection maps

$$\pi_r : S_{d(2d-5)} \otimes S_d \longrightarrow S_{d(2d-4)-2r}$$

induced by the decomposition (1).

Define $\Phi_r(M, N)$ to be the image of $1 \in \mathbf{C}$ via the map $\pi_r \circ f_3 \circ f_2 \circ f_1$. This is a joint covariant of M, N of order $d(2d-4) - 2r$. We will describe it in coordinates. For $0 \leq i \leq d$, define

$$w_i = (-1)^i \frac{1}{i!(d-i)!}\,\Theta\Big(\bigwedge_{\substack{s=0 \\ s\neq i}}^{d} \psi_{M,N}(x_0^s\,x_1^{d-s})\Big),$$

which is an element of $S_{d(2d-5)}$. Then

$$(f_3 \circ f_2 \circ f_1)(1) = \sum_{i=0}^{d} w_i \otimes x_0^i\,x_1^{d-i}, \quad \text{and} \quad \Phi_r = \sum_{i=0}^{d} (w_i, x_0^i\,x_1^{d-i})_r.$$

All of this is straightforward and follows by chasing through the f_i. Each Φ_r has total degree d in the coefficients of M, N (because w_i does).

Theorem 3.5. *Let M, N be orders $2d-2, 2d-6$ respectively. Then the pair (M, N) satisfies the (equivalent) conditions* (i)–(iii) *if and only if*

$$\Phi_r(M, N) = 0 \quad \text{for } 0 \le r \le d.$$

Proof. If (iii) holds, then $f_1 = 0$, which shows the 'only if' part.
Conversely, assume that all the Φ_r vanish. Then $(f_3 \circ f_2 \circ f_1)(1) = 0$, which implies that all the w_i vanish. By the fundamental property of Wronskians, the forms

$$\psi_{M,N}(x_0^s\, x_1^{d-s}), \quad 0 \le s \le d,\ s \ne i$$

are linearly dependent for any i. But then the map $\wedge^d \psi_{M,N}$ is zero on every basis element of $\wedge^d S_d$, hence it is zero. This implies (iii). □

3.5. The incomplete Plücker imbedding

The fact that $\mathcal{R}$ is determined by T_1, T_3 has the following geometric interpretation. Assume $d \ge 3$, and let $\mathcal{G} = G(2, S_d)$ denote the Grassmannian of two-dimensional subspaces in S_d. (See [6, Lecture 6] for generalities on Grassmannians.) The line bundle $\mathcal{O}_{\mathcal{G}}(1)$ has global sections

$$H^0(\mathcal{G}, \mathcal{O}_{\mathcal{G}}(1)) \simeq \wedge^2 S_d \simeq S_2(S_{d-1}) \simeq \bigoplus_{r=0}^{\frac{d-1}{2}} S_{2d-4r-2}.$$

The usual Plücker imbedding is given by the complete linear system $|\mathcal{O}_{\mathcal{G}}(1)|$. Consider the subspace $W = S_{2d-2} \oplus S_{2d-6} \subseteq H^0(\mathcal{O}_{\mathcal{G}}(1))$.

Proposition 3.6. *The map*

$$\mu : \mathcal{G} \longrightarrow \mathbb{P}\, W, \quad \mathbb{P}\, \Pi_{A,B} \longrightarrow [\, T_1 \oplus T_3\,]$$

is an imbedding.

The usual conventions ([7, Ch. II, §7]) dictate that the imbedding is in $\mathbb{P}W^*$, but note the self-duality in §2.3.

Proof. We have already shown that μ is a set-theoretic injection. To complete the proof, it suffices to show that it is an injection on tangent spaces at every point (cf. [7, Ch. II, Prop. 7.3]). The Zariski tangent space to $\mathcal{G}$ at $\Pi = \Pi_{A,B}$ is canonically isomorphic to $\mathrm{Hom}(\Pi, S_d/\Pi)$ (see [6, Lecture 16]). Let $\alpha : \Pi \longrightarrow S_d/\Pi$ be a tangent vector, and say

$$\alpha(A) = Q + \Pi, \quad \alpha(B) = P + \Pi,$$

for some forms P, Q of order d.

The tangent space to $\mathbb{P}W$ at $[T_1 \oplus T_3]$ is isomorphic to $W/\langle T_1 \oplus T_3 \rangle$. Let $d\mu : T_{\mathcal{G},\Pi} \longrightarrow T_{\mathbb{P},\mu(\Pi)}$ denote the induced map on tangent spaces. Then $d\mu(\alpha)$ is the element

$$((A, P)_1 + (Q, B)_1) \oplus ((A, P)_3 + (Q, B)_3) \in W$$

considered modulo $T_1 \oplus T_3$. (To see this, let ϵ be an 'infinitesimal'. Now expand $(A+\epsilon\, Q, B+\epsilon\, P)_i, i=1,3$, and set $\epsilon^2=0$.)

We would like to show that $d\mu$ is injective, hence suppose that $d\mu(\alpha)=0$. Then there exists a constant c such that

$$\begin{aligned}(A,P)_1+(Q,B)_1 &= c\,(A,B)_1,\\ (A,P)_3+(Q,B)_3 &= c\,(A,B)_3.\end{aligned}$$

Substitute $P+c\,B$ for P (which does not change α), then

$$(A,P)_1=(B,Q)_1, \quad (A,P)_3=(B,Q)_3.$$

If the first pair is zero, then P,Q are respectively constant multiples of A,B, hence $\alpha=0$. If not, then $\Pi_{A,P}=\Pi_{Q,B}$ by Corollary 3.3. But this implies $\Pi_{A,B}=\Pi_{P,Q}$, again forcing $\alpha=0$. □

Remark 3.7. Let $\mathfrak{a}\subseteq \mathrm{Sym}^\bullet W$ denote the ideal generated by the coefficients of $\Phi_0,\dots,\Phi_d$, and J the homogeneous ideal of the image $\mu(\mathcal{G})\subseteq \mathbb{P}W$. Since $\mathfrak{a}$ defines the image set-theoretically, $(\sqrt{\mathfrak{a}})_{\mathrm{sat}}=J$. In general $\mathfrak{a}$ and J are different ideals (e.g., for $d=3$, the former is generated in degree 3 and the latter in degree 2). I do not know if one can state a more precise relation between them.

4. Generic reduction formulae

4.1.

We begin with the example which eventually led to the main result of this section. Let A,B be of order 2. The series $\begin{pmatrix} A & B & A\\ 2 & 2 & 2\\ 0 & 1 & 1\end{pmatrix}$ implies the relation

$$((A,B)_1,A)_1+\frac{1}{2}\,(A,B)_2\,A=\frac{1}{2}\,(A,A)_2\,B;$$

which can be rewritten as

$$-\frac{2}{(A,A)_2}(A,T_1)_1=B-\frac{(A,B)_2}{(A,A)_2}\,A.$$

Hence, given $\mathcal{R}$ (which involves only T_1 in this case) and A, the function

$$(A,T_1)\longrightarrow -\frac{2}{(A,A)_2}(A,T_1)_1 \tag{18}$$

recovers B up to an additive multiple of A. (Since $\mathcal{R}(A,B+\mathbf{k}\,A)=\mathcal{R}(A,B)$, the last proviso is indispensable.) We will show that there exist such formulae for every d.

We may call (18) a reduction formula in the following sense. If we are given a linear second order o.d.e., together with one of its solutions, then a second solution can be found by the method of 'reduction of order' (see [14, §44]). In our case, we are to find B, given the equation $\psi_{T_1,T_3}(F)=0$ with one solution A. However, this analogy is inexact in two respects:

- our formula will involve all the $\{T_{2r+1}\}$, and not merely T_1, T_3,
- the process is algebraic and involves no integration.

Moreover, the formula is generic in the sense that it is only defined over an open subset, e.g., the set $\{A \in \mathbb{P}^2 : (A, A)_2 \neq 0\}$ above.

4.2.

Throughout this section we assume that A, B are order d forms whose coefficients are algebraically independent indeterminates. Write

$$A = \sum_{p=0}^{d} \binom{d}{p} a_p \, x_0^{d-p} \, x_1^p. \tag{19}$$

Let $\mathbf{J}$ be an invariant of A of degree (say) n. We define its first evectant (cf. [17]) to be

$$\mathcal{E}_{\mathbf{J}} = \frac{(-1)^d}{n} \sum_{q=0}^{d} (-1)^q \, \frac{\partial \mathbf{J}}{\partial a_q} \, x_0^q \, x_1^{d-q}, \tag{20}$$

it is a covariant of degree-order $(n-1, d)$.

Lemma 4.1. *We have an identity* $(\mathcal{E}_{\mathbf{J}}, A)_d = \mathbf{J}$.

Proof. Substitute (19) and (20) in formula (2). We get a nonzero term whenever $p = q$ and $i = d - p$, hence

$$\begin{aligned}(\mathcal{E}_{\mathbf{J}}, A)_d &= \frac{(-1)^{2d}}{n\,(d\,!)^2} \sum_{p=0}^{d} (p!(d-p)!)^2 \binom{d}{d-p}^2 a_p \, \frac{\partial \mathbf{J}}{\partial a_p} \\ &= \frac{1}{n} \sum_p a_p \, \frac{\partial \mathbf{J}}{\partial a_p} = \mathbf{J},\end{aligned}$$

the last equality is by Euler's theorem. □

Now our generic reduction formula is as follows. Let

$$\beta(A, \mathcal{R}) = -\frac{1}{\mathbf{J}} \sum_{r \geq 0} c_r \, (\mathcal{E}_{\mathbf{J}}, T_{2r+1})_{d-2r-1}, \tag{21}$$

with the c_r as in (8).

Theorem 4.2. *With notation as above,*

$$\beta(A, \mathcal{R}) = B - \frac{(\mathcal{E}_{\mathbf{J}}, B)_d}{\mathbf{J}} \, A.$$

Hence, as long as A stays away from the hypersurface $\{\mathbf{J} = 0\}$, we can recover B from A and $\mathcal{R}(A, B)$.

Remark 4.3. If d is even, then we can take $\mathbf{J}$ to be the unique degree two invariant $(A, A)_2$. There is no invariant in degrees ≤ 3 if d is odd, but then there exists a degree four invariant $\mathbf{J} = ((A, A)_{d-1}, (A, A)_{d-1})_2$.

4.3.

The proof of the theorem will emerge from the discussion below. The element $A \wedge B \in \wedge^2 S_d$ defines a map

$$\sigma_{A\wedge B} : S_d \longrightarrow S_d, \quad F \longrightarrow (F,B)_d\, A - (F,A)_d\, B.$$

We identify the codomain of $\sigma = \sigma_{A\wedge B}$ with S_d^* as in Subsection 2.3.

Lemma 4.4. *With the convention above, σ is skew-symmetric, i.e.,*

$$\delta_{\sigma(F)}(G) = -\delta_{\sigma(G)}(F), \quad \textit{for } F, G \in S_d.$$

Proof. Unwinding the definitions, this becomes

$$(F,B)_d(A,G)_d - (F,A)_d(B,G)_d = -\{(G,B)_d(A,F)_d - (G,A)_d(B,F)_d\},$$

which is clear. □

Lemma 4.5. *With notation as above,*

$$\sigma(F) = [\,(F,\mathcal{R})^{\mathbf{x}}_{d-1}\,]_{\mathbf{y}:=\mathbf{x}}\,. \tag{22}$$

The right hand side of this identity is interpreted as follows: calculate the $(d-1)$-th transvectant of F and $\mathcal{R}$ as $\mathbf{x}$-forms (treating the $\mathbf{y}$ in $\mathcal{R}$ as constants). This produces an $\mathbf{xy}$-form of order $(1, d-1)$; finally replacing $\mathbf{y}$ by $\mathbf{x}$ gives a form of order d.

Proof. We will calculate both sides symbolically. Let $A = \alpha_x^d,\ B = \beta_x^d,\ F = f_x^d$, then

$$\begin{aligned}
\mathcal{R} &= \frac{A(\mathbf{x})B(\mathbf{y}) - A(\mathbf{y})B(\mathbf{x})}{\omega} = \frac{\alpha_x^d\,\beta_y^d - \alpha_y^d\,\beta_x^d}{\omega} \\
&= \frac{(\alpha_x\,\beta_y - \alpha_y\,\beta_x)\sum\limits_{i=0}^{d-1}(\alpha_x\,\beta_y)^{d-1-i}(\alpha_y\,\beta_x)^i}{\omega} \\
&= (\alpha\,\beta)\sum_i (\alpha_x\,\beta_y)^{d-1-i}(\alpha_y\,\beta_x)^i.
\end{aligned}$$

Now calculate the $(d-1)$-th transvectant of F with each summand in the last expression (treating α_y, β_y as constants). Using the algorithm of [5, §49],

$$(f_x^d, \alpha_x^{d-1-i}\,\beta_x^i)_{d-1} = (-1)^{d-1}(\alpha\, f)^{d-1-i}\,(\beta\, f)^i\, f_x.$$

Hence,

$$[\,(F,\mathcal{R})^{\mathbf{x}}_{d-1}\,]_{\mathbf{y}:=\mathbf{x}} = (-1)^{d-1}(\alpha\,\beta)\, f_x \sum_i (\alpha\, f)^{d-1-i}\,(\beta\, f)^i\,\alpha_x^i\,\beta_x^{d-1-i}. \tag{23}$$

Now directly from the definition,

$$\begin{aligned}\sigma(F) &= \{(f\,\beta)^d\,\alpha_x^d - (f\,\alpha)^d\,\beta_x^d\}\\ &= (-1)^d\{(\beta\,f)^d\,\alpha_x^d - (\alpha\,f)^d\,\beta_x^d\}\\ &= (-1)^d\{(\beta\,f)\,\alpha_x - (\alpha\,f)\,\beta_x\}\sum_i (\alpha\,f)^{d-1-i}\,(\beta\,f)^i\,\alpha_x^i\,\beta_x^{d-1-i}.\end{aligned}$$

Since $(\beta\,f)\alpha_x - (\alpha\,f)\beta_x = -(\alpha\,\beta)f_x$, the last expression is identical to (23). □

Lemma 4.6. *Let* $\mathbf{T}$ *be an arbitrary form of order* $2d-4r-2$. *Then*

$$[\,(F,\omega^{2r}\,T^{\mathfrak{p}})^{\mathbf{x}}_{d-1}\,]_{\mathbf{y}:=\mathbf{x}} = (F,\mathbf{T})_{d-2r-1}.$$

Proof. Let $\mathbf{T} = t_x^{2d-4r-2}$, so that $T^{\mathfrak{p}} = t_x^{d-2r-1}\,t_y^{d-2r-1}$. Then make a calculation as in the previous lemma. □

Now substitute the Taylor series (10) into the right hand side of (22), and use the previous lemma. This gives the formula

$$\sigma(F) = \sum_{r\geq 0} c_r\,(F,T_{2r+1})_{d-2r-1}. \tag{24}$$

Now specialize to $F = \mathcal{E}_{\mathbf{J}}$. Then

$$\sigma(\mathcal{E}_{\mathbf{J}}) = (\mathcal{E}_{\mathbf{J}},B)_d\,A - (\mathcal{E}_{\mathbf{J}},A)_d\,B = (\mathcal{E}_{\mathbf{J}},B)_d\,A - \mathbf{J}\,B,$$

hence

$$\beta(A,\mathcal{R}) = -\frac{1}{\mathbf{J}}\,\sigma(\mathcal{E}_{\mathbf{J}}) = B - \frac{(\mathcal{E}_{\mathbf{J}},B)_d}{\mathbf{J}}\,A.$$

This completes the proof of Theorem 4.2. □

5. Formulae for T_5 and T_7

We have observed that T_1, T_3 determine the higher odd transvectants T_{2r+1}. However this dependence is rather indirect, and it is unclear if one can give a formula for the latter in terms of the former. In this section we give such explicit formulae for T_5 and T_7.

5.1. The Plücker relations

Let

$$\mathcal{G} \subseteq \mathbb{P}(\wedge^2 S_d) = \mathbb{P}(\bigoplus_{r\geq 0} S_{2d-4r-2})$$

be the usual Plücker imbedding, and let $\mathbb{I}$ denote the homogeneous ideal of the image. It is well-known that $\mathbb{I}$ is generated by its quadratic part $\mathbb{I}_2$, usually called the module of Plücker relations.

Lemma 5.1. *As* $SL(V)$*-modules,* $\mathbb{I}_2 \simeq \wedge^4 S_d$.

Proof. Consider the short exact sequence

$$0 \to \mathbb{I}_2 \to H^0(\mathcal{O}_{\mathbb{P}(\wedge^2 S_d)}(2)) \to H^0(\mathcal{O}_{\mathcal{G}}(2)) \to 0.$$

(The exactness on the right comes from the projective normality of the imbedding.) Using the plethysm formula of [10, §I.8, Example 9], the middle term is isomorphic to

$$S_2(\wedge^2 S_d) \simeq S_{(2,2)}(S_d) \oplus \wedge^4 S_d.$$

By the Borel-Weil theorem (see [13, p. 687]), $H^0(\mathcal{O}_{\mathcal{G}}(2)) \simeq S_{(2,2)}(S_d)$. This completes the proof. □

Each Plücker relation corresponds to an algebraic identity between the $\{T_{2r+1}\}$. To be more precise, let $\{M_{2r+1} : r \geq 0\}$ be generic forms of orders $2d-4r-2$, and $S_e \stackrel{\xi}{\hookrightarrow} \mathbb{I}_2$ an inclusion of $SL(V)$-modules. Then ξ corresponds to a joint covariant $\Xi\,(M_1, M_3, \dots)$ of order e and total degree two in the $\{M_{2r+1}\}$, such that

$$\Xi\,(T_1, T_3, \dots) = 0, \qquad \text{for any } A, B \text{ of order } d.$$

Example 5.2. Assume $d = 4$. In this case $\mathbb{I}_2 \simeq S_4$, so we look for an order 4 covariant in M_1, M_3. There are three 'monomials' of total degree 2 and order 4, namely $(M_1, M_1)_4, (M_1, M_3)_2, M_3^2$. Our covariant must be a linear combination of these, i.e.,

$$\Xi(M_1, M_3) = \alpha_1\,(M_1, M_1)_4 + \alpha_2\,(M_1, M_3)_2 + \alpha_3\,M_3^2,$$

for some constants α_i.

Now specialize to $A = x_0^4, B = x_1^4$, and use formula (2) to calculate T_1, T_3 and Ξ explicitly. Since $\Xi(T_1, T_3)$ must vanish identically, its coefficients give 5 linear equations for the α_i. Solving these (they must admit a nontrivial solution), we deduce that

$$[\alpha_1 : \alpha_2 : \alpha_3] = [25 : -10 : -4],$$

which determines Ξ (of course, up to a scalar). This 'method of undetermined coefficients' (specializing the forms followed by solving linear equations) will be liberally used in the sequel.

Example 5.3. For $d = 3$, the Grassmannian is a quadric hypersurface defined by

$$\Xi(M_1, M_3) = (M_1, M_1)_4 - \frac{1}{6}\,M_3^2.$$

5.2.

We begin with a technical lemma about the irreducible submodules of $\mathbb{I}_2$.

Lemma 5.4. *If $d \geq 4$, then there exists exactly one copy each of the modules S_{4d-12}, S_{4d-16} inside $\mathbb{I}_2$.*

Proof. There are isomorphisms

$$\mathbb{I}_2 \simeq \wedge^4 S_d \simeq S_4(S_{d-3}) \simeq S_{d-3}(S_4),$$

where the second isomorphism is from (11), and the third is Hermite reciprocity. Hence we may as well work with $S_{d-3}(S_4)$. Now the following are in bijective correspondence (see [9] for details):

- inclusions $S_e \subseteq S_{d-3}(S_4)$ of $SL(V)$-modules,
- covariants of degree-order $(d-3, e)$ (distinguished up to scalars) for binary quartics.

Fortunately, a complete set of generators for the covariants of binary quartics is known (see [5, §89]). It contains five elements, conventionally called f, H, t, i, j, having degree-orders

$$(1,4),\ (2,4),\ (3,6),\ (2,0),\ (4,0).$$

(It is unnecessary for us to know how they are defined.) Each covariant of quartics is a polynomial in the elements of this set.

Now it is elementary to see that only one expression of degree-order $(d-3, 4d-12)$ is possible, namely f^{d-3}. Similarly, the only possible expression for degree-order $(d-3, 4d-16)$ is $f^{d-5}\,H$. Hence there is exactly one copy each of S_{4d-12} and S_{4d-16}. $\square$

5.3.

We will find the joint covariant Ξ corresponding to $S_{4d-12} \subseteq \mathbb{I}_2$. We look for degree two monomials of order $4d-12$ in the $\{T_{2r+1}\}$; any such monomial must be of the form

$$(T_{2a+1}, T_{2b+1})_s,$$

where

- $(2d-4a-2)+(2d-4b-2)-2s = 4d-12$,
- $a, b \le \lfloor \frac{d-1}{2} \rfloor$,
- $s \le \min\{2d-4a-2, 2d-4b-2\}$,
- if $a = b$, then s is even.

The first condition comes from the order, the rest are forced by properties (3), (4) of transvectants. Sifting through these conditions gives only four possibilities, namely

$$(T_1, T_1)_4,\ (T_1, T_3)_2,\ T_3^2,\ T_1 T_5.$$

Hence we have an identical relation of the form

$$\alpha_1\,(T_1, T_1)_4 + \alpha_2 (T_1, T_3)_2 + \alpha_3\,T_3^2 - \alpha_4\,T_1\,T_5 = 0.$$

Specialize A, B successively to the pairs

$$(x_0^d, x_1^d),\ (x_0^{d-1}x_1, x_1^d),\ (x_0^{d-2}x_1^2, x_1^d),\ (x_0^{d-1}x_1, x_0\,x_1^{d-1}),$$

and use the method of undetermined coefficients. Up to a scalar, the solution is

$$\begin{array}{ll} \alpha_1 = -\frac{2(2d-3)^2}{d(d-2)} & \alpha_2 = \frac{4(2d-3)(d-3)}{d(d-2)} \\ \alpha_3 = 1 & \alpha_4 = \frac{(d-3)(d-4)(2d-3)^2}{d(2d-5)(2d-7)(d-2)}. \end{array}$$

This gives a formula for T_5.

Theorem 5.5. *Assume $d \geq 5$, then*

$$T_5 = \frac{1}{T_1}\,(\,\frac{\alpha_1}{\alpha_4}\,(T_1, T_1)_4 + \frac{\alpha_2}{\alpha_4}\,(T_1, T_3)_2 + \frac{\alpha_3}{\alpha_4}\,T_3^2).$$

We can make a similar argument with S_{4d-16}, which leads to a formula for T_7. Define

$$\begin{array}{ll} \beta_1 = -\frac{8(2d-5)(2d-7)(2d-3)}{d(d-1)(4d-13)} & \beta_2 = -\frac{60(2d-7)(2d-5)}{d(d-1)(4d-13)} \\ \beta_3 = \frac{12(2d-3)(d-5)}{d(4d-13)} & \beta_4 = \frac{20(2d-5)(2d-7)(d-3)}{(d-1)(4d-13)(2d-3)} \\ \beta_5 = 1 & \beta_6 = \frac{(d-5)(d-6)(2d-3)(2d-5)}{d(d-1)(2d-9)(2d-11)} \end{array}$$

Theorem 5.6. *For $d \geq 7$,*

$$T_7 = \frac{1}{T_1}(\,\frac{\beta_1}{\beta_6}\,(T_1, T_1)_6 + \frac{\beta_2}{\beta_6}\,(T_1, T_3)_4 + \frac{\beta_3}{\beta_6}\,(T_1, T_5)_2 + \frac{\beta_4}{\beta_6}\,(T_3, T_3)_2 + \frac{\beta_5}{\beta_6}\,T_3\,T_5).$$

(Of course we can substitute for T_5 using the previous result, but this would make the formula very untidy.)

This method breaks down for higher transvectants, so a new idea will be needed for the general case. My colleague A. Abdesselam, when shown the formulae above, remarked that the coefficients look very similar to those appearing in the classical hypergeometric series. Perhaps there is something to this suggestion.

6. Open problems

This section contains a series of miscellaneous calculations and examples, all of them for small specific values of d. They should serve simultaneously as a source of open questions and further lines of inquiry.

6.1. The Jacobian predicate

Let A, M be forms of orders $d, 2d-2$. Consider the following predicate

$$J(A, M) : \text{there exists an order } d \text{ form } B \text{ such that } (A, B)_1 = M.$$

If $J(A, M)$ holds, then $(A, M)_2 = \mathbf{k}\; T_3\, A$, hence A must divide $(A, M)_2$. We will see below that this condition is sufficient for $d = 2, 3$, but not for $d = 4$.

Proposition 6.1. *Assume $d = 2$. Then*

$$J(A, M) \iff (A, M)_2 = 0.$$

Proof. The forward implication is clear.

For the converse, assume $(A, M)_2 = 0$. Then $\begin{pmatrix} A & M & A \\ 2 & 2 & 2 \\ 0 & 1 & 1 \end{pmatrix}$ implies that $((A, M)_1, A)_1 = -\frac{1}{2}(A, A)_2 M$. If $(A, A)_2 \neq 0$, then let

$$B = \frac{2}{(A, A)_2}(A, M)_1.$$

If $(A, A)_2 = 0$, then by a change of variable, we may assume $A = x_0^2$. Then $(A, M)_2 = 0$ implies that $M = c_1 x_0^2 + c_2 x_0 x_1$. Now let $B = c_1 x_0 x_1 + \frac{c_2}{2} x_1^2$. In either case, $(A, B)_1 = M$. $\square$

Proposition 6.2. *Assume $d = 3$, then*

$$J(A, M) \iff ((A, M)_2, A)_1 = 0.$$

Proof. By Lemma 2.1, $((A, M)_2, A)_1 = 0$ iff $(A, M)_2 = \mathbf{k}\, A$. This shows the forward implication.

Conversely, assume that $(A, M)_2 = c\, A$ for some constant c. I claim that the map

$$\psi_{M,6\,c} : S_3 \longrightarrow S_3, \quad F \longrightarrow (M, F)_2 - c\, F$$

is skew-symmetric. Indeed,

$$\delta_{\psi_{M,6\,c}(F)}(G) = ((M, F)_2, G)_3 - c\,(F, G)_3.$$

Using $\begin{pmatrix} M & F & G \\ 4 & 3 & 3 \\ 1 & 2 & 2 \end{pmatrix}$, this can be transformed into

$$-((M, G)_2, F)_3 + c\,(G, F)_3 = -\delta_{\psi_{M,6\,c}(G)}(F).$$

This proves the claim, and implies that the rank of $\psi_{M,6\,c}$ must be even. Suppose that A and another form B span its kernel. Then by Proposition 3.4, $(A, B)_1 = M$ (after multiplying B by a constant if necessary). $\square$

Example 6.3. Assume $d = 4$, and let $A = (x_0 x_1)^2$, $M = (x_0 x_1)^3$. Then A divides $(A, M)_2 = \mathbf{k}\, (x_0 x_1)^3$. However there exists no B such that $(A, B)_1 = M$. Indeed,

$$(A, B)_1 = \mathbf{k}\, x_0\, x_1 (x_1 B_{x_1} - x_0 B_{x_0}) = (x_0\, x_1)^3$$

would imply $x_1 B_{x_1} - x_0 B_{x_0} = \mathbf{k}\,(x_0 x_1)^2$. But then $B = \mathbf{k}\,(x_0 x_1)^2$, which is absurd.

The two propositions above suggest the following natural problem:

Problem 6.4. Find a (finite) number of joint covariants of A, M which simultaneously vanish iff $J(A, M)$ holds.

6.2. The resultant

Let $\mathrm{Res}(A, B)$ denote the resultant of A, B. Up to a scalar, it is equal to the discriminant of $\mathcal{R}(A, B)$ (regarded as a quadratic form). Since the latter implicitly depends only on T_1, T_3, the following problem is natural:

Problem 6.5. Give an explicit formula (in a reasonable sense) for $\mathrm{Res}(A, B)$ as a joint invariant of T_1 and T_3.

For instance, if $d = 2$ then $\mathbf{k}\,\mathrm{Res}(A, B) = (T_1, T_1)_2$.

Proposition 6.6. *If $d = 3$, then*

$$\mathbf{k}\,Res(A, B) = T_3\,(T_1, T_1)_4 - 6\,(T_1, (T_1, T_1)_2)_4.$$

Proof. By construction, $\mathrm{Res} = \mathrm{Res}(A, B)$ is joint invariant of total degree 3 in T_1, T_3. Every joint invariant is a linear combination of compound transvectants (see [2, p. 92]), hence Res is a linear combination of terms of the form

$$(X_1, (X_2, X_3)_a)_b,$$

where a, b are integers, and each X_i stands for either T_1 or T_3. Since the total order must be zero, $\sum\limits_i \mathrm{ord\ X_i} = \mathrm{2(a + b)}$. Using properties (3), (4), we are left with only two possibilities, namely

$$(T_3, (T_1, T_1)_4)_0, \quad (T_1, (T_1, T_1)_2)_4.$$

Now specialize to $A = x_0x_1(x_0 - x_1), B = x_0(x_0 + x_1)(x_0 + 2x_1)$ and use the method of undetermined coefficients. □

Gordan [4] has given a formula for the resultant in terms of all the odd order transvectants $\{T_{2r+1} : r \geq 0\}$.

6.3. The minimal equation for T_3

Consider the following equivalence relation on pairs (A, B) of independent order d forms:

$$(A, B) \sim (\alpha A + \beta B, \gamma A + \delta B) \quad \text{if } \alpha\delta - \beta\gamma = 1.$$

An equivalence class determines and is determined by T_1, T_3. Let $\mathcal{F}$ denote the set of equivalence classes, and consider the map

$$\pi : \mathcal{F} \longrightarrow \mathbb{A}^{2d-1}, \quad (A, B) \longrightarrow T_1.$$

It is known that π has finite fibres, and the cardinality of the general fibre is equal to the Catalan number $\rho(d) = \frac{1}{d}\binom{2d-2}{d-1}$ (see [3, Theorem 1.3]).

Now assume $d = 4$, then $\rho(4) = 5$. Let A, B be forms of order 4 with indeterminate coefficients, and write

$$T_1 = \sum_{i=0}^{6} \binom{6}{i} u_i\, x_0^{6-i} x_1^i, \quad T_3 = \sum_{j=0}^{2} \binom{2}{j} v_j\, x_0^{2-j} x_1^j,$$

where u_i, v_j are functions of the coefficients of A, B. The map π corresponds to a degree 5 field extension $K \subseteq L$, where

$$K = \mathbf{C}(u_0, \ldots, u_6), \quad L = K(v_0, v_1, v_2).$$

We recall the concept of a seminvariant of a form: it is an expression in the coefficients of the form which remains unchanged by a substitution

$$x_0 \longrightarrow x_0 + c\,x_1, \; x_1 \longrightarrow x_1; \quad c \in \mathbf{C}. \tag{25}$$

An alternative is to define it as the leading coefficient of a covariant (see [5, §32]). Let

$$v_0^5 + \sum_{i=1}^{5} l_i\, v_0^{5-i} = 0, \qquad l_i \in K, \tag{26}$$

denote the unique minimal equation of v_0 over K. Firstly, since v_0 is a seminvariant of T_3 and substitutions in (25) must leave (26) unchanged, all the l_i are seminvariants of T_1. Secondly, by the main theorem of [5, §33], any algebraic relation between the seminvariants lifts to a relation between the corresponding covariants. That is to say, we must have an identity

$$T_3^5 + \sum_{i=1}^{5} \Lambda_i\, T_3^{5-i} = 0, \tag{27}$$

where Λ_i are covariants of T_1, and (27) reduces to (26) by the substitution $x_0 := 1, x_1 := 0$. By homogeneity, Λ_i must have degree-order $(i, 2i)$.

6.4.

A complete set of generators for the ring of covariants of order 6 forms is given in [5, §134]. It is then a routine matter to identify the Λ_i by the method of undetermined coefficients. I omit all calculations and merely state the result. Define the following covariants of T_1.

$$\begin{aligned}
&q_{20} = (T_1, T_1)_6, \quad q_{24} = (T_1, T_1)_4, \quad q_{28} = (T_1, T_1)_2,\\
&q_{32} = (T_1, q_{24})_4, \quad q_{36} = (T_1, q_{24})_2, \quad q_{38} = (T_1, q_{24})_1,\\
&q_{44} = (T_1, q_{32})_2.
\end{aligned}$$

These are all taken from the table in [5, p. 156], but the notation is modified so that q_{ab} is of degree-order (a, b). There can be no covariant of degree-order $(1, 2)$, hence $\Lambda_1 = 0$. The others are

$$\begin{aligned}
\Lambda_2 &= -\frac{125}{8}\, q_{24}\\
\Lambda_3 &= \frac{625}{24}\, q_{36} + \frac{125}{36}\, T_1\, q_{20}\\
\Lambda_4 &= \frac{3125}{48}\, q_{24}^2 - \frac{625}{96}\, q_{20}\, q_{28} - \frac{3125}{96}\, T_1\, q_{32}\\
\Lambda_5 &= \frac{3125}{64}\, T_1\, q_{44} + \frac{3125}{64}\, q_{32}\, q_{28} - \frac{3125}{16}\, q_{36}\, q_{24} - \frac{3125}{192}\, T_1\, q_{20}\, q_{24}.
\end{aligned}$$

Problem 6.7. Find the equation analogous to (27) for arbitrary d.

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Received August 1, 2004