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On Boundary Arcs Joining Antipodal Points of a Planar Convex Body

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Abstract.
Using notions of Minkowski geometry (i.e., of the geometry of finite dimensional Banach spaces) we find new characterizations of centrally symmetric convex bodies, equiframed curves, bodies of constant width and certain convex bodies with modified constant width property. In particular, we show that straightforward extensions of some properties of bodies of constant Euclidean width are also valid for bodies of constant Minkowskian width if the underlying Minkowskian circle is an equiframed curve. All obtained characterizations are restricted to the case of the plane and involve certain measures of boundary arcs that join antipodal points of a planar convex body.

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1. Introduction

We refer to the next section for all standard notations from Euclidean and convex geometry. It should be noticed that our standard notations are mostly borrowed

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from [29]. Let B be a convex body in $\mathbb{E}^2$ centered at the origin o. Then $\mathcal{M}^2(B)$ denotes the *Minkowski plane* (i.e., two-dimensional real normed space) with unit ball B. The norm of $\mathcal{M}^2(B)$ is denoted by $\|\,.\,\|_B$. Basic information on Minkowski spaces can be found in [31] and the surveys [27] and [25]. A homothetical copy of B is called a *Minkowskian ball* in $\mathcal{M}^2(B)$, the corresponding homothety coefficient the *Minkowskian radius* of this Minkowskian ball. The *isoperimetrix* $\tilde{B}$ in $\mathcal{M}^2(B)$ is defined as the dual body of B rotated by the angle $\frac{\pi}{2}$. Obviously, the body B is the isoperimetrix in the Minkowski plane $\mathcal{M}^2(\tilde{B})$, i.e., $\tilde{\tilde{B}} = B$. Let K be an arbitrary convex body in $\mathcal{M}^2(B)$. If $u \in \mathcal{M}^2(B) \setminus \{o\}$, then the Minkowskian distance between the two different supporting lines orthogonal to the direction u is called the *Minkowskian width* of K at direction u (notation: $w_{K,B}(u)$). It is known that

$$w_{K,B}(u) = \frac{w_K(u)}{h_B(u)}. \tag{1}$$

See [18] and [5] for results on cross-section measures in Minkowski spaces as well as [19] for computational problems for Minkowskian cross-section measures.

Given a Euclidean unit vector u in $\mathbb{E}^2$, let $\alpha_B(u) > 0$ be such that $4\alpha_B(u)$ is the minimal area of a parallelogram which is circumscribed about B and has two opposite sides orthogonal to u. The quantity $\alpha_B(u)$ is expressed analytically by

$$\alpha_B(u) = h_B(u) r_B(v), \tag{2}$$

where v is a Euclidean unit vector orthogonal to u. Clearly, (2) yields

$$\alpha_B(u) = \frac{h_B(u)}{h_{\tilde{B}}(u)}. \tag{3}$$

It is easy to see that the boundary of every planar convex body can be parameterized by a Lipschitz function. Indeed, let K be a planar convex body in $\mathbb{E}^2$, let $\operatorname{perim}(K)$ be the Euclidean perimeter of $\operatorname{bd} K$, and $p(t),\ t \in [0, \operatorname{perim}(K)]$, be a parameterization of $\operatorname{bd} K$ by arc length (that is, t is the length of the arc $\{p(s) : 0 \le s \le t\}$). Since $|t_1 - t_2|$ is the length of a boundary arc of K joining $p(t_1)$ with $p(t_2)$ we infer that $|p(t_1) - p(t_2)| \le |t_1 - t_2|$ for every $t_1, t_2 \in [0, \operatorname{perim}(K)]$, which implies that $p(t)$ is a Lipschitz function. In this paper a *parameterization* $p(t),\ t \in [t_0, t_1]$ $(t_0 < t_1)$, of a curve $\gamma \subseteq \mathbb{E}^2$ is defined as a Lipschitz continuous function such that for every point x from $\{p(t) : t_0 < t < t_1\}$ the pre-image $\{t \in [t_0, t_1] : p(t) = x\}$ of the point x is a segment in $\mathbb{R}$ probably degenerate to a singleton, i.e., for t ranging from t_1 to t_2, the point $p(t)$ moves on γ from $p(t_0)$ to $p(t_1)$ according to some orientation of γ.

Two boundary points p and q of a convex body $K \subseteq \mathbb{E}^2$ are said to be *antipodal* if they lie in different parallel supporting lines of K. A chord of K joining two antipodal points of K is called an *affine diameter* of K. A parameterization $p(t),\ t \in [0, 2\pi]$, of $\operatorname{bd} K$ is called *antipodality preserving* if $p(t)$ is antipodal to $p(t + \pi)$ for every $t \in [0, \pi]$. It turns out that the following lemma holds.

Lemma 1. *The boundary of every planar convex body possesses an antipodality preserving parameterization.*

Given a Minkowski plane $\mathcal{M}^2(B)$, a convex body $K \subseteq \mathbb{E}^2$, and an antipodality preserving parameterization $p(t)$ of $\operatorname{bd} K$, we introduce the following functions of t. (For definiteness we assume that $p(t)$ has counterclockwise orientation.) For $t \in [0, 2\pi]$ let $u(t)$ be an outward Euclidean unit normal of K at the point $p(t)$ (if K is non-smooth, $u(t)$ is discontinuous). By $w_B(t)$ we shall denote the Minkowskian width of $K \subseteq \mathcal{M}^2(B)$ at direction $u(t)$. Let $\gamma(t)$ denote the counterclockwise oriented boundary arc of K starting at $p(t)$ and terminating at $p(t+\pi)$, see Figure 1. By $l_B(t)$ we denote the length of $\gamma(t)$ measured in $\mathcal{M}^2(B)$, and by $v(t)$ the area of $\operatorname{conv} \gamma(t)$ (see Figure 1). Further on, let $\alpha_B(t) := \alpha_B(u(t))$, see Figure 2. The above functions depending on t can be extended periodically. Hence they will be considered for t ranging over the whole real axis.

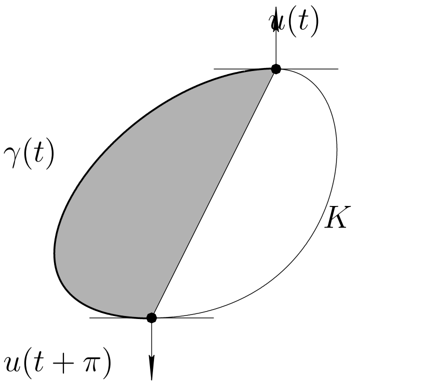

Figure 1: $\operatorname{conv} \gamma(t)$ is filled by gray color

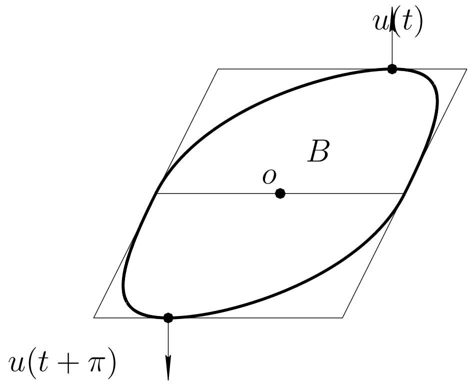

Figure 2: Illustration to $\alpha_B(t)$

The following theorem presents two differential relations involving the functions $v(t), w_B(t), l_B(t), l_{\tilde{B}}(t)$, and $\alpha_B(t)$. The abbreviation a.e. stands for *almost everywhere*. It is used to mark those relations which are fulfilled for almost all values (in the sense of measure) of the parameter involved in the relation.

Theorem 2. *Let K be an arbitrary convex body in a Minkowski plane $\mathcal{M}^2(B)$, and $p(t)$ be an antipodality preserving parameterization of* $\operatorname{bd} K$. *Then the quantities $w_B(t), l_B(t), l_{\tilde{B}}(t), \alpha_B(t)$ and $v(t)$, associated with $p(t)$, are related by the following two equalities:*

$$w_B(t) l'_{\tilde{B}}(t) - 2v'(t) \overset{a.e.}{=} 0 \qquad (t \in \mathbb{R}), \tag{4}$$

$$\alpha_B(t) w_B(t) l'_B(t) - 2v'(t) \overset{a.e.}{=} 0 \qquad (t \in \mathbb{R}). \tag{5}$$

A convex body $K \subseteq \mathcal{M}^2(B)$ is said to be of *constant Minkowskian width* if K has the same Minkowskian width at any direction. We introduce a modification of the notion constant width, which is involved in the statements of our main results. A vector $u \in \mathbb{E}^2 \setminus \{o\}$ is an *extreme normal* of K if u is either left or right or unique normal of K at some point $p \in \operatorname{bd} K$ (cf. also [29, p. 74]). Any quantity $w_{K,B}(u)$, where u is an extreme normal of K, is said to be a *tangential width* of

K in $\mathcal{M}^2(B)$. A convex body $K \in \mathcal{M}^2(B)$ is said to be of *constant tangential width* in $\mathcal{M}^2(B)$ if its tangential width in $\mathcal{M}^2(B)$ is determined uniquely, i.e., $w_{K,B}(u) = \text{const}$ for each extreme normal $u \in \mathbb{S}^1$ of DK (where DK denotes the difference body of K). Given an antipodality preserving parameterization $p(t)$ of $\operatorname{bd} K$, it is not hard to prove that $K \in \mathcal{M}^2(B)$ is of constant tangential width in $\mathcal{M}^2(B)$ if and only if $w_B(t) \overset{a.e.}{=} \text{const}$. It is known that K is of constant width $\lambda > 0$ in $\mathcal{M}^2(B)$ if and only if $DK = \lambda B$. In contrast to this, K is of constant tangential width in $\mathcal{M}^2(B)$ if and only if DK is a tangential body of λB (for the definition of tangential body see [29, pp. 69–70]). Clearly, strictly convex bodies of constant tangential width are necessarily of constant width in the classical sense.

The results on bodies of constant width in Euclidean and Minkowski spaces are surveyed in [10], [13], [21, Section 5], and [25]. Let $\lambda > 0$ and p_1, p_2, p_3 be points in $\mathcal{M}^2(B)$ such that $\|p_i - p_j\| = \lambda$ for $i, j \in \{1, 2, 3\},\ i \neq j$. Then the intersection of three Minkowskian circles of radius λ centered at p_1, p_2, and p_3 is called a *Minkowskian Reuleaux triangle.* Reuleaux triangles are the best known non-trivial examples of planar bodies of constant width (we shall need Reuleaux triangles in the proof of one of our theorems). Applying Theorem 2 we obtain the following characterizations of planar convex bodies of constant width and constant tangential width.

Theorem 3. *Let $K \subseteq \mathbb{E}^2$ be a planar convex body, and $p(t)$ be an antipodality preserving parameterization of* $\operatorname{bd} K$. *Then for functions $l_{\tilde{B}}(t), l_B(t), w_B(t), \alpha_B(t)$, and $v(t)$, associated with $p(t)$, the following statements hold:*

I. *The body K is of constant tangential width in $\mathcal{M}^d(B)$ if and only if*

$$w_B(t) l_{\tilde{B}}(t) - 2v(t) \overset{a.e.}{=} \text{const} \qquad (t \in \mathbb{R}). \tag{6}$$

II. *The body K is of constant width in $\mathcal{M}^2(B)$ if and only if*

$$w_B(t) l_{\tilde{B}}(t) - 2v(t) \overset{a.e.}{=} \frac{1}{2} w_B(t)^2 V(B) - V(K) \qquad (t \in \mathbb{R}). \tag{7}$$

III. *The body K is of constant tangential width in $\mathcal{M}^2(\tilde{B})$ if and only if*

$$\alpha_B(t) w_B(t) l_B(t) - 2v(t) \overset{a.e.}{=} \text{const} \qquad (t \in \mathbb{R}).$$

IV. *The body K is of constant width in $\mathcal{M}^2(\tilde{B})$ if and only if*

$$\alpha_B(t) w_B(t) l_B(t) - 2v(t) \overset{a.e.}{=} \frac{1}{2} \alpha_B(t)^2 w_B(t)^2 V(\tilde{B}) - V(K) \qquad (t \in \mathbb{R}).$$

Let $\mathcal{M}^2(B)$ be a Minkowski plane with strictly convex B and let K be a planar strictly convex body. Further on, let $p(t)$ denote the antipodality preserving parameterization of K for which $u(t) = (\cos t, \sin t)$. Then, by Theorem 3 (Part I), K is of constant width in $\mathcal{M}^2(B)$ if and only if $w_B(t) l_{\tilde{B}}(t) - 2v(t) = \text{const}$. In the Euclidean plane this characterization of planar bodies of constant width was proved in [10, p. 41], the necessity for the Euclidean plane was derived in [11].

Other recent characterizations of constant Minkowskian width are presented in [2], [1], [4], and [7].

We also derive the following characterization of planar centrally symmetric convex bodies.

Theorem 4. *A convex body K in a Minkowski plane $\mathcal{M}^2(B)$ is centrally symmetric if and only if every chord of K that bisects the Minkowskian perimeter of K is necessarily an affine diameter of K.*

The Euclidean version of the above theorem was established in [32] for the case that K is strictly convex and $\operatorname{bd} K$ has continuous curvature and in [22] for an arbitrary convex body $K \subseteq \mathbb{E}^2$. Also, in [22] it was stated without proof that Theorem 4 holds for an arbitrary Minkowski plane. Further characterizations of centrally symmetric convex bodies are collected in [21, Section 4] and [9, §14].

Theorem 4 can be applied for deriving characterizations of centrally symmetric members within classes of special convex bodies (e.g., characterizations of unit Minkowskian balls within the class of planar bodies of constant Minkowskian width). Some characterizations of Minkowskian unit balls within the class of planar convex bodies of constant Minkowskian width that were obtained in [14] are straightforward corollaries of Theorem 4 (see also [20] for the Euclidean version of the mentioned results from [14]). The statement of Theorem 4 for an arbitrary convex body in $\mathbb{E}^2$ was obtained by several authors using various methods, cf. [8], [30], and [24]. The statement of Theorem 4 has recently been carried over to convex bodies in Minkowski spaces of any dimension (cf. [3]). The Euclidean version of some of the results in Minkowski spaces obtained in [3] was presented in [28].

Let $\mathcal{M}^2(B)$ be a Minkowski plane such that the isoperimetrix $\tilde{B}$ is a homothetical copy of B. Then the homothetical copies of $\operatorname{bd} B$ are called *Radon curves,* cf. [31, p. 128]. If every boundary point p of B is touched by a circumscribed parallelogram of B having minimal area, then the homothetical copies of $\operatorname{bd} B$ are called *equiframed curves,* cf. [26]. It is known that every Radon curve is necessarily equiframed, and that the converse is not true in general.

Let $p(t)$ be an arbitrary antipodality preserving parameterization of a planar convex body $K \subseteq \mathcal{M}^2(B)$. In view of the results from [26, Section 4], $\operatorname{bd} B$ is an equiframed curve if and only if $\alpha_B(t) \overset{a.e.}{=} \text{const}$ $(t \in \mathbb{R})$.

The following theorem presents some characterizations of equiframed curves and related characteristic properties of bodies of constant width and constant tangential width in Minkowski planes whose unit Minkowskian circle is an equiframed curve. See also [16] and [17] for a related characterization of Radon and equiframed curves and [15] for a characterization of ellipsoids in terms of equiframed sections.

Theorem 5. *Let $\mathcal{M}^2(B)$ be an arbitrary Minkowski plane. Then the following statements hold true.*

I. *The following conditions are equivalent:*

(i) *The boundary of B is an equiframed curve.*

(ii) *There exists a constant* $\alpha > 0$ *such that for every convex body* K *in* $\mathcal{M}^2(B)$ *and every antipodality preserving parameterization* $p(t)$ *of* $\operatorname{bd} K$ *we have*

$$\alpha w_B(t) l_B'(t) - 2v'(t) \overset{a.e.}{=} 0 \qquad (t \in \mathbb{R}). \tag{8}$$

(iii) *For every convex body* K *and every antipodality preserving parameterization* $p(t)$ *of* $\operatorname{bd} K$ *there exists a constant* $\alpha_K > 0$ *such that*

$$\alpha_K w_B(t) l_B'(t) - 2v'(t) \overset{a.e.}{=} 0 \qquad (t \in \mathbb{R}). \tag{9}$$

(iv) *For some convex body* K *in* $\mathcal{M}^2(B)$ *whose boundary possesses an antipodality preserving parameterization* $p(t)$ *with* $l_B'(t) \neq 0$ *almost everywhere and for some* $\alpha_K > 0$ *we have:*

$$\alpha_K w_B(t) l_B(t) - 2v(t) \overset{a.e.}{=} \text{const} \qquad (t \in \mathbb{R}). \tag{10}$$

II. *If* $\operatorname{bd} B$ *is an equiframed curve,* $\alpha := \alpha_B(t)$*, and* K *is an arbitrary convex body in* $\mathcal{M}^2(\tilde{B})$*, then*

(a) *the equality*

$$\alpha l_B(t) w_B(t) - 2v(t) \overset{a.e.}{=} \text{const} \qquad (t \in \mathbb{R})$$

holds if and only if K *is of constant tangential width in* $\mathcal{M}^2(\tilde{B})$*;*

(b) *the equality*

$$\alpha l_B(t) w_B(t) - 2v(t) \overset{a.e.}{=} \frac{1}{2}\alpha^2 w_B(t)^2 V(\tilde{B}) - V(K) \qquad (t \in \mathbb{R})$$

holds if and only if K *is of constant width in* $\mathcal{M}^2(\tilde{B})$.

2. Preliminaries

The Euclidean plane is denoted by $\mathbb{E}^2$, o stands for the origin and $|\,.\,|$ for the norm in $\mathbb{E}^2$. We also use $|\,.\,|$ to denote the absolute value of scalars. The length in $\mathbb{E}^2$ is denoted by μ. In analytic expressions the elements of $\mathbb{E}^2$ will be identified with pairs of real numbers. By $\mathbb{S}^1$ we denote the unit circle in $\mathbb{E}^2$. Given $x, y \in \mathbb{E}^2$, by $[x, y]$ we denote the segment joining x and y. A convex body in $\mathbb{E}^2$ is a compact, convex set with nonempty interior, cf. [9] and [29]. The area of a convex body $K \subseteq \mathbb{E}^2$ is denoted by $V(K)$. The Minkowski sum $K_1 + K_2$ of two convex bodies K_1 and K_2 in $\mathbb{E}^2$ is given by $K_1 + K_2 := \{x_1 + x_2 : x_1 \in K_1,\ x_2 \in K_2\}$. If K is a convex body in $\mathbb{E}^2$, then the Minkowski sum $K + (-K)$ is denoted by DK and called the *difference body* of K. The abbreviations conv and bd stand for convex hull and boundary, respectively. The support and width functions of a convex body $K \subseteq \mathbb{E}^2$ are given by the formulas $h_K(u) := \max\{\langle x, u\rangle : x \in K\}$ and $w_K(u) := h_K(u) + h_K(-u)$, $u \in \mathbb{E}^2$), respectively. If we assume that the origin is contained in the interior of K, then the *radius function* of K is introduced by $r_K(u) := \max\{\alpha > 0 : \alpha u \in K\}$, $u \in \mathbb{E}^2 \setminus \{o\}$, and the *dual body* of K is defined by $K^* := \{u \in \mathbb{E}^2 : h_K(u) \le 1\}$. It can be shown that $K^{**} = K$ and $r_{K^*}(u) h_K(u) = 1$ for every $u \in \mathbb{E}^2 \setminus \{o\}$.

Let I be a segment in $\mathbb{R}$. Then a scalar function $f(t),\ t \in I$, is said to be a *Lipschitz function* if for some $\alpha \ge 0$ and every $t_1, t_2 \in I$ we have $|f(t_1) - f(t_2)| \le \alpha |t_1 - t_2|$. It is known that Lipschitz functions possess derivatives which are Lebesgue integrable functions, cf. [23]. Analogously, a vector function $p(t),\ t \in I$, with values in $\mathbb{E}^2$ is called a *Lipschitz function* if for some $C \ge 0$ and every $t_1, t_2 \in I$ we have

$$|p(t_1) - p(t_2)| \le C|t_1 - t_2|.$$

It is not hard to verify that $p(t)$ is a Lipschitz vector function if and only if both its coordinates are Lipschitz scalar functions.

For a Lebesgue measurable subset ω of $\mathbb{S}^1$ the *reverse spherical image* $\tau(K, \omega)$ of ω with respect to a convex body $K \subseteq \mathbb{E}^2$ is the set of those boundary points p of K that have an outward normal belonging to ω (cf. [29, p. 78]). Let $S_K(\omega)$ denote the length (i.e., one-dimensional measure) of $\tau(K, \omega)$. It is known that for any measurable $\omega \subseteq \mathbb{S}^1$ and any convex bodies K_1 and K_2 in $\mathbb{E}^2$ we have

$$S_{K_1+K_2}(\omega) = S_{K_1}(\omega) + S_{K_2}(\omega), \tag{11}$$

cf. [29, Section 4.3]

Given $x, y \in \mathbb{E}^2$, $\det(x, y)$ stands for the determinant of the 2×2 matrix with columns x and y (in that order). Geometrically, $\det(x, y)$ is twice the signed area of the triangle $\operatorname{conv}\{o, x, y\}$ with the sign determined by the orientation of the system of vectors x, y.

Let us consider an arbitrary convex body $K \subseteq \mathbb{E}^2$ and a parameterization $x(t),\ t \in [t_0, t_1]$ $(t_0 < t_1)$, of $\operatorname{bd} K$. Then by Green's formula the area of K can be expressed by

$$V(K) = \frac{1}{2} \int_{t_0}^{t_1} \det(x(t), x'(t))\, dt. \tag{12}$$

If $x(t)$ and $y(t)$ are Lipschitz vector functions taking values in $\mathbb{E}^2$, then it is well known that

$$\frac{d}{dt} \det(x(t), y(t)) \overset{a.e.}{=} \det(x'(t), y(t)) + \det(x(t), y'(t)) \qquad (t \in \mathbb{R}). \tag{13}$$

Let T be a triangle in a Minkowski plane $\mathcal{M}^2(B)$, I be a side of T, and u be a Euclidean normal of I. By $\tilde{a}$ we denote the length of I measured in $\mathcal{M}^2(\tilde{B})$, and by h the quantity $w_{T,B}(u)$, i.e., the Minkowskian width of T at direction u measured in $\mathcal{M}^2(B)$. The quantity h is called the *Minkowskian height of* T corresponding to the base I. Then the area of T can be given by the Minkowskian formula

$$V(T) = \frac{1}{2} h \tilde{a}, \tag{14}$$

for the proof see [31, Section 4.6] and [6, Section 5]. We remark that (14) is a Minkowskian analogue of the Euclidean formula $\frac{1}{2}$ *height* $\times$ *base* for the area of a triangle.

The *mixed area* $V(K_1, K_2)$ is a functional depending on planar convex bodies K_1 and K_2 in $\mathbb{E}^2$ and determined by the equality

$$V(K_1 + K_2) = V(K_1) + 2V(K_1, K_2) + V(K_2).$$

One can show that for any positive scalars α_1, α_2 and any K_1 and K_2 we have $V(\alpha_1 K_1, \alpha_2 K_2) = \alpha_1 \alpha_2 V(K_1, K_2)$. Furthermore, it is easy to see that for every planar convex body $K \subseteq \mathbb{E}^2$ we have $V(K, K) = V(K)$. More detailed information on the theory of mixed volumes and mixed areas can be found in the monograph [29] and [12, Chapter 4]. The Minkowskian perimeter $\mathrm{perim}_B(K)$ of K is the Minkowskian length of $\mathrm{bd}\, K \subseteq \mathcal{M}^2(B)$. It is known that

$$\mathrm{perim}_B(K) = 2V(K, \tilde{B}), \tag{15}$$

see equality (4.8) in [31]. On can show that the Minkowskian perimeters of K and DK are related by

$$\mathrm{perim}_B(DK) = 2\,\mathrm{perim}_B(K). \tag{16}$$

The Euclidean perimeter of $K \subseteq \mathbb{E}^2$ will be denoted by $\mathrm{perim}(K)$.

3. The proofs

The following proof of Lemma 1 is rather technical and relatively long, so it can be skipped by the readers not interested in technical details.

Proof of Lemma 1. Let q_0 be a boundary point of DK and $q(t)$, $t \in [0, \pi]$, be a parameterization of a boundary arc of DK such that $q(t)$ starts at q_0, terminates at $-q_0$, has the counterclockwise orientation with respect to DK and the scaled parameterization $q(\frac{\pi}{\mathrm{perim}(K)} s)$, $s \in [0, \mathrm{perim}(K)]$, is a parameterization by the Euclidean arc length. Clearly, $q(t)$ is a Lipschitz function and, moreover, for every $t_1, t_2 \in [0, \pi]$ we have $|p(t_1) - p(t_2)| \le \frac{\mathrm{perim}(K)}{\pi} \cdot |t_1 - t_2|$. For every $t \in [0, \pi]$ let $I(t)$ denote the minimal (with respect to inclusion) exposed face of DK containing the point $q(t)$. Since DK is two-dimensional, $I(t)$ is a segment probably degenerate to a singleton. Let $a_1(t)$ and $a_2(t)$ be the endpoints of $I(t)$ chosen so that for $a_1(t) \ne a_2(t)$ the vectors $a_1(t), a_2(t)$ (in that order) form a right system. Let $b_1(t), b_2(t), b_1(t+\pi), b_2(t+\pi)$ be the uniquely determined boundary points of K such that $a_i(t) = b_i(t) - b_i(t+\pi)$, where $i = 1, 2$. Then then segments $J(t) := [b_1(t), b_2(t)]$ and $J(t+\pi) := [b_1(t+\pi), b_2(t+\pi)]$ are exposed faces of K such that $I(t) = J(t) - J(t+\pi)$. If $a_1(t) \ne a_2(t)$, the segment $I(t)$ is split by $q(t)$ in the ratio $\lambda(t) : (1 - \lambda(t))$, where $\lambda(t) := \frac{|q(t) - a_1(t)|}{|a_2(t) - a_1(t)|}$. Consequently, $q(t) = (1 - \lambda(t)) \cdot a_1(t) + \lambda(t) \cdot a_2(t)$. For the case when $a_1(t) = a_2(t)$ the scalar $\lambda(t)$ can be defined arbitrarily. Let us introduce the points $p(t)$ and $p(t+\pi)$ such that $J(t)$ and $J(t+\pi)$ are split by $p(t)$ and $p(t+\pi)$, respectively, in the same ratio as $I(t)$ by the point $q(t)$. More precisely, we put

$$\begin{aligned} p(t) &:= (1 - \lambda(t)) \cdot b_1(t) + \lambda(t) \cdot b_2(t) \\ p(t+\pi) &:= (1 - \lambda(t)) \cdot b_1(t+\pi) + \lambda(t) \cdot b_2(t+\pi). \end{aligned}$$

By construction $p(t)$ is antipodal to $p(t+\pi)$ for every $t \in [0,\pi]$. Thus, it only remains to verify that $p(t)$ is a Lipschitz function.

First we extend $p(t), b_1(t)$, and $b_2(t)$ periodically to the whole real axis. We also extend $\lambda(t)$ by the equality $\lambda(t+\pi) := \lambda(t)$ $(t \in [0,\pi])$ to the segment $[0,2\pi]$ and then periodically to the whole real axis. The function $q(t)$ is extended to $[0,2\pi]$ by $q(t+\pi) := -q(t)$ $(t \in [0,\pi])$ and then periodically to $\mathbb{R}$. The functions $I(t), a_1(t)$, and $a_2(t)$ are extended in the same manner as $q(t)$.

In order to show that $p(t)$ is a Lipschitz function we take arbitrary distinct values t_1, t_2 in $[0,2\pi]$ and show that $|p(t_1) - p(t_2)| \le \alpha \cdot |t_1 - t_2|$, where α is a constant independent on t_1 and t_2. Without loss of generality let $t_1 < t_2$. Then $\operatorname{bd} K$ is split by the points $p(t_1)$ and $p(t_2)$ into two boundary arcs $p(I_1)$ and $p(I_2)$, where $I_1 := [t_1, t_2]$ and $I_2 := [t_2, 2\pi + t_1]$. Then the length of one of the above two segments does not exceed π; for definiteness we assume that this holds for I_1. Consequently, the segments I_1 and $I_1' := I_1 + \pi = [t_1+\pi, t_2+\pi]$ are disjoint. Let us show that the length of the boundary arc $q(I_1)$ of DK is equal to the total length of the boundary arcs $p(I_1)$ and $p(I_1')$ of K. For $t \in \mathbb{R}$ let $u(t)$ denote the outward Euclidean unit vector such that $\tau_K(\{u(t)\}) = J(t)$. Let ω denote the set of those Euclidean unit vectors that lie inside the angle between $u(t_1)$ and $u(t_2)$ (i.e., lie in the positive hull of $u(t_1)$ and $u(t_2)$). Then the reverse spherical image $\tau_K(\omega)$ can be represented by $\tau_K(\omega) = p(I_1) \cup [b_1(t_1), p(t_1)] \cup [p(t_2), b_2(t_2)]$, see Figure 3.

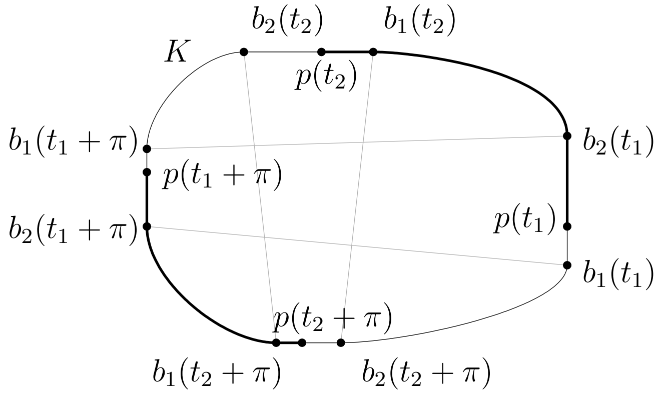

Figure 3

Hence

$$S_K(\omega) = \mu(p(I_1)) + |b_1(t_1) - p(t_1)| + |p(t_2) - b(t_2)|$$

or, equivalently,

$$S_K(\omega) = \mu(p(I_1)) + \lambda(t_1) \cdot |b_1(t_1) - b_2(t_1)| + (1 - \lambda(t_2)) \cdot |b_1(t_2) - b_2(t_2)|. \quad (17)$$

In the same manner we can derive the following two equalities:

$$\begin{aligned} S_K(-\omega) &= \mu(p(I_1')) + \lambda(t_1) \cdot |b_1(t_1+\pi) - b_2(t_1+\pi)| \\ &\quad + (1 - \lambda(t_2)) \cdot |b_1(t_2+\pi) - b_2(t_2+\pi)|, \quad (18) \\ S_{DK}(\omega) &= \mu(q(I_1)) + \lambda(t_1) \cdot |a_1(t_1) - a_2(t_1)| \\ &\quad + (1 - \lambda(t_2)) \cdot |a_1(t_2) - a_2(t_2)|. \quad (19) \end{aligned}$$

Summing up (17) with (18) and then applying (11) (for bodies K and $-K$) we get

$$S_{DK}(\omega) = \mu(p(I_1))+\mu(p(I_1'))+\lambda(t_1)\cdot(|b_1(t_1)-b_2(t_1)|+|b_1(t_1+\pi)-b_2(t_1+\pi)|)+ (1-\lambda(t_2))\cdot(|b_1(t_2)-b_2(t_2)|+|b_1(t_2+\pi)-b_2(t_2+\pi)|).$$

In view of the equality $I(t) = J(t) - J(t+\pi)$ the latter amounts to

$$\begin{aligned} S_{DK}(\omega) &= \mu(p(I_1)) + \mu(p(I_1')) + \lambda(t_1)\cdot|a_1(t_1) - a_2(t_1)| \\ &\quad +(1-\lambda(t_2))\cdot|a_1(t_2)-a_2(t_2)|. \end{aligned}$$

Thus, taking into account (19), we arrive at

$$\mu(q(I_1)) = \mu(p(I_1)) + \mu(p(I_1')).$$

Consequently, we have

$$|p(t_1)-p(t_2)| \le \mu(p(I_1)) \le \mu(p(I_1)) + \mu(p(I_1')) = \mu(q(I_1)) \le \frac{\mathrm{perim}(K)}{\pi}\cdot|t_1-t_2|,$$

which shows that $p(t)$ is a Lipschitz function. □

Proof of Theorem 2. It can be proved that for an arbitrary Minkowski plane $\mathcal{M}^2(B)$ we have

$$l_B(t) = \int_t^{t+\pi} \|p'(s)\|_B\, ds.$$

Consequently, differentiating the above equality we get

$$l_B'(t) \overset{a.e.}{=} \|p'(t+\pi)\|_B - \|p'(t)\|_B \qquad (t\in\mathbb{R}). \tag{20}$$

Now let us evaluate $v(t)$, which was defined as $V(\mathrm{conv}\,\gamma(t))$. The domain $\mathrm{conv}\,\gamma(t)$ is bounded by the boundary arc $\gamma(t)$ and the segment $[p(t), p(t+\pi)]$. The arc $\gamma(t)$ is parameterized by $p(s)$, where $s\in[t,t+\pi]$, while $[p(t),p(t+\pi)]$ can be parameterized by the vector function $(1-s)\cdot p(t) + s\cdot p(t+\pi)$ with s ranging over $[0,1]$. Thus, arranging the parameterization of $\mathrm{bd\,conv}\,\gamma(t)$ from the parameterizations of $\gamma(t)$ and $[p(t),p(t+\pi)]$ indicated above, and then using Green's formula for this specific parameterization, we get

$$2v(t) = \int_t^{t+\pi} \det(p(s),p'(s))\,ds - \det(p(t),p(t+\pi)).$$

Differentiating with respect to t we arrive at

$$\begin{aligned} 2v'(t) &\overset{a.e.}{=} \det(p(t+\pi),p'(t+\pi)) \\ &\quad - \det(p(t),p'(t)) - \det(p'(t),p(t+\pi)) - \det(p(t),p'(t+\pi)), \end{aligned}$$

which is equivalent to

$$2v'(t) \overset{a.e.}{=} \det(p(t+\pi)-p(t),p'(t+\pi)) + \det(p(t+\pi)-p(t),p'(t)). \tag{21}$$

We express the determinants appearing in (21) by areas of the triangles $T(t) := \mathrm{conv}\{o, p'(t), p(t+\pi) - p(t)\}$ and reformulate (21) in the form

$$2v'(t) \overset{a.e.}{=} 2V(T(t+\pi)) - 2V(T(t)).$$

For $t \in [0, 2\pi]$ the Minkowskian height of the triangle $T \subseteq \mathcal{M}^2(B)$ with respect to the base $[o, p'(t)]$ is obviously equal to $w_B(t)$. Therefore, we use (14) and get

$$2v'(t) \overset{a.e.}{=} \|p'(t+\pi)\|_{\tilde{B}} w_B(t) - \|p'(t)\|_{\tilde{B}} w_B(t).$$

Further on, applying (20) for B replaced by $\tilde{B}$, we obtain the equality

$$2v'(t) \overset{a.e.}{=} l'_{\tilde{B}}(t) w_B(t),$$

which is equivalent to (4).

Now let us prove (5). Interchanging in (4) B and $\tilde{B}$ we get $w_{\tilde{B}}(t) l'_B(t) - 2v'(t) = 0$. Consequently, in order to prove (5) it is sufficient to obtain the equality

$$w_{\tilde{B}}(t) = \alpha_B(t) w_B(t), \tag{22}$$

which can be done as follows:

$$w_{\tilde{B}}(t) \overset{(1)}{=} \frac{w_K(u(t))}{h_{\tilde{B}}(u(t))} = \frac{w_K(u(t))}{h_B(u(t))} \cdot \frac{h_B(u(t))}{h_{\tilde{B}}(u(t))} \overset{(1),(3)}{=} w_B(t)\alpha_B(t). \qquad \square$$

Proof of Theorem 3. I. Suppose K is of constant tangential width in $\mathcal{M}^2(B)$. Then $w_B(t) \overset{a.e.}{=} \mathrm{const}$ $(t \in \mathbb{R})$. Consequently, integrating (4) we get the necessity. Conversely, let $c(K) = w_B(t) l_{\tilde{B}}(t) - 2v(t)$ for some $c(K) \in \mathbb{R}$ and almost all $t \in \mathbb{R}$. Replacing t by $t + \pi$ we get $c(K) = w_B(t+\pi) l_{\tilde{B}}(t+\pi) - 2v(t+\pi)$. Summing up the above two expressions for $c(K)$ and applying the trivial equalities $w_B(t) \overset{a.e.}{=} w_B(t+\pi)$, $l_{\tilde{B}}(t) + l_{\tilde{B}}(t) = \mathrm{perim}_{\tilde{B}}(K)$, $v(t) + v(t+\pi) = V(K)$, we arrive at

$$c(K) \overset{a.e.}{=} \frac{1}{2} w_B(t) \,\mathrm{perim}_{\tilde{B}}(K) - V(K) \quad (t \in \mathbb{R}).$$

In view of (15) and (16) the latter amounts to

$$c(K) \overset{a.e.}{=} \frac{1}{2} w_B(t) V(DK, B) - V(K) \quad (t \in \mathbb{R}), \tag{23}$$

and it follows that $w_B(t) \overset{a.e.}{=} \mathrm{const}$, i.e., K of constant tangential width in $\mathcal{M}^2(B)$. From the above considerations we see that (23) is valid for every K of constant tangential width in $\mathcal{M}^2(B)$ and an appropriate $c(K) \in \mathbb{R}$.

II. Let K be of constant width λ in $\mathcal{M}^2(B)$. Every body of constant width in $\mathcal{M}^2(B)$ is necessarily of constant tangential width in $\mathcal{M}^2(B)$. Hence (23) holds. Applying $DK = \lambda B$ and the properties of mixed area we transform (23) to (7). Conversely, if (7) holds, let us sum up (7) together with (7) applied for t replaced by $t+\pi$. We obtain $w_B(t)\,\mathrm{perim}_{\tilde{B}}(K) - 2V(K) \overset{a.e.}{=} w(t)^2 V(B) - 2V(K)$, and, in view of $\mathrm{perim}_{\tilde{B}}(K) = V(DK, B)$, we have

$$V(DK, B) \overset{a.e.}{=} w_B(t) V(B). \tag{24}$$

Hence $w_B(t) = \lambda$ for $\lambda := V(DK, B)/V(B)$ and almost all $t \in \mathbb{R}$, and by this K is of constant tangential width in $\mathcal{M}^2(B)$. The latter is equivalent to the fact that DK is a tangential body of λB. Then

$$V(DK, B) \geq V(\lambda B, B) = \lambda V(B) \overset{a.e.}{=} w_B(t) V(B) \tag{25}$$

where the inequality from (25) is degenerate to equality if and only if $DK = \lambda B$ (i.e., if and only if K is of constant Minkowskian width in $\mathcal{M}^2(B)$). But, in view of (24), the equality in (25) is attained. Hence K is indeed of constant width in $\mathcal{M}^2(B)$.

III, IV. In view of (22) Parts III and IV follow directly from Parts I and II, respectively. □

Proof of Theorem 4. The necessity is obvious. Let us show the suffciency. Assume that every chord bisecting the Minkowskian perimeter of K is necessarily an affine diameter. Let $p_0(t)$, $t \in [0, \mathrm{perim}_B(K)]$, be a parameterization of $\mathrm{bd}\, K$ by the Minkowskian arc length with respect to $\mathcal{M}^2(B)$. Then, by assumption, the parameterization $p(t) := p_0(\frac{\mathrm{perim}_B(K)}{2\pi} \cdot t)$, $t \in [0, 2\pi]$, is antipodality preserving. Furthermore, it is easy to see, that for $l_B(t)$, associated with $p(t)$, we have $l_B(t) = \mathrm{const} = \frac{1}{2}\,\mathrm{perim}_B(K)$. Without loss of generality we suppose that

$$p(\pi) = -p(0). \tag{26}$$

We have $l'_B(t) = 0$ and hence by (20)

$$\|p'(t+\pi)\|_B \overset{a.e.}{=} \|p'(t)\|_B \qquad (t \in \mathbb{R}). \tag{27}$$

Obviously, $p'(t)$ and $p'(t+\pi)$ are tangent vectors of K at points $p(t)$ and $p(t+\pi)$, respectively. The points $p(t)$ and $p(t+\pi)$ lie in two different supporting lines of K. Therefore $p'(t)$ and $p'(t+\pi)$ are parallel and have opposite directions (almost everywhere). In view of (27), the latter implies that

$$p'(t+\pi) \overset{a.e.}{=} -p'(t) \qquad (t \in \mathbb{R}). \tag{28}$$

Integrating (28) from 0 to $s \in [0, \pi]$ and taking into account (26), we get the equality $p(s+\pi) = -p(s)$ for an arbitrary $s \in [0, \pi]$, which shows the central symmetry of K. □

Proof of Theorem 5. The implication (i) ⇒ (ii) follows directly from (5) (we recall that $\mathrm{bd}\, B$ is equiframed if and only if $\alpha_B(t) \overset{a.e.}{=} \mathrm{const}$).

The implication (ii) ⇒ (iii) is trivial.

Let us show (iii) $\Rightarrow$ (iv). We wish to construct a convex body K having constant Minkowskian width and find an antipodality preserving parameterization $p(t)$ of K such that $l'_B(t) \neq 0$ almost everywhere. Assuming that (iii) is fulfilled we may take K to be a Minkowski sum of a Reuleaux triangle of Minkowskian width $\lambda > 0$ and a Minkowskian ball of radius $r > 0$. Then the boundary of K can be represented as the union $\gamma_1 \cup \gamma_2$, where γ_1 and γ_2 are the unions of Minkowskian circle arcs of radius $r_1 := r$ and $r_2 := \lambda + r$, respectively. We choose the parameterization $p(t)$ of $\operatorname{bd} K$ such that $\|p'(t)\|_B = r_i$ almost everywhere for $p(t) \in \gamma_i$ $(i = 1, 2)$.

It is easy to see that for such $p(t)$ we have $|l'_B(t)| = |r_1 - r_2| = \lambda \neq 0$ almost everywhere. Since $w_B(t) = \text{const}$, the integration of (9) yields (10).

Let us show (iv) $\Rightarrow$ (i). We assume that

$$c_1(K) = \alpha_K w_B(t) l_B(t) - 2v(t) \tag{29}$$

for some $c_1(K) \in \mathbb{R}$ and almost every $t \in \mathbb{R}$. Replacing t by $t + \pi$ we transform (29) to $c_1(K) \overset{a.e.}{=} \alpha_K l_B(t + \pi) w_B(t) - 2v(t + \pi)$. Summing up (29) and the latter equality we get

$$2c_1(K) := \alpha_K w_B(t) \operatorname{perim}_B(K) - 2V(K),$$

which implies that $w_B(t) \overset{a.e.}{=} \lambda$ for some $\lambda > 0$, i.e., K is of constant tangential width λ in $\mathcal{M}^2(B)$. Thus, (29) amounts to

$$c_1(K) = \alpha_K \lambda l_B(t) - 2v(t),$$

and differentiating we get $\alpha_K \lambda l'_B(t) - 2v'(t) \overset{a.e.}{=} 0$. On the other hand, by (5) we obtain $\lambda \alpha_B(t) l'_B(t) - 2v'(t) \overset{a.e.}{=} 0$. Taking the difference of the above two equalities we arrive at

$$\lambda(\alpha_B(t) - \alpha_K) l'_B(t) = 0$$

Since $l'_B(t) \neq 0$ almost everywhere, we obtain that $\alpha_B(t) \overset{a.e.}{=} \alpha_K$. Consequently, $\operatorname{bd} B$ is an equiframed curve.

Part II of the theorem follows directly from Parts III and IV of Theorem 3. $\square$

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