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On Brocard Points in the Isotropic Plane

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Abstract. The isotropic analogue of Brocard's theorem was first obtained by K. Strubecker [7]. Later this analogue was also discussed by J. Lang [3] and H. Sachs [5]. In the present paper we derive some new results related to Brocard points in the isotropic plane. Namely, an expression for the distance between the Brocard points and certain relations between the circumradii of the corresponding Brocard circles are given. Also we present a statement which is dual to Brocard's theorem.

Keywords: Brocard lines, Brocard points, duality, dual numbers, Galilean plane, isotropic geometry

1. Introduction

The *isotropic (Galilean) plane* (see [7] and [5]) is defined as a projective plane with an absolute which consists of a line f (the *absolute line*) and a point F (the *absolute point*) lying on f. We can choose a basic coordinate system Oxy such that the axis Oy coincides with the absolute line f and the infinite point of Oy coincides with the absolute point F. Then the direction, determined by the axis Oy, is said to be the *isotropic* (or *special*) direction.

A line in the isotropic plane is called *isotropic* if it is parallel to the isotropic direction, and two points P_1 and P_2 are said to be *parallel* if the line P_1P_2 is isotropic.

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The (oriented) *distance* from the point $P_1(x_1, y_1)$ to the point $P_2(x_2, y_2)$ is defined by

$$\overline{P_1P_2} = x_2 - x_1.$$

Let $g_1 : y = k_1x + n_1$ and $g_2 : y = k_2x + n_2$ be two non-isotropic lines. Then the (oriented) *angle* from g_1 to g_2 is defined by

$$\angle(g_1, g_2) = k_2 - k_1.$$

A conic k in the isotropic plane, which contains the absolute point F and touches the absolute line at F, is called an *isotropic circle (cycle).*

Any isotropic circle can be presented by the equation

$$k : y = Rx^2 + \alpha x + \beta,$$

where the real number $R \neq 0$ is said to be the *radius of* k, and $\alpha, \beta \in \mathbb{R}$.

For any *admissible* triangle $\triangle ABC$ (i.e., a triangle whose three pairs of vertices form non-parallel points, in each case) there exists a unique circumscribed isotropic circle.

If $\triangle ABC$ is an admissible triangle, $\angle(AC, AB) = \alpha$, $\angle(BA, BC) = \beta$, $\angle(CB, CA) = \gamma$, and R is the circumradius of $\triangle ABC$, then the following relations hold[1]:

$$\overline{BC} + \overline{CA} + \overline{AB} = 0, \tag{1}$$

$$\alpha + \beta + \gamma = 0, \tag{2}$$

$$\frac{\overline{BC}}{\alpha} = \frac{\overline{CA}}{\beta} = \frac{\overline{AB}}{\gamma} = -\frac{1}{R}. \tag{3}$$

For further basic notions of isotropic geometry in the plane we refer to [5] and [7].

2. Dual numbers and a lemma

The so-called *dual numbers* used in the following were introduced by E. Study, but the first extensive discussions of applications of these numbers in geometry go back to J. Grünwald [2].

Any dual number z can be written in the form

$$z = x + \varepsilon y, \tag{4}$$

where x and y are real numbers and $\varepsilon^2 = 0$ (see [5], [2], [9], and [10]). The number x is called the *modulus* of the dual number z and denoted by $|z|$, i.e., $|z| = x$.

Any dual number (4) with $x \neq 0$ can also be written in the form $z = |z|\,(1 + \varepsilon\varphi)$, where $\varphi = \dfrac{y}{x}$ is called the *argument* of z, denoted by $\arg z$.

Following K. Strubecker [6], we will now identify points of the isotropic plane with dual numbers, similar to the identification of points of the Euclidean plane with complex numbers.

[1] The last relation is called the *law of sines* in the isotropic plane.

Thus, the dual numbers $z_1 = x_1 + \varepsilon y_1$ and $z_2 = x_2 + \varepsilon y_2$ determine two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ whose distance is given by

$$\overline{P_1P_2} = |z_2 - z_1| \ (= x_2 - x_1).$$

Furthermore, let a, b and c be three dual numbers determining an admissible triangle $\triangle ABC$ with angles

$$\angle(AC, AB) = \alpha, \quad \angle(BA, BC) = \beta, \quad \angle(CB, CA) = \gamma. \tag{5}$$

In terms of dual numbers, the relations (1) and (3) read as follows:

$$|c - a| + |a - c| + |b - a| = 0,$$
$$\frac{|c-b|}{\alpha} = \frac{|a-c|}{\beta} = \frac{|b-a|}{\gamma} = -\frac{1}{R}. \tag{6}$$

We continue with a technical lemma.

Lemma 1. *For any three dual numbers a, b and c, that determine an admissible triangle $\triangle ABC$ with angles as given in* (5)*, the following relation holds:*

$$\frac{a-c}{a-b} = \frac{\beta}{\gamma}(-1 + \varepsilon\alpha).$$

Proof. By [10] we have

$$\angle(ab, ac) = \arg\frac{a-c}{a-b},$$

implying

$$\frac{a-c}{a-b} = \left|\frac{a-c}{a-b}\right| (1 + \varepsilon \arg\frac{a-c}{a-b}) = \frac{|a-c|}{|a-b|}\,(1 - \varepsilon\alpha).$$

Using the law of sines in the version (6), we get

$$\frac{a-c}{a-b} = -\frac{\beta}{\gamma}\,(1 - \varepsilon\alpha) = \frac{\beta}{\gamma}\,(-1 + \varepsilon\alpha). \qquad \square$$

3. An extension of the isotropic analogue of Brocard's theorem

The following statement, which is the isotropic analogue of Brocard's theorem in the Euclidean plane (see, e.g., the survey [4]) was first given by K. Strubecker [7]. Later also J. Lang [3] and H. Sachs (see [5], p. 44) discussed this isotropic analogue.

Theorem 1. *Let $\triangle ABC$ be an admissible triangle and $\angle(AC, AB) = \alpha, \angle(BA, BC) = \beta, \angle(CB, CA) = \gamma$. Let $k_1 = (A; BC)$ be the circle which passes through the vertices A and B and is tangent to the line BC at B, and define $k_2 = (B; CA)$ and $k_3 = (C; AB)$ analogously. Similarly, let $\widehat{k}_1 = (A; CB)$, $\widehat{k}_2 = (B; AC)$, and*

$\widehat{k}_3 = (C; BA)$. *Then the circles* k_1, k_2, *and* k_3 *intersect at a point* W^1, *called the first Brocard point, and* $\widehat{k}_1, \widehat{k}_2$, *and* $\widehat{k}_3$ *intersect at a point* W^2, *called the second Brocard point. In addition, the following relations hold (see Fig. 1):*

$$\angle BAW^1 = \angle CBW^1 = \angle ACW^1 = -\angle CAW^2 = -\angle ABW^2 = -\angle BCW^2,$$

$$-\frac{1}{\omega_1} = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}, \tag{7}$$

where $\omega_1 = \angle BAW^1$ *and*

$$\frac{\overline{AW^1}}{\overline{AW^2}} : \frac{\overline{BW^1}}{\overline{BW^2}} : \frac{\overline{CW^1}}{\overline{CW^2}} = \frac{\overline{CA}}{\overline{AB}} : \frac{\overline{AB}}{\overline{BC}} : \frac{\overline{BC}}{\overline{CA}}. \tag{8}$$

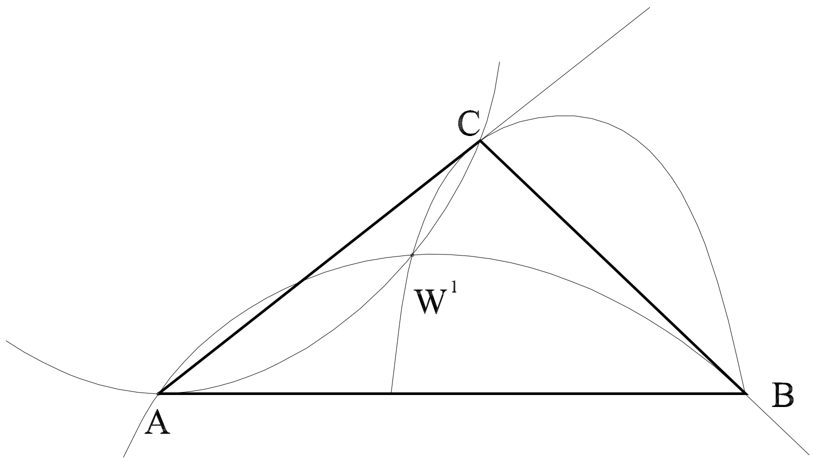

Figure 1

It should be noticed that J. Tölke [8] extended parts of that theorem by characterizing Brocard points in the isotropic plane among more general pairs of points.

Now we are ready to present our extension of Theorem 1.

Theorem 2. *Let* $\triangle ABC$ *be an admissible triangle in the isotropic plane with angles* $\angle(AC, AB) = \alpha, \angle(BA, BC) = \beta, \angle(CB, CA) = \gamma$, R *be the radius of the circumcircle of* $\triangle ABC$, W^1 *be the first Brocard point of* $\triangle ABC$, *and* W^2 *be its second Brocard point. Then the following relations hold:*

$$\frac{1}{\overline{W^1W^2}} = R\,(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}), \tag{9}$$

$$\frac{R_1}{\hat{R}_1} : \frac{R_2}{\hat{R}_2} : \frac{R_3}{\hat{R}_3} = \left(\frac{\hat{R}_2}{R_3}\right)^2 : \left(\frac{\hat{R}_3}{R_1}\right)^2 : \left(\frac{\hat{R}_1}{R_2}\right)^2 = \left(\frac{\overline{CA}}{\overline{AB}}\right)^2 : \left(\frac{\overline{AB}}{\overline{BC}}\right)^2 : \left(\frac{\overline{BC}}{\overline{CA}}\right)^2, \tag{10}$$

$$\begin{aligned} R_1R_2\hat{R}_1\hat{R}_2 &= (R_3\hat{R}_3)^2, \\ R_2R_3\hat{R}_1\hat{R}_2 &= (R_1\hat{R}_1)^2, \\ R_3R_1\hat{R}_3\hat{R}_1 &= (R_2\hat{R}_2)^2, \end{aligned} \tag{11}$$

where $R_1, R_2, R_3, \hat{R}_1, \hat{R}_2$, *and* $\hat{R}_3$ *are the radii of the Brocard circles* k_1, k_2, k_3, $\hat{k}_1, \hat{k}_2$, *and* $\hat{k}_3$, *respectively.*

Proof. Denote by $a = a_1 + \varepsilon a_2$, $b = b_1 + \varepsilon b_2$, $c = c_1 + \varepsilon c_2$, w^1, and w^2 the dual numbers of the points A, B, C, W^1, and W^2, respectively. Let $\omega_1 = \angle BAW^1$. Applying (2) and Lemma 1 with respect to $\triangle BCW^1$ (see Fig. 2), we get

$$\frac{b - w^1}{b - c} = \frac{\gamma + \omega_1}{-\gamma}[-1 + \varepsilon(-\omega_1)]. \tag{12}$$

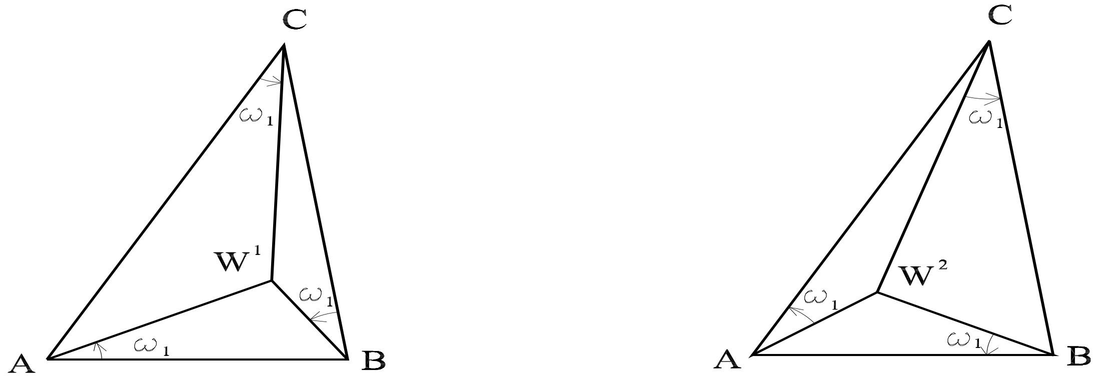

Figure 2

Analogously, for $\triangle CAW^2$ we get the relation

$$\frac{c - w^2}{c - a} = \frac{-\omega_1}{-\gamma}[-1 + \varepsilon(\gamma + \omega_1)]. \tag{13}$$

From (12) and (13) we obtain

$$w^1 = (-\frac{\omega_1}{\gamma} - \varepsilon\omega_1\frac{\gamma + \omega_1}{\gamma})b + (1 + \frac{\omega_1}{\gamma} + \varepsilon\omega_1\frac{\gamma + \omega_1}{\gamma})c$$

and

$$w^1 = (-\frac{\omega_1}{\gamma} + \varepsilon\omega_1\frac{\gamma + \omega_1}{\gamma})a + (1 + \frac{\omega_1}{\gamma} - \varepsilon\omega_1\frac{\gamma + \omega_1}{\gamma})c.$$

Therefore

$$\overline{W^1W^2} = |w^2 - w^1| = \frac{\omega_1}{\gamma}\,(b_1 - a_1) = \omega_1\frac{\overline{AB}}{\gamma}.$$

Together with the law of sines in the version (3) and (7), we deduce (9) from this.

Further on, we calculate

$$\overline{CW^1} = |w^1 - c| = -\frac{\omega_1}{\gamma}\,b_1 + (1 + \frac{\omega_1}{\gamma})\,c_1 - c_1 = \omega_1\frac{\overline{BC}}{\gamma},$$

$$\overline{CW^2} = |w^2 - c| = -\frac{\omega_1}{\gamma}\,a_1 + (1 + \frac{\omega_1}{\gamma})\,c_1 - c_1 = \omega_1\frac{\overline{AC}}{\gamma}.$$

Analogously,

$$\overline{AW^1} = \omega_1\frac{\overline{CA}}{\alpha}, \quad \overline{AW^2} = \omega_1\frac{\overline{BA}}{\alpha}, \quad \overline{BW^1} = \omega_1\frac{\overline{AB}}{\beta}, \quad \overline{BW^2} = \omega_1\frac{\overline{CB}}{\beta}.$$

Now we apply (3) to $\triangle ABW^1$, $\triangle BCW^1$, $\triangle CAW^1$, $\triangle CAW^2$, $\triangle ABW^2$, $\triangle CBW^2$, and so we get

$$R_1 = \frac{\beta}{\overline{AB}}, \quad R_2 = \frac{\gamma}{\overline{BC}}, \quad R_3 = \frac{\alpha}{\overline{CA}}, \quad \hat{R}_1 = \frac{\gamma}{\overline{CA}}, \quad \hat{R}_2 = \frac{\alpha}{\overline{AB}}, \quad \hat{R}_3 = \frac{\beta}{\overline{BC}} \tag{14}$$

and

$$\overline{AW^1} = \frac{\omega_1}{R_3}, \quad \overline{BW^1} = \frac{\omega_1}{R_1}, \quad \overline{CW^1} = \frac{\omega_1}{R_2},$$
$$\overline{AW^2} = -\frac{\omega_1}{\hat{R}_2}, \quad \overline{BW^2} = -\frac{\omega_1}{\hat{R}_3}, \quad \overline{CW^2} = -\frac{\omega_1}{\hat{R}_1}.$$

Hence

$$\frac{\overline{AW^1}}{\overline{AW^2}} : \frac{\overline{BW^1}}{\overline{BW^2}} : \frac{\overline{CW^1}}{\overline{CW^2}} = \frac{\hat{R}_2}{R_3} : \frac{\hat{R}_3}{R_1} : \frac{\hat{R}_1}{R_2},$$

and by (8) the second equality of (10) is proved.

Moreover, (3) and (14) imply

$$\frac{R_1}{\hat{R}_1} : \frac{R_2}{\hat{R}_2} : \frac{R_3}{\hat{R}_3} = \frac{\beta\,\overline{CA}}{\gamma\,\overline{AB}} : \frac{\gamma\,\overline{AB}}{\alpha\,\overline{BC}} : \frac{\alpha\,\overline{BC}}{\beta\,\overline{CA}} = \left(\frac{\overline{CA}}{\overline{AB}}\right)^2 : \left(\frac{\overline{AB}}{\overline{BC}}\right)^2 : \left(\frac{\overline{BC}}{\overline{CA}}\right)^2,$$

i.e., (10) is established.

Finally we remark that (11) follows immediately from (10). □

4. Brocard lines

One nice property of isotropic geometry is the principle of duality, which asserts that every theorem remains true if we consistently interchange the words 'point' and 'line', 'distance' between two points and 'angle' between two lines, 'lie on' and 'pass through', 'join' and 'intersection', 'collinear' and 'concurrent', see [10], p. 54–65. This principle of duality can also be applied to Brocard's theorem. Namely, the following statement is a direct corollary of Theorem 1:

Corollary 1. *If the lines p_1, q_1, and r_1 through the vertices A, B, and C of the triangle $\triangle ABC$ with angles $\angle(CA, AB) = \alpha$, $\angle(AB, BC) = \beta$, $\angle(BC, CA) = \gamma$ are determined by*

$$\angle(BA, p_1) = \angle(CB, q_1) = \angle(AC, r_1) = \omega_1,$$

and the lines p_2, q_2, and r_2 through the vertices A, B, and C are determined by

$$\angle(CA, p_2) = \angle(AB, q_2) = \angle(BC, r_2) = -\omega_1,$$

where ω_1 satisfies the equation

$$-\frac{1}{\omega_1} = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma},$$

then the lines p_1, q_1, and r_1 meet at some point W^1 (the first Brocard point), and the lines p_2, q_2, and r_2 meet at some point W^2 (the second Brocard point).

Dualizing this statement, we establish a new theorem.

Theorem 3. *Let the points P_1, Q_1, and R_1 lie on the lines BC, CA, and AB, and the points P_2, Q_2, and R_2 lie also on the lines BC, CA, and AB such that*

$$\overline{CP_1} = \overline{AQ_1} = \overline{BR_1} = x, \qquad \overline{BP_2} = \overline{CQ_2} = \overline{AR_2} = -x,$$

where x satisfies the relation

$$-\frac{1}{x} = \frac{1}{\overline{AB}} + \frac{1}{\overline{BC}} + \frac{1}{\overline{CA}}.$$

Then the points P_1, Q_1, and R_1 are collinear, and the points P_2, Q_2, and R_2 are also collinear.

Remark 1. The line through P_1, Q_1, and R_1 we will call the *first Brocard line*, and the line through P_2, Q_2, and R_2 is said to be the *second Brocard line* of the triangle.

Finally we will ask for the angle between the Brocard lines.

The dual of the circumcircle Z of the triangle $\triangle ABC$ (i.e., the circle considered as a set of points, containing the vertices A, B, C) is the incircle of $\triangle ABC$ (considered as the envelope of a set of tangents, containing the prolonged sides BC, CA, AB), see [10], p. 106. Furthermore, the radius r of the incircle is equal to $\frac{R}{4}$, where R is the circumradius (see [5], p. 28). Thus, applying the principle of duality to (9) we establish

Corollary 2. *For the angle φ between the Brocard lines of the triangle $\triangle ABC$ the relation*

$$\frac{1}{\varphi} = \frac{R}{4}\left(\frac{1}{\overline{AB}} + \frac{1}{\overline{BC}} + \frac{1}{\overline{CA}}\right)$$

holds, where R is the circumradius of $\triangle ABC$.

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