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Borel Ideals in Three Variables

Maria Grazia Marinari **Luciana Ramella** *

Dipartimento di Matematica, Università
v. Dodecaneso 35, I-16146 Genova, Italy
e-mail: marinari@dima.unige.it ramella@dima.unige.it

1. Introduction

Borel ideals, are special monomial ideals, occurring as generic initial ideals of homogeneous ideals $\mathfrak{a} \subseteq \mathbf{P}(\mathbf{n}) := \mathbf{k}[X_1, \ldots, X_n]$, widely studied after Galligo's and Bayer-Stillman's results ([6] and [1]). More precisely (under the action of $Gl(\mathbf{n}, \mathbf{k})$ on $\mathbf{P}(\mathbf{n})$: $g(X_j) = \sum_{i=1}^{n} g_{ij} X_i$, $g = (g_{ij}) \in Gl(\mathbf{n}, \mathbf{k})$), given any term-ordering $<$ and homogeneous ideal $\mathfrak{a} \subseteq \mathbf{P}(\mathbf{n})$, there exists a non-empty open subset U of $Gl(\mathbf{n}, \mathbf{k})$ such that as g ranges in U, $gin(\mathfrak{a}) := in(g(\mathfrak{a}))$ is constant. Moreover, $gin(\mathfrak{a})$ is fixed by the group $\mathbf{B}$ of upper-triangular invertible matrices, if $X_1 > \cdots > X_n$, while $gin(\mathfrak{a})$ is fixed by the group $\mathbf{B}'$ of lower-triangular invertible matrices if $X_1 < \cdots < X_n$. Monomial ideals $\mathfrak{a} \subseteq \mathbf{P}(\mathbf{n})$, can be studied via the associated order-ideal $\mathcal{N}(\mathfrak{a})$ consisting of all the terms (= monic monomials) 'outside' $\mathfrak{a}$ and called *sous-éscalier* of $\mathfrak{a}$ ([6], [8] and [10]). For a Borel ideal $\mathfrak{b} \subseteq \mathbf{P}(\mathbf{n})$, $\mathcal{N}(\mathfrak{b})$ is fixed by $\mathbf{B}'$ if $X_1 > \cdots > X_n$, and by $\mathbf{B}$ if $X_1 < \cdots < X_n$. Studying Borel ideals through their sous-éscaliers, following A. Galligo ([7]), we consider $X_1 < \cdots < X_n$.

In Section 2 we fix our notation. In Section 3 we introduce the Borel subsets of the multiplicative semigroup of terms in $\mathbf{P}(\mathbf{n})$, illustrating some of their features and giving a 'general construction' to produce Borel subsets of assigned cardinality in each degree. In Section 4 we describe the Borel ideals $\mathfrak{b} \subset \mathbf{P}(\mathbf{n})$; in particular, basing on the combinatorics of $\mathcal{N}(\mathfrak{b})$, we associate to every 0-dimensional $\mathfrak{b} \subseteq \mathbf{P}(\mathbf{n})$, generated in degrees $\leq s+1$, an n by $s+1$ matrix $\tilde{\mathcal{M}}(\mathfrak{b})$ with non-negative integral entries $\tilde{m}_{i,j}(\mathfrak{b})$. Since on $\tilde{\mathcal{M}}(\mathfrak{b})'s$ rows one reads the Hilbert functions of sections of $\mathbf{P}(\mathbf{n})/\mathfrak{b}$ with linear spaces (see Definition 4.10 and Remark 4.12 a)),

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inspired by [2], $\tilde{\mathcal{M}}(\mathfrak{b})$ is called *sous-éscalier sectional matrix.* In Section 5, given any O-sequence $\mathbf{h} = (1, n, h_2, \dots, h_d, \dots, h_s)$ of positive integers we introduce an equivalence relation $\sim$ on the set $\mathcal{B}^n_{\mathbf{h}}$ of 0-dimensional Borel ideals corresponding to $\mathbf{h}$ via: $\mathfrak{b} \sim \mathfrak{b}'$ if they have the same sous-éscalier sectional matrix. We also introduce a poset structure on $\mathcal{B}^n_{\mathbf{h}}/\sim$, by means of the partial order relation $\prec$ defined via: $\bar{\mathfrak{b}} \prec \bar{\mathfrak{b}'}$ if $\bar{\mathfrak{b}} \neq \bar{\mathfrak{b}'}$ and $\tilde{m}_{i,j}(\mathfrak{b}) \leq \tilde{m}_{i,j}(\mathfrak{b}')$ for each representatives $\mathfrak{b}, \mathfrak{b}'$. The *Lex-segment ideal* $\mathcal{L}(\mathbf{h})$ gives the unique maximal element of $\mathcal{B}^n_{\mathbf{h}}/\sim$. In the 3-variable case, by the combinatorial character of $\prec$, we construct the *generalized rev-lex segment* ideal $\pounds(\mathbf{h})$ and prove our main results: for any O-sequence $\mathbf{h} = \underline{(1, 3, h_2, \dots, h_s)} \in \mathbb{N}^{*(s+1)}$, the poset $\mathcal{B}^3_{\mathbf{h}}/\sim$ has a 'natural' lattice structure and $\pounds(\mathbf{h})$ is its unique minimal element; if $n \geq 4$, $\mathcal{B}^n_{\mathbf{h}}/\sim$, only admits a poset structure having, in general, several different minimal elements (see Theorem 5.6 and Example 5.3).

We are grateful to D. Bayer for suggesting us to investigate this subject.

2. Notation

In this section we fix our notation, recalling some general facts which will be used. For each positive integer n, $\mathbf{P(n)}$ is the polynomial ring in the variables $X_1, \dots, X_n$ over a field $\mathbf{k}$ of characteristic 0. If $n \leq 4$, X, Y, Z, T replace ordinately $X_1, \dots, X_4$. For $1 \leq i \leq n$, $\mathbf{P(i)} := \mathbf{k}[X_1, \dots, X_i]$ and $\mathbf{P'(i)} := \mathbf{k}[X_{n-i+1}, \dots, X_n]$ are thought as subrings of $\mathbf{P(n)} = \mathbf{P'(n)}$.

For every $j \in \mathbb{N}$ let $\mathbf{P(n)}_j$ denote the j-homogeneous part of $\mathbf{P(n)}$ and similarly, for M a subset of $\mathbf{P(n)}$ let M_j denote the degree j part.

The multiplicative semigroup of terms $\mathbf{T(n)}$ is the set of monic monomials $\mathbf{X^a} := X_1^{a_1} \cdot X_2^{a_2} \cdots X_n^{a_n}$ with $a_i \in \mathbb{N}$, for $1 \leq i \leq n$, $\mathbf{T(i)}$ and $\mathbf{T'(i)}$ denote respectively the terms involving the set of variables $\{X_1, \dots, X_i\}$ and $\{X_{n-i+1}, \dots, X_n\}$. For each subset N of $\mathbf{T(n)}$, we let $N(\mathbf{i})$ be the intersection $N \cap \mathbf{T(i)}$ and $N'(\mathbf{i})$ the intersection $N \cap \mathbf{T'(i)}$. If no confusion can arise, we ordinately write $\mathbf{P}$, $\mathbf{T}$, $\mathbb{T}$ and $\mathbb{T}'$ for $\mathbf{P(n)}$, $\mathbf{T(n)}$, $\mathbf{T(n-1)}$ and $\mathbf{T'(n-1)}$. On $\mathbf{T}$ among the possible term-orderings, we will consider the lexicographic (l), degree-lexicographic (dl) and degree-reverse-lexicographic (drl) with $X_1 < \dots < X_n$. The following decompositions (in increasing order) hold for all $j \in \mathbb{N}^*$ and $n \neq 1$, (see [10]):

$$(\bullet) \qquad \mathbf{T}_j = \mathbb{T}_j \sqcup X_n \mathbb{T}_{j-1} \sqcup \dots \sqcup X_n^{j-1} \mathbb{T}_1 \sqcup X_n^j \mathbb{T}_0 = \bigsqcup_{r=0}^{n} X_n^r \mathbb{T}_{j-r} \text{ (w.r.t. dl)}$$

$$(\bullet\bullet) \qquad \mathbf{T}_j = X_1 \mathbf{T}_{j-1} \sqcup \mathbb{T}'_j = \bigsqcup_{i=1}^{n} X_i \mathbf{T}'(\mathbf{n} - \mathbf{i} + \mathbf{1})_{j-1} \text{ (w.r.t. drl).}$$

For each $i, j \in \mathbb{N}^*$, $1 \leq i \leq n$, $1 \leq \omega \leq \binom{i+j-1}{j}$, the set of the ω smallest terms of $\mathbf{T(i)}_j$ w.r.t. l (resp. rl) is denoted $\mathbf{L}_{i,\omega,j}$ (resp. $\Lambda_{i,\omega,j}$), and called *ω-(initial)-l-segment* (resp. *ω-(initial)-rl-segment*) of $\mathbf{T(i)}_j$.

As usual, the leading term (w.r.t. the given term-ordering) of an $f \in \mathbf{P}$ is denoted $T(f) \in \mathbf{T}$; for a homogeneous ideal $\mathfrak{a} \subset \mathbf{P}$, $T(\mathfrak{a}) := \{T(f) : f \in \mathfrak{a}\}$ is a semigroup ideal and $in(\mathfrak{a}) \subset \mathbf{P}$ is the generated monomial ideal. We call *sous-éscalier of* $\mathfrak{a}$ the order ideal $\mathcal{N}(\mathfrak{a}) := \mathbf{T} \setminus T(\mathfrak{a})$.

For each subset N of $\mathbf{T}$ and positive integers i, j with $0 \leq i \leq n-1$ we denote by $\lambda_{i,j}(N)$ the number of egree j terms of N involving the variables $X_{i+1}, \dots, X_n$:

$$\lambda_{i,j}(N) := \#(N'(\mathbf{n}-\mathbf{i})_j), \tag{1}$$

it may be useful to conventionally put $\lambda_{n,j}(N) := 0$. If $N \subseteq \mathbf{T}_{\bar{j}}$ for some $\bar{j} \in \mathbb{N}^*$, then $\lambda_{i,j}(N) = 0$ for all $j \neq \bar{j}$, thus we write $\lambda_i(N)$ instead of $\lambda_{i,\bar{j}}(N)$.
For $t = X_1^{a_1} \cdot X_2^{a_2} \cdots X_n^{a_n} \in \mathbf{T}$, $N \subseteq \mathbf{T}$, $i, j \in \mathbb{N}^*$ with $1 \leq i \leq n$, we put

$$\mu(t) := \min\{\ell \in \{1, \dots, n\} \;:\; a_\ell \neq 0\}, \tag{2}$$

$$\nu_{i,j}(N) := \#\{t \in N_j \;:\; \mu(t) = i\}. \tag{3}$$

As for $1 \leq i \leq n$ we have $t \in \mathbf{T}'(\mathbf{n}-\mathbf{i})$ iff $\mu(t) \geq i+1$ for all $N \subseteq \mathbf{T}$, it holds:

$$\nu_{i,j}(N) = \lambda_{i-1,j}(N) - \lambda_{i,j}(N). \tag{4}$$

If $N \subseteq \mathbf{T}_j$ for some $j \in \mathbb{N}^*$ we set $N_{(0)} := N$ and, for all $\ell \in \mathbb{N}^*$

$$N_{(\ell)} := \mathbf{T}_{j+\ell} \setminus \{X_1, \dots, X_n\} \cdot (\mathbf{T}_{j+\ell-1} \setminus N_{(\ell-1)}), \tag{5}$$

calling it *potential expansion* of N in $\mathbf{T}_{j+\ell}$.
By definition, for each homogeneous ideal $\mathfrak{a} \subseteq \mathbf{P}$, as $\mathbf{T}_j \setminus \mathcal{N}(\mathfrak{a})_j = \mathfrak{a} \cap \mathbf{T}_j$, one has

$$(\mathcal{N}(\mathfrak{a})_j)_{(1)} = \mathbf{T}_{j+1} \setminus T\{\mathfrak{a}_j \mathbf{P}_1\}, \tag{6}$$

and, since $\mathfrak{a}_j \mathbf{P}_1 \subseteq \mathfrak{a}_{j+1}$, one also has

$$\mathcal{N}(\mathfrak{a})_{j+1} \subseteq (\mathcal{N}(\mathfrak{a})_j)_{(1)}. \tag{7}$$

For a monomial ideal $\mathfrak{a} \subseteq \mathbf{P}(\mathbf{n})$, $G(\mathfrak{a})$ denotes its minimal system of generators. If $\mathfrak{a}$ is generated in degrees $\leq s+1$, with initial degree $d \in \mathbb{N}^*$, then

$$\#G(\mathfrak{a})_j = \#(\mathcal{N}(\mathfrak{a})_{j-1})_{(1)} - \#(\mathcal{N}(\mathfrak{a})_j) \;\text{ holds for every } d \leq j \leq s+1. \tag{8}$$

Note that, in the 0-dimensional case, one has in particular $G(\mathfrak{a})_{s+1} = (\mathcal{N}(\mathfrak{a})_s)_{(1)}$.

3. Borel subsets of T

In this section we give the notion of Borel subset of $\mathbf{T}$ and some useful properties.

Definition 3.1. *A subset B of $\mathbf{T}$ is Borel if $t \in B$ and $X_j \mid t$ imply $X_i t / X_j \in B$ for all $i < j$.*

Remark 3.2. a) For a Borel $B \subseteq \mathbf{T}(\mathbf{i})_j$ it holds $X_i^j \in B$ iff $B = \mathbf{T}(\mathbf{i})_j$, if B has cardinality $\omega < \binom{i+j-1}{j}$, then $\lambda_0(B) = \omega$ and $\lambda_{i-1}(B) = 0$. So, if $i = 3$, only $\lambda_1(B)$ is meaningful.
b) For each $i, j \in \mathbb{N}^*$, $1 \leq i \leq n$, $1 \leq \omega \leq \binom{i+j-1}{j}$, $\mathbf{L}_{i,\omega,j}$ and $\Lambda_{i,\omega,j}$ are Borel subset of $\mathbf{T}(\mathbf{i})_j$, moreover $\mathbf{L}_{i,\omega,j} = \Lambda_{i,\omega,j}$ iff $\omega \in \{1, 2, \binom{i+j-1}{j} - 2, \binom{i+j-1}{j} - 1, \binom{i+j-1}{j}\}$ and $\mathbf{L}_{1,1,j} = \Lambda_{1,1,j} = \{X_1^j\}$, $\mathbf{L}_{2,\omega,j} = \Lambda_{2,\omega,j} = \{X_1^j, \dots X_1^{j-\omega+1} X_2^{\omega-1}\}$.

Notation 3.3. For all $a, j \in \mathbb{N}^*$, $a\{j\}$ means the j-binomial expansion of a,

$$a = \binom{k(j)}{j} + \binom{k(j-1)}{j-1} + \cdots + \binom{k(r)}{r}$$

with $k(j) > k(j-1) > \cdots > k(r) \geq r \geq 1$.
Moreover, for all $\ell \in \mathbb{Z}$ we let:

$$(a\{j\})^{\ell} := \binom{k(j)+\ell}{j} + \binom{k(j-1)+\ell}{j-1} + \cdots + \binom{k(r)+\ell}{r},$$

where $\binom{k(j-m)+\ell}{j-m} = 0$ if $k(j-m)+\ell < j-m$ for some $0 \leq m \leq j-r$. In particular $(a\{j\})^{1-n} = 1$ if $a = \binom{n+j-1}{j}$, $(a\{j\})^{1-n} = 0$ if $a < \binom{n+j-1}{j}$.

Lemma 3.4. *For each $j \in \mathbb{N}^*$, $0 \leq i \leq n-1$ and $1 \leq \omega \leq \binom{n+j-1}{j} - 1$ it holds:*

$$\lambda_i(\mathbf{L}_{n,\omega,j}) = (\omega\{j\})^{-i}.$$

Proof. We prove by induction on $n \geq 2$ and $j \in \mathbb{N}^*$ that $\lambda_1(\mathbf{L}_{n,\omega,j}) = (\omega\{j\})^{-1}$, this if $n = 2$ is trivial for all $j \in \mathbb{N}^*$. Assume our contention for $m \leq n-1, h \leq j-1$ and deduce it for n and j. For $1 \leq \omega \leq \binom{n+j-1}{j} - 1$ we set $\sigma(\omega) := -1$ and $\alpha(\omega) := \omega$ if $\omega \leq \binom{n+j-2}{j}$, otherwise we set $\alpha(\omega) := \omega - \sum_{\ell=0}^{\sigma(\omega)} \binom{n+j-2-\ell}{j-\ell}$ with $\sigma(\omega)$ defined via:
$\binom{n+j-2}{j} + \cdots + \binom{n+j-2-\sigma(\omega)}{j-\sigma(\omega)} < \omega \leq \binom{n+j-2}{j} + \cdots + \binom{n+j-2-\sigma(\omega)}{j-\sigma(\omega)} + \binom{n+j-2-\sigma(\omega)-1}{j-\sigma(\omega)-1}$.
As $\sum_{\ell=0}^{j} \binom{n+j-2-\ell}{j-\ell} = \binom{n+j-1}{j} > \omega$, we have $\sigma(\omega) = j-1$ iff $\omega = \binom{n+j-1}{j} - 1$, i.e. for all $\omega \neq \binom{n+j-1}{j} - 1$, it holds $j - \sigma(\omega) - 1 \geq 1$. By ($\bullet$) of Section 2, every $\tau \in \mathbf{L}_{n,\omega,j}$ is not divisible by $X_n^{\sigma(\omega)+2}$, thus, $\sigma(\omega) = -1$ implies $\mathbf{L}_{n,\omega,j} \subseteq \mathbb{T}_j$, and the inductive hypothesis on n applies. Otherwise, $j - \sigma(\omega) - 1 \geq j-1$ and $\mathbf{L}_{n,\omega,j} = \bigsqcup_{\ell=0}^{\sigma(\omega)} X_n^{\ell}\mathbb{T}_{j-\ell} \sqcup X_n^{\sigma(\omega)+1}\mathbf{L}_{n-1,\alpha(\omega),j-\sigma(\omega)-1}$.
As $\omega\{j\} = \sum_{\ell=0}^{\sigma(\omega)} \binom{n+j-2-\ell}{j-\ell} + \alpha(\omega)\{j - \sigma(\omega) - 1\}$, we end by the inductive hypothesis on j. Similarly for $i > 1$.

Remark 3.5. a) One computes $\lambda_i(\Lambda_{n,\omega,j})$ similarly (for this reason we gave our proof of Lemma 3.4 different from [11], Theorem 5.5). For each $j, \omega \in \mathbb{N}^*$ and $1 \leq \omega \leq \binom{n+j-1}{j} - 1$, by ($\bullet\bullet$) of Section 2, we have:

$$\lambda_1(\Lambda_{n,\omega,j}) = \begin{cases} 0 & \text{if } \omega \leq \binom{n+j-2}{j-1} \\ \omega - \binom{n+j-2}{j-1} & \text{otherwise} \end{cases}.$$

Defining $\rho(\omega)$ via: $\binom{n+j-2}{j-1} + \cdots + \binom{n+j-2-\rho(\omega)}{j-1} \leq \omega < \binom{n+j-2}{j-1} + \cdots + \binom{n+j-2-\rho(\omega)-1}{j-1}$, $\sum_{\ell=0}^{n-1} \binom{n+j-2-\ell}{j-1} = \binom{n+j-1}{j} > \omega$ implies $\rho(\omega) \leq n-2$ and again by ($\bullet\bullet$) of Section 2 we have

$$\lambda_i(\Lambda_{n,\omega,j}) = \begin{cases} 0 & \text{if } \rho(\omega) < i-1, \\ \omega - \sum\limits_{\ell=0}^{i-1} \binom{n+j-2-\ell}{j-1} & \text{if } \rho(\omega) \geq i-1. \end{cases}$$

Note that $\rho(\omega)+1$ is the greatest i between 0 and $n-2$ with $\lambda_i(\Lambda_{n,\omega,j}) \neq 0$ (i.e. $\rho(\omega)+2$ gives the greatest i between 1 and $n-1$, for which $X_i^j \in \Lambda_{n,\omega,j}$.)

b) Moreover, by Remark 3.2 b) and Lemma 3.4, for all Borel subsets $B \subseteq \mathbf{T}_j$, consisting of ω elements, with $3 \leq \omega \leq \binom{n+j-1}{j} - 3$, we have

$$\lambda_i(\mathbf{L}_{n,\omega,j}) \geq \lambda_i(B) \geq \lambda_i(\Lambda_{n,\omega,j}). \tag{9}$$

Lemma 3.6. *If $B \subseteq \mathbf{T}_j$ is Borel then $B_{(1)}$ is so, with cardinality $\sum\limits_{i=1}^{n} \lambda_{i-1}(B)$.*

Proof. Note that for each $r \in \mathbb{N}^*$, $C \subseteq \mathbf{T}_r$ is Borel iff

$$t \in \mathbf{T}_r \setminus C \text{ and } X_\ell | t \text{ imply } X_i t / X_\ell \in \mathbf{T}_r \setminus B \text{ for all } i > \ell. \tag{10}$$

By definition, $B_{(1)} = \mathbf{T}_{j+1} \setminus \{X_1, \ldots, X_n\} \cdot (\mathbf{T}_j \setminus B)$ and we will show that (10) is verified by $\{X_1, \ldots, X_n\} \cdot (\mathbf{T}_j \setminus B)$. Namely, $\bar{t} \in \{X_1, \ldots, X_n\} \cdot (\mathbf{T}_j \setminus B)$ implies $\bar{t} = X_\alpha t$ for some $1 \leq \alpha \leq n$ and $t \in \mathbf{T}_j \setminus B$. Clearly $X_\alpha | \bar{t}$ and for all $i > \alpha$ we have $X_i \bar{t}/X_\alpha = X_i t \in \{X_1, \ldots, X_n\} \cdot (\mathbf{T}_j \setminus B)$. If $X_\ell | \bar{t}$ for $\ell \neq \alpha$, then $X_i t / X_\ell \in (\mathbf{T}_j \setminus B)$ for all $i > \ell$, since B is Borel, so $X_i \bar{t}/X_\ell = X_i X_\alpha t / X_\ell \in \{X_1, \ldots, X_n\} \cdot (\mathbf{T}_j \setminus B)$ for all $i > \ell$. Moreover, $X_i \cdot B'(n-i+1) \cap \{X_1, \ldots, X_n\} \cdot (\mathbf{T}_j \setminus B) = \emptyset$ for each $i, 1 \leq i \leq n-1$ and for all $\tau \in B_{(1)}$ it holds $\tau \in X_{\mu(\tau)} \cdot B'(\mathbf{n} - \mu(\tau) + \mathbf{1})$. Thus

$$B_{(1)} = \bigcup_{i=1}^{n} X_i \cdot B'(\mathbf{n} - \mathbf{i} + \mathbf{1}) \tag{11}$$

and the union is disjoint because of (••) of Section 2. Thus, by the definition of $\lambda_{i-1}(B)$:

$$\#(B_{(1)}) = \sum_{i=1}^{n} \lambda_{i-1}(B). \tag{12}$$

Theorem 3.7. *If $B \subseteq \mathbf{T}_j$ is Borel, then for every $\ell \in \mathbb{N}^*$, $B_{(\ell)} \subseteq \mathbf{T}_{j+\ell}$ is so and $\#B_{(\ell)} = \# \bigsqcup\limits_{1 \leq i_1 \leq \cdots \leq i_\ell \leq n} X_{i_1} X_{i_2} \cdots X_{i_\ell} \cdot B'(\mathbf{n} - \mathbf{i}_\ell + \mathbf{1}) = \sum\limits_{i=1}^{n} \binom{i+\ell-2}{\ell-1} \lambda_{i-1}(B)$.*

Proof. Clearly $B_{(\ell)}$ is Borel being defined iteratively as $(B_{(\ell-1)})_{(1)}$ (see (5) of Section 2). Since $B_{(2)} = (B_{(1)})_{(1)}$ and, by the proof of Lemma 3.6, $B_{(1)} = \bigsqcup\limits_{i=1}^{n} X_i \cdot B'(\mathbf{n} - \mathbf{i} + \mathbf{1})$, one has $B_{(2)} = \bigsqcup\limits_{i=1}^{n} X_i \cdot B'_{(1)}(\mathbf{n} - \mathbf{i} + \mathbf{1}) = \bigsqcup\limits_{i=1}^{n} X_i [\bigsqcup\limits_{r=1}^{n} X_r B'(\mathbf{n} - \mathbf{r} + \mathbf{1})]'(\mathbf{n} - \mathbf{i} + \mathbf{1})$.

Since $t \in \mathbf{T}'(\mathbf{n} - \mathbf{i} + \mathbf{1})$ iff $\mu(t) \geq i$ (see (4) of Section 2), one has

$$[X_r B'(\mathbf{n} - \mathbf{r} + \mathbf{1})]'(\mathbf{n} - \mathbf{i} + \mathbf{1}) = \begin{cases} \emptyset & \text{if } r < i, \\ X_r B'(\mathbf{n} - \mathbf{r} + \mathbf{1}) & \text{if } r \geq i. \end{cases}$$

Thus, $B_{(2)} = \bigsqcup_{i=1}^{n} X_i[\bigsqcup_{r=i}^{n} X_r B'(\mathbf{n-r+1})] = \bigsqcup_{1\le i\le r\le n} X_i X_r B'(\mathbf{n-r+1})]$. Assume our contention for $\ell \in \mathbb{N}^*$, and deduce it for $\ell+1$. As $B_{(\ell+1)} = (B_{(\ell)})_{(1)}$, one has

$$\begin{aligned}
B_{(\ell+1)} &= \bigsqcup_{i=1}^{n} X_i[\bigsqcup_{1\le r_1\le\cdots\le r_\ell\le n} X_{r_1}X_{r_2}\cdots X_{r_\ell}\cdot B'(\mathbf{n}-\mathbf{r}_\ell+\mathbf{1})]'(\mathbf{n-i+1})\\
&= \bigsqcup_{i=1}^{n} X_i[\bigsqcup_{i\le r_1\le\cdots\le r_\ell\le n} X_{r_1}X_{r_2}\cdots X_{r_\ell}\cdot B'(\mathbf{n}-\mathbf{r}_\ell+\mathbf{1})]\\
&= \bigsqcup_{1\le i_1\le\cdots\le i_\ell\le i_{\ell+1}\le n} X_{i_1}X_{i_2}\cdots X_{i_{\ell+1}}\cdot B'(\mathbf{n}-\mathbf{i}_{\ell+1}+\mathbf{1}).
\end{aligned}$$

By counting how many times $B'(\mathbf{n-i+1})$, i running from 1 to n, contributes to the above union, one gets $\#(B_{(\ell)})$. $B = B'(\mathbf{n})$ only occurs multiplied by X_1^ℓ, $B'(\mathbf{n-1})$ occurs $\binom{\ell}{\ell-1}$ times (multiplied by the $t \in \mathbf{T(2)}_\ell$ divisible by X_2) and $B'(\mathbf{n-i+1})$ occurs $\binom{i+\ell-2}{\ell-1}$ times (multiplied by the $t \in \mathbf{T(i)}_\ell$ divisible by X_i). Thus, as claimed, $\#(B_{(\ell)}) = \sum_{i=1}^{n}\binom{i+\ell-2}{\ell-1}\lambda_{i-1}(B)$.

Theorem 3.7 shows how to construct Borel subsets of given cardinality in each degree (for Lex-segments see [8], for the general case see [9]).

General Construction 3.8. Fix $d < s \in \mathbb{N}^*$ and $1 \le \omega \le \binom{n+d-1}{d} - 1$, for all $0 \le j \le d-1$, we let $B_j := \mathbf{T}_j$ and $B_d \subseteq \mathbf{T}_d$ a Borel subset of cardinality $\omega_0 := \omega$. We also let $B_{d+\ell} \subseteq (B_{d+\ell-1})_{(1)}$ be a Borel subset of cardinality ω_ℓ for all $1 \le \ell \le s-d$ and $\omega_\ell \le \#(B_d)_{(\ell)}$, and $B_j = \emptyset$ for all $j > s$. As clearly $(B_j)_{(1)} = \mathbf{T}_{j+1}$, for all $0 \le j \le d-1$, we have $B_{r+1} \subseteq (B_r)_{(1)}$, for each $r \in \mathbb{N}$. Thus, $\mathbb{N} := \bigsqcup_{r\in\mathbb{N}} B_r$ is an order ideal and a Borel subset of $\mathbf{T}$, with $\#N = \sum_{i=0}^{d-1}\binom{n+i-1}{i} + \sum_{\ell=0}^{s-d}\omega_\ell$.

Remark 3.9. a) From Lemma 3.4 and Lemma 3.6, we get:

- $\mathbf{L}_{n,\eta,j+1} \subseteq (\mathbf{L}_{n,\omega,j})_{(1)}$ for every $\eta \le \#((\mathbf{L}_{n,\omega,j})_{(1)})$, yet
- $\Lambda_{n,\eta,j+1} \subseteq (\Lambda_{n,\omega,j})_{(1)}$ only for $\eta \le \omega$.

b) For each r between 0 and $n-2$, we have $\lambda_r(B_{(1)}) = \sum_{i=r}^{n-2}\lambda_i(B)$.

If $n = 3$, $\#(B_{(\ell)}) = \lambda_0(B) + \ell\lambda_1(B)$, i.e. $\lambda_1(B_{(\ell)}) = \lambda_1(B)$ for each $\ell \in \mathbb{N}$.

4. Borel ideals

In this and next section, $\mathbf{h} := (1, n, \ldots, h_d, \ldots, h_s) \in \mathbb{N}^{*(s+1)}$ is the O-sequence of a homogeneous 0-dimensional ideal $\mathfrak{a} \subseteq \mathbf{P}$ with initial degree $d \le s$ and generators in degrees $\le s+1$ (i.e. $H_{\mathbf{P}/\mathfrak{a}}(j) = h_j$ for $0 \le j \le s$ and $H_{\mathbf{P}/\mathfrak{a}}(j) = 0$ for $j \ge s+1$. In particular we will say that such an $\mathbf{h}$ is not *increasing* if $\Delta(\mathbf{h}) := (1, n-1, \ldots, h_d - h_{d-1}, \ldots, h_s - h_{s-1}) = (1, n-1, \ldots, \Delta(\mathbf{h})_d, \ldots, \Delta(\mathbf{h})_s) \in \mathbb{Z}^{s+1}$ satisfies $\Delta(\mathbf{h})_j \le 0$, for all $j \ge d+1$ (n.b. for $n \ge 2$, $\Delta(\mathbf{h})_j = \binom{n+j-2}{j} > 0$ if $1 \le j \le d-1$; no assumption is made on $\Delta(\mathbf{h})_d$).

Notation 4.1. The *l-segment ideal* associated to $\mathbf{h}$ is $\mathcal{L}(\mathbf{h})$ with $\mathcal{N}(\mathcal{L}(\mathbf{h}))_j = \mathbf{T}_j$ if $0 \le j \le d-1$, $\mathcal{N}(\mathcal{L}(\mathbf{h}))_j = \mathbf{L}_{n,h_j,j}$ if $d \le j \le s$ and $\mathcal{N}(\mathcal{L}(\mathbf{h}))_j = \emptyset$ if $s+1 \le j$ (see [8]). For $\mathbf{h}$ non-increasing, the associated *rl-segment ideal* is $\Lambda(\mathbf{h})$ with $\mathcal{N}(\Lambda(\mathbf{h}))_j = \mathbf{T}_j$ if $0 \le j \le d-1$, $\mathcal{N}(\Lambda(\mathbf{h}))_j = \Lambda_{n,h_j,j}$ if $d \le j \le s$ and $\mathcal{N}(\Lambda(\mathbf{h}))_j = \emptyset$ if $s+1 \le j$ (see [3] and [10]).

Definition 4.2. *A monomial ideal $\mathfrak{b} \subseteq \mathbf{P}$ is Borel if $\mathcal{N}(\mathfrak{b})_j$ is so, for all $j \in \mathbb{N}$ and $\mathcal{B}^n_{\mathbf{h}}$ is the set of 0-dimensional Borel ideals $\mathfrak{b} \subseteq \mathbf{P}(\mathbf{n})$ corresponding to $\mathbf{h}$.*

For $n = 2$ all notions coincide. If $n \ge 3$, then l-segment and rl-segment ideals are Borel, yet there are Borel ideals neither l-segment nor rl-segment. For a Borel ideal $\mathfrak{b} \subseteq \mathbf{P}$ of initial degree $d \in \mathbb{N}^*$ $X_n^d \in G(\mathfrak{b})$, thus $\nu_{n,d}(\mathfrak{b}) = 1$ and $\nu_{n,j}(\mathfrak{b}) = 0$ for all $j \ne d$.

Remark 4.3. a) $\mathcal{B}^n_{\mathbf{h}} \ne \emptyset$ as it contains $\mathcal{L}(\mathbf{h})$; if $\Delta(\mathbf{h})_j > 0$ for some $j \ge d+1$, by Remark 3.9 a) there isn't corresponding rl-segment ideal.

b) If $\mathfrak{b} \in \mathcal{B}^n_{\mathbf{h}}$, as $G(\mathfrak{b})_{s+1} = (\mathcal{N}(\mathfrak{b})_s)_{(1)}$ and $\nu_{n,j}(\mathfrak{b}) = 0$, for all $j \ne d$, Lemma 3.6 applied to $\mathcal{N}(\mathfrak{b})_s$ implies $\nu_{i,s+1}(\mathfrak{b}) = \lambda_{i-1,s}(\mathcal{N}(\mathfrak{b}))$ for each i in the range between 1 and n. Moreover, for each ℓ in the range between 0 and $s-(d+\ell)$:

$$h_{d+\ell+1} = \#((\mathcal{N}(\mathfrak{b})_{d+\ell})_{(1)} \setminus G(\mathfrak{b})_{d+\ell+1}) = \sum_{i=0}^{n-2} \lambda_{i,d+\ell}(\mathcal{N}(\mathfrak{b})) - \#(G(\mathfrak{b})_{d+\ell+1}).$$

c) By Theorem 3.7 for constructing $\mathfrak{b} \in \mathcal{B}^n_{\mathbf{h}}$ one needs, for r varying from 0 to $s-d$, Borel subsets $B_r \subseteq \mathbf{T}_{d+r}$ of cardinality h_{d+r}, with the following constraints:

1. $B_{r+1} \subseteq (B_r)_{(1)}$,
2. $\#(B_{(\ell)}) \ge h_{d+r+\ell}$ for each ℓ in the range between 0 and $s-(d+\ell)$.

Lemma 4.4. *A monomial ideal $\mathfrak{a} \subseteq \mathbf{P}$ corresponding to $\mathbf{h}$ satisfies $\lambda_{1,j}(\mathcal{N}(\mathfrak{a})) \ge \Delta(\mathbf{h})_j$, for all j in the range between 0 and s.*

Proof. By ($\bullet\bullet$) of Section 2, $\mathcal{N}(\mathfrak{a})_j = (\mathcal{N}(\mathfrak{a})_j \cap X_1\mathbf{T}_{j-1}) \sqcup (\mathcal{N}(\mathfrak{a})_j)'(\mathbf{n-1})$. Letting

$$\xi_j := \#(\mathcal{N}(\mathfrak{a})_j \cap X_1\mathbf{T}_{j-1}),$$

we have $h_j = \#(\mathcal{N}(\mathfrak{a})_j) = \xi_j + \lambda_{1,j}(\mathcal{N}(\mathfrak{a}))$. Moreover, $\mathfrak{a}_{j-1}\mathbf{P}_1 \subseteq \mathfrak{a}_j$ implies $\mathfrak{a}_{j-1}X_1 \subseteq \mathfrak{a}_j \cap X_1\mathbf{T}_{j-1}$ or, which is the same, $\mathcal{N}(\mathfrak{a})_j \cap X_1\mathbf{T}_{j-1} \subseteq \mathcal{N}(\mathfrak{a})_{j-1}X_1$, i.e. $\xi_j \le h_{j-1}$. So $\Delta(\mathbf{h})_j := h_j - h_{j-1} = \xi_j + \lambda_{1,j}(\mathcal{N}(\mathfrak{a})) - h_{j-1} \le \lambda_{1,j}(\mathcal{N}(\mathfrak{a}))$.

Corollary 4.5. *A $\mathfrak{b} \in \mathcal{B}^n_{\mathbf{h}}$ satisfies $\lambda_{1,j}(\mathcal{N}(\mathfrak{b})) = \Delta(\mathbf{h})_j$ exactly for those j in the range between 0 and s, such that $G(\mathfrak{b})_j$ does not contain any term divisible by X_1.*

Proof. As clearly $G(\mathfrak{b})_j = \emptyset$ for each $0 \le j \le d-1$, only $j = d+\ell$, $0 \le \ell \le s-d$, matter. Moreover, from Lemma 4.4 one infers that $\lambda_{1,d+\ell}(\mathcal{N}(\mathfrak{b})) = \Delta(\mathbf{h})_{d+\ell}$ iff $\xi_{d+\ell} = h_{d+\ell-1}$. As $G(\mathfrak{b})_{d+\ell} = (\mathcal{N}(\mathfrak{b})_{d+\ell-1})_{(1)} \setminus \mathcal{N}(\mathfrak{b})_{d+\ell}$ and $(\mathcal{N}(\mathfrak{b})_{d+\ell-1})_{(1)} = \bigsqcup_{j=1}^{n-1} X_j(\mathcal{N}(\mathfrak{b})_{d+\ell-1})'(\mathbf{n-j+1})$, this means exactly $X_1 \nmid t$, for all $t \in G(\mathfrak{b})_{d+\ell}$.

If $n = 3$, we can say more and therefore, from now on, unless otherwise noticed, $\mathbf{T} := \mathbf{T}(\mathbf{3})$ endowed with drl and $\mathbf{h} := (1, 3, h_2, \ldots, h_d, \ldots, h_s)$, as if $d = s+1$ then $\mathcal{B}^3_{\mathbf{h}} = \{(X,Y,Z)^s\}$, one can take $d \le s$. We begin giving the following definition:

Definition 4.6. ([9], [10]) *For $j \in \mathbb{N}^*$ and $1 \le i \le j+1$, $0 \le a \le j+1$ we set:*

$$\ell_{ij} := \{X^{j-i+1}Z^{i-1}, X^{j-i}YZ^{i-1}, \dots, Y^{j-i+1}Z^{i-1}\} \text{ and } R_{a,j} := \bigsqcup_{i=1}^{a} \ell_{ij}.$$

Note that: $R_{0,j} = \emptyset$, $R_{j+1,j} = \mathbf{T(3)}_j$, and $R_{a,j}$ is the (initial)-l-segment $\mathbf{L}_{3,\frac{a(2j-a+3)}{2},j}$. If $B \subseteq \mathbf{T(3)}_j$ is Borel, then $\#(B \cap \ell_{ij}) > \#(B \cap \ell_{i+1j})$ for every $1 \le i \le j+1$; if $B \cap \ell_{\bar{i}j} \neq \emptyset$, for some $1 \le \bar{i} \le j+1$, a full segment (from the left) of $\ell_{\bar{i}j}$ lies in B.

Definition 4.7. *For each $0 \le \ell \le s-d$, the increasing character of* $\mathbf{h}$ *in degree $d+\ell$ is $a_\ell := \max\{0, \max_{i \ge d+\ell}\{\Delta(\mathbf{h})_i\}\}$.*

In Definition 4.7 we have $d \ge a_0 \ge a_1 \ge \cdots \ge a_{s-d} := \max\{0, \Delta(\mathbf{h})_s\}$. Thus, $a_\ell = 0$ for some $0 \le \ell \le s-d$, implies $a_{\ell+r} = 0$ for all $0 \le r \le s-(d+\ell)$.

We point out that bonds of Remark 4.3 c) reduce (by Remark 3.9 b)) to:

- $B_{\ell+1} \subseteq (B_\ell)_{(1)}$,
- $\lambda_1(B_\ell) \ge a_\ell$, for all $0 \le \ell \le s-d$.

Definition 4.8. 1. *Denoting $m(\mathbf{h}) \le s-d$ the index of the last positive increasing character of* $\mathbf{h}$, *we introduce $\bar{\mathbf{h}} \in \mathbb{N}^{*(d+m(\mathbf{h})+1)}$, defined by:*

$$\bar{h}_j = \begin{cases} \Delta(\mathbf{h})_j = j+1 & \text{if } 0 \le j \le d-1, \\ a_{j-d} & \text{if } d \le j \le d+m(\mathbf{h}). \end{cases}$$

2. *Following our General Construction* 3.8, *we define the order ideal $\mathbb{L}(\mathbf{h}) := \bigsqcup_{j \in \mathbb{N}} \mathbb{L}(\mathbf{h})_j$, where:*
   - $\mathbb{L}(\mathbf{h})_j := \mathbf{T}_j$ *if* $0 \le j \le d-1$,
   - $\mathbb{L}(\mathbf{h})_j := R_{\bar{\mathbf{h}}_j,j} \sqcup \{t_1, \dots, t_{b(j)}\}$ *if $d \le j \le d+m(\mathbf{h})$, with $b(j) := h_j - \frac{\bar{h}_j(2j-\bar{h}_j+3)}{2}$ and $t_1 < \cdots < t_{b(j)}$ the smallest terms of $(\mathbb{L}(\mathbf{h})_{j-1})_{(1)} \setminus R_{\bar{h}_j,j}$,*
   - $\mathbb{L}(\mathbf{h})_j := \{t_1, \dots, t_{h_j}\}$ *if $d+m(\mathbf{h})+1 \le j \le s$, with $t_1 < \cdots < t_{h_j}$ the smallest terms of $(\mathbb{L}(\mathbf{h})_{j-1})_{(1)}$,*
   - $\mathbb{L}(\mathbf{h})_j := \emptyset$ *if* $j > s$.
3. *The generalized-rl-segment-ideal $\pounds(\mathbf{h}) \in \mathcal{B}^3_{\mathbf{h}}$ is the monomial ideal with sous-éscalier $\mathbb{L}(\mathbf{h})$.*

Let $s_1 < \cdots < s_{h_j} \in \mathcal{N}(\pounds(\mathbf{h}))_j$ and $\tau_1 < \cdots < \tau_{h_j} \in \mathcal{N}(\mathfrak{b})_j$, for $\mathfrak{b} \in \mathcal{B}^3_{\mathbf{h}}$, $0 \le j \le s$, be the respective elements, one has $s_r \le \tau_r$ for all $1 \le r \le h_j$. As $\lambda_{1,d+\ell}(\mathcal{N}(\mathfrak{b})) \ge a_\ell$ for each $\mathfrak{b} \in \mathcal{B}^3_{\mathbf{h}}$, the trace of $\mathcal{N}(\pounds(\mathbf{h}))_{d+\ell}$ in $\mathbb{T}'_{d+\ell}$, ℓ varying from 0 to $s-d$, is minimal among the elements of $\mathcal{B}^3_{\mathbf{h}}$. If $\mathbf{h}$ is not-increasing, then $\mathbb{L}(\mathbf{h}) = \Lambda(\mathbf{h})$.

Remark 4.9. a) The sequence $\overline{\mathbf{h}}$ of Definition 4.8 a) is an O-sequence being the Hilbert function of the Borel ideal:

$$(\pounds(\mathbf{h}), X_1)/(X_1) \subseteq \mathbf{P}'(\mathbf{2}).$$

b) Letting, for all homogeneous ideal $\mathfrak{a} \subseteq \mathbf{P}$ and i in the range between 1 and n, $\mathfrak{a}[i] := (\mathfrak{a}, X_1, \dots, X_i)/(X_1, \dots, X_i)$, we have:

$$\lambda_{i,j}(\mathcal{N}(\mathfrak{a})) := \#((\mathcal{N}(\mathfrak{a})_j)'(\mathbf{n}-\mathbf{i})) = H_{\mathbf{P}/\mathfrak{a}[i]}(j).$$

Drawing inspiration from [2], we associate to every 0-dimensional Borel ideal $\mathfrak{b} \subseteq \mathbf{P}$, generated in degrees $\leq s+1$, a matrix in $M_{n,s+1}(\mathbb{N})$, defined as follows:

Definition 4.10. *The sous-éscalier sectional matrix (ses-matrix) of a 0-dimensional Borel ideal $\mathfrak{b} \subseteq \mathbf{P}$, generated in degrees $\leq s+1$, is $\tilde{\mathcal{M}}(\mathfrak{b}) = (\tilde{m}_{i,j}(\mathfrak{b})) \in M_{n,s+1}(\mathbb{N})$:*

$$\tilde{m}_{i,j}(\mathfrak{b}) := \lambda_{i-1,j-1}(\mathcal{N}(\mathfrak{b})),\ 1 \leq i \leq n,\ 1 \leq j \leq s+1.$$

In general different ideals in $\mathcal{B}^n_{\mathbf{h}}$ can share the same ses-matrix.

Example 4.11. If $\mathbf{h} = (1,3,4,3)$, then both $\Lambda(\mathbf{h}) = (Z^2, YZ, Y^3, XY^2, X^3Z, X^3Y, X^4)$ and $\mathcal{L}(\mathbf{h}) = (Z^2, YZ, Y^3, X^2Z, X^2Y^2, X^3Y, X^4)$ are in $\mathcal{B}^3_{\mathbf{h}}$. Note that

$$\tilde{\mathcal{M}}(\Lambda(\mathbf{h})) = \begin{pmatrix} 1 & 3 & 4 & 3 \\ 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} = \tilde{\mathcal{M}}(\mathcal{L}(\mathbf{h})).$$

Remark 4.12. a) By Definition 4.10 and Remark 4.9 a) we have:

- $(\tilde{m}_{i,1}(\mathfrak{b}), \ldots, \tilde{m}_{i,s+1}(\mathfrak{b})) = (H_{\mathbf{P}/\mathfrak{b}[i-1]}(0), \ldots, H_{\mathbf{P}/\mathfrak{b}[i-1]}(s))$ as i ranges between 1 and n, in particular, for each i the $0 \neq \tilde{m}_{i,j}(\mathfrak{b})$ form an O-sequence.
- $(\tilde{m}_{1,j}(\mathfrak{b}), \ldots, \tilde{m}_{n,j}(\mathfrak{b})) = (H_{\mathbf{P}/\mathfrak{b}[0]}(j-1), \ldots, H_{\mathbf{P}/\mathfrak{b}[n-1]}(j-1)), 1 \leq j \leq s+1.$

b) Given any O-sequence $\mathbf{h} = (1, 3, \binom{4}{2}, \ldots, \binom{d+1}{d-1}, h_d, \ldots, h_s)$, by Lemma 3.4, the second row of $\tilde{\mathcal{M}}(\mathcal{L}(\mathbf{h}))$ is

$$(1\ \ 2\ \ 3\ \ \cdots\ \ d\ \ (h_d\{d\})^{-1}\ \ \cdots\ \ (h_s\{s\})^{-1}),$$

while, by construction, the second row of $\tilde{\mathcal{M}}(\pounds(\mathbf{h}))$ is

$$(1\ \ 2\ \ 3\ \ \cdots\ \ d\ \ a_0\ \ \cdots\ \ a_{m(\mathbf{h})}\ \ 0\ \ \cdots\ \ 0).$$

In general, for all O-sequence $\mathbf{h} = (1, n, \ldots, \binom{n+d-2}{d-1}, h_d, \ldots, h_s)$, the i-th row of $\tilde{\mathcal{M}}(\mathcal{L}(\mathbf{h}))$ is $(1, n-i+1, \binom{n-i+2}{2}, \ldots, \binom{n+d-(i+1)}{d-1}, (h_d\{d\})^{-(i-1)}, \ldots, (h_s\{s\})^{-(i-1)})$.

A well-known result of Eliahou-Kervaire [4] gives a handy formula for the graded Betti numbers of a Borel ideal $\mathfrak{b} \subseteq \mathbf{P}$. Namely, if $X_1 < \cdots < X_n$, it holds:

$$\beta_{q,j+q}(\mathfrak{b}) = \sum_{\substack{t \in G(\mathfrak{b}) \\ deg\, t = j}} \binom{n-\mu(t)}{q} = \sum_{i=1}^{n} \binom{n-i}{q} \nu_{i,j}(\mathfrak{b}), \qquad (*)$$

(where $\mu(t)$ is defined in (2) and $\nu_{i,j}(\mathfrak{b})$ stays for $\nu_{i,j}(G(\mathfrak{b}))$ (defined in (3))).
In particular, for a 0-dimensional Borel ideal $\mathfrak{b} \subseteq \mathbf{P}$ of initial degree d and generated in degrees $\leq s+1$, j and q in $(*)$ vary respectively between d and $s+1$, 0 and $n-1$.

Proposition 4.13. *Two 0-dimensional Borel ideals $\mathfrak{b}, \mathfrak{b}' \subseteq \mathbf{P}(\mathbf{n})$ have the same ses-matrix iff they have the same graded Betti numbers.*

Proof. Let $\mathfrak{b}, \mathfrak{b}'$ have either the same ses-matrix or graded Betti numbers, and let d (resp. $s+1$) be the initial (resp. greatest) degree of generators. Denoting $\star$ either of $\mathfrak{b}$ and $\mathfrak{b}'$, from $(*)$ above we get linear relations between the $\beta_{-,-}(\star)'s$ and $\nu_{+,+}(\star)'s$, more precisely, for j varying between d and $s+1$, we get a Gauss-reduced linear system of n equations, q varying between $n-1$ and 0. Namely, $q = n-1$ implies $\beta_{n-1,j+n-1}(\star) = \sum_{i=1}^{n} \binom{n-i}{n-1} \nu_{i,j}(\star) = \nu_{1,j}(\star)$, substituting it in $q = n-2$ we get $\beta_{n-2,j+n-2}(\star) = \sum_{i=1}^{n} \binom{n-i}{n-2} \nu_{i,j}(\star) = (n-1)\nu_{1,j}(\star) + \nu_{2,j}(\star) = (n-1)\beta_{n-1,j+n-1}(\star) + \nu_{2,j}(\star)$, i.e. $\nu_{2,j}(\star) = \beta_{n-2,j+n-2}(\star) - (n-1)\beta_{n-1,j+n-1}(\star)$ and so on, until $q = 0$. Thus $\beta_{q,j+q}(\star)$'s are function of $\nu_{i,j}(\star)$'s, $0 \le q \le n-1$, $d \le j \le s+1$, $1 \le i \le n$, and conversely. As $\nu_{n,d}(\star) = 1$ and $\nu_{n,j}(\star) = 0$, for all j between $d+1$ and $s+1$, dependency relations between the $\beta_{-,-}(\star)$'s hold. By (8) and (11) we have also

$$\nu_{i,j}(\star) = \#(\mathcal{N}(\star)'((\mathbf{n}-\mathbf{i}+\mathbf{1})_{j-1})_{(1)} - \lambda_{i,j}(\mathcal{N}(\star)) - \sum_{h=1}^{n-1} \nu_{i+h,j}(\star)$$

from which, recalling that $\lambda_{n-1,j}(\mathcal{N}(\star)) = 0$ for $d \le j \le s$, we get the following relations between $\nu_{i,j}(\star)$'s and $\lambda_{i,j}(\mathcal{N}(\star))$'s:

- $\nu_{i,d}(\star) = \binom{n-i+d-1}{d-1} + \lambda_{i,d}(\mathcal{N}(\star)) - \lambda_{i-1,d}(\mathcal{N}(\star))$, $\quad n-1 \ge i \ge 1$,
- $\nu_{i,j}(\star) = \lambda_{i-1,j-1}(\mathcal{N}(\star)) - \lambda_{i-1,j}(\mathcal{N}(\star)) + \lambda_{i,j}(\mathcal{N}(\star))$, $n-1 \ge i \ge 1$, $d < j \le s$,
- $\nu_{i,s+1}(\star) = \lambda_{i-1,s}(\mathcal{N}(\star))$ $\quad$ for $1 \le i \le n-1$,

which allow to express the last ones in terms of the first ones.

We get our contention taking into account $(\diamond)$ and $\bullet$'s and recalling that $\tilde{m}_{i,j}(\star) := \lambda_{i-1,j-1}(\mathcal{N}(\star))$ for i and j respectively in the range between 1 and n, 1 and $s+1$.

Remark 4.14. Let $n = 3$, $\mathbf{h}$ an O-sequence and $\mathfrak{b}, \mathfrak{b}' \in \mathcal{B}^3_{\mathbf{h}}$, then:

a) $\nu_{3,d}(\mathfrak{b}) = 1$, $\nu_{2,d}(\mathfrak{b}) = d - \lambda_{1,d}(\mathcal{N}(\mathfrak{b}))$, $\nu_{1,d}(\mathfrak{b}) = \binom{d+1}{2} - h_d + \lambda_{1,d}(\mathcal{N}(\mathfrak{b}))$;

b) as $G(\mathfrak{b})_{d+\ell+1} = (X\mathcal{N}(\mathfrak{b})_{d+\ell} \sqcup Y((\mathcal{N}(\mathfrak{b})_{d+\ell})'(2)) \setminus \mathcal{N}(\mathfrak{b})_{d+\ell+1}$, for $\ell, 1 \le \ell \le s-d$, we have $\nu_{3,d+\ell+1}(\mathfrak{b}) = 0$, $\nu_{2,d+\ell+1}(\mathfrak{b}) = \lambda_{1,d+\ell}(\mathcal{N}(\mathfrak{b})) - \lambda_{1,d+\ell+1}(\mathcal{N}(\mathfrak{b}))$, and $\nu_{1,d+\ell+1}(\mathfrak{b}) = h_{d+\ell} - h_{d+\ell+1} + \lambda_{1,d+\ell+1}(\mathcal{N}(\mathfrak{b}))$;

c) if $0 \to L_2 \to L_1 \to L_0 \to \mathbf{P} \to \mathbf{P}/\mathfrak{b} \to 0$ is the minimal free resolution of $(\mathbf{P}/\mathfrak{b})$, with $L_i = \bigoplus_{j=d}^{s+1} \mathbf{P}(-j-i)^{\beta_{i,j+i}}$, letting $0 = h_{s+1} = \lambda_{0,s+1}(\mathcal{N}(\mathfrak{b}))$, since $\lambda_{1,j}(\mathcal{N}(\mathfrak{b})) = \beta_{0,j+1} - h_j + h_{j+1}$ for j in the range between d and s, the above found values, inserted in the [4]'s formula $(*)$ give:

$$\beta_{0,j} = \begin{cases} \binom{d+2}{2} - h_d & \text{if } j = d \\ h_{j-1} - h_j + \lambda_{1,j-1}(\mathcal{N}(\mathfrak{b})) & \text{if } j \text{ varies from } d+1 \text{ to } s+1 \end{cases}$$

$$\beta_{1,j+1} = \begin{cases} d(d+2) - 3h_d + h_{d+1} + \beta_{0,d+1} & \text{if } j = d \\ h_{j-1} - 2h_j + h_{j+1} + \beta_{0,j+1} + \beta_{0,j} & \text{if } j \text{ varies from } d+1 \text{ to } s+1 \end{cases}$$

$\beta_{2,j+2} = h_{j-1} - 2h_j + h_{j+1} + \beta_{0,j+1}$ if j varies between d and $s+1$.

From the above consideration we get

$$\beta_{q,j+q}(\mathfrak{b}) \ge \beta_{q,j+q}(\mathfrak{b}') \text{ if and only if } \tilde{m}_{q,j+q}(\mathfrak{b}) \ge \tilde{m}_{q,j+q}(\mathfrak{b}').$$

If $n \neq 3$, graded Betti numbers of a 0-Borel ideal are not characterized only in terms of its $\mathbf{h}$ and $\beta_{0,j}$'s.

Example 4.15. In $\mathbf{P} := \mathbf{P}(4)$ (with $X < Y < Z < T$) let $\mathfrak{a}$ and $\mathfrak{b}$ be the two Borel ideals:

$\mathfrak{a} = (X^6, X^5Y, X^4Y^2, X^3Y^3, X^2Y^4, XY^5, Y^6, X^5Z, X^4YZ, X^3Y^2Z, X^2Y^3Z, XY^4Z, Y^5Z, X^4Z^2, X^3YZ^2, X^2Y^2Z^2, XY^3Z^2, Y^4Z^2, X^3Z^3, X^2YZ^3, XY^2Z^3, Y^3Z^3, X^2Z^4, XYZ^4, Y^2Z^4, Z^5, X^5T, X^4YT, X^3Y^2T, X^2Y^3T, XY^4T, Y^5T, X^4ZT, X^3YZT, X^2Y^2ZT, XY^3ZT, Y^4ZT, X^2Z^2T, YZ^2T, Z^3T, X^2T^2, XYT^2, Y^2T^2, XZT^2, YZT^2, Z^2T^2, XT^3, YT^3, ZT^3, T^4)$,

$\mathfrak{b} = (X^6, X^5Y, X^4Y^2, X^3Y^3, X^2Y^4, XY^5, Y^6, X^5Z, X^4YZ, X^3Y^2Z, X^2Y^3Z, XY^4Z, Y^5Z, X^4Z^2, X^3YZ^2, X^2Y^2Z^2, XY^3Z^2, Y^4Z^2, X^3Z^3, X^2YZ^3, XY^2Z^3, Y^3Z^3, X^2Z^4, XYZ^4, Y^2Z^4, XZ^5, YZ^5, Z^6, X^5T, X^4YT, X^3Y^2T, X^2Y^3T, Y^4T, X^4ZT, X^3YZT, X^2Y^2ZT, Y^3ZT, X^3Z^2T, YZ^2T, Z^3T, X^2T^2, XYT^2, Y^2T^2, XZT^2, YZT^2, Z^2T^2, XT^3, YT^3, ZT^3, T^4)$.

We have $\mathfrak{a}, \mathfrak{b} \in \mathcal{B}^4_{\mathbf{h}}$ with $\mathbf{h} = (1, 4, 10, 20, 23, 29)$, $\mathbf{P}/\mathfrak{a}$ and $\mathbf{P}/\mathfrak{b}$ have the same $\beta_{0,j}$'s, but different $\beta_{i,j+i}$ for $i \geq 1$, indeed their minimal free resolutions are:

$0 \to \mathbf{P}^4(-7) \oplus \mathbf{P}(-8) \oplus \mathbf{P}^{29}(-9) \to \mathbf{P}^{16}(-6) \oplus \mathbf{P}^3(-7) \oplus \mathbf{P}^{94}(-8) \to \mathbf{P}^{23}(-5) \oplus \mathbf{P}^4(-6) \oplus \mathbf{P}^{101}(-7) \to \mathbf{P}^{12}(-4) \oplus \mathbf{P}^2(-5) \oplus \mathbf{P}^{36}(-6) \to \mathbf{P} \to \mathbf{P}/\mathfrak{a} \to 0$,

$0 \to \mathbf{P}^4(-7) \oplus \mathbf{P}^{29}(-9) \to \mathbf{P}^{16}(-6) \oplus \mathbf{P}^2(-7) \oplus \mathbf{P}^{93}(-8) \to \mathbf{P}^{23}(-5) \oplus \mathbf{P}^4(-6) \oplus \mathbf{P}^{100}(-7) \to \mathbf{P}^{12}(-4) \oplus \mathbf{P}^2(-5) \oplus \mathbf{P}^{36}(-6) \to \mathbf{P} \to \mathbf{P}/\mathfrak{b} \to 0$.

5. The poset structure

In this section, for each $\mathbf{h} = (1, n, h_2, \ldots, h_d, \ldots, h_s)$, with $n \geq 3$, we first introduce an equivalence relation $\sim$ on the set $\mathcal{B}^n_{\mathbf{h}}$ of 0-dimensional Borel ideals of $\mathbf{P}(\mathbf{n})$ corresponding to $\mathbf{h}$. Then we define a partial order $\prec$ on $\mathcal{B}^n_{\mathbf{h}}/\sim$, which endows it with a poset structure, some features of which are studied.

Definition 5.1.
1. *Two Borel ideals $\mathfrak{b}$, $\mathfrak{b}' \in \mathcal{B}^n_{\mathbf{h}}$ are equivalent (in symbol $\mathfrak{b} \sim \mathfrak{b}'$) if share the same ses-matrix (see Definition* 4.10*).*
2. *Two equivalence classes $\bar{\mathfrak{b}}$, $\bar{\mathfrak{b}'} \in \mathcal{B}^n_{\mathbf{h}}/\sim$ satisfy $\bar{\mathfrak{b}} \prec \bar{\mathfrak{b}'}$ if $\bar{\mathfrak{b}} \neq \bar{\mathfrak{b}'}$ and for all representatives $\mathfrak{b}, \mathfrak{b}'$ the ses-matrices's entries satisfy $m_{i,j}(\mathfrak{b}) \leq m_{i,j}(\mathfrak{b}')$.*

Remark 5.2. a) In Example 4.15, the rl-segment ideal $\Lambda(\mathbf{h})$ and the l-segment ideal $\mathcal{L}(\mathbf{h})$ are distinct but $\Lambda(\mathbf{h}) \sim \mathcal{L}(\mathbf{h})$.

b) By Proposition 4.13, equivalent elements of $\mathcal{B}^n_{\mathbf{h}}$ have the same minimal free resolution. After Proposition 4.13, tedious but easy computations show that $\bar{\mathfrak{b}} \prec \bar{\mathfrak{b}'} \in \mathcal{B}^n_{\mathbf{h}}/\sim$ implies $\beta_{i,j}(\mathfrak{b}) \leq \beta_{i,j}(\mathfrak{b}')$, for all $\mathfrak{b} \in \bar{\mathfrak{b}}$, $\mathfrak{b}' \in \bar{\mathfrak{b}'}$.

c) The ses-matrices of Example 4.15 are not comparable w.r.t. $\prec$, indeed:

$$\tilde{\mathcal{M}}(\mathfrak{a}) = \begin{pmatrix} 1 & 4 & 10 & 20 & 23 & 29 \\ 1 & 3 & 6 & 10 & 7 & 7 \\ 1 & 2 & 3 & 4 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix} \quad \text{and}$$

$$\tilde{\mathcal{M}}(\mathfrak{b}) = \begin{pmatrix} 1 & 4 & 10 & 20 & 23 & 29 \\ 1 & 3 & 6 & 10 & 7 & 6 \\ 1 & 2 & 3 & 4 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

Yet, as their minimal free resolutions show: $\beta_{i,j}(\mathfrak{b}) \leq \beta_{i,j}(\mathfrak{a})$, for all $i = 0,1,2,3$, $j = 4,5,6$, this means that $\prec$ is a partial order stronger than the one given by the graded Betti numbers.

d) By (9) and Remark 3.9 a), the class $\overline{\mathcal{L}(\mathbf{h})}$ of the l-segment ideal corresponding to $\mathbf{h}$ is the maximum element of $\mathcal{B}^n_{\mathbf{h}}/\sim$.

e) If $\mathbf{h}$ is not increasing, (9) and Remark 3.9 b) imply that the la class $\overline{\Lambda(\mathbf{h})}$ of the rl-segment ideal corresponding to $\mathbf{h}$ is the minimum element of $\mathcal{B}^n_{\mathbf{h}}/\sim$.

f) If $n = 3$, by Remark 4.9 b) the class $\overline{\pounds(\mathbf{h})}$ of the generalized-rl-segment ideal corresponding to $\mathbf{h}$ is the minimum element of $\mathcal{B}^n_{\mathbf{h}}/\sim$, whatsoever $\mathbf{h}$ could be.

Altogether we have that $\mathcal{B}^3_{\mathbf{h}}/\sim$, endowed with $\prec$, is a poset with universal extremes $\mathbf{0} = \overline{\pounds(h)}$ and $\mathbf{1} = \overline{\mathcal{L}(h)}$.

Example 5.3. Let $\mathbf{h} = (1,4,10,20,35,46,59)$. Then $\Delta(\mathbf{h}) = (1,3,6,10,15,11,13)$, $in.deg.(\mathbf{h}) = 5$, as $\Delta(\mathbf{h})_6 = 13 > 0$, does not exist rl-segment ideal. By Lemma 4.4 $\tilde{m}_{2,6}(\mathfrak{b}) \geq 11$ and $\tilde{m}_{2,7}(\mathfrak{b}) \geq 13$ for all $\mathfrak{b} \in \mathcal{B}^4_{\mathbf{h}}$. We have to look for $\lambda_{1,6}(\mathcal{N}(\mathfrak{b})) = 13$, by Proposition 3.6 and Remark 4.3 e), $\lambda_{1,5}(\mathcal{N}(\mathfrak{b})) + \lambda_{2,5}(\mathcal{N}(\mathfrak{b})) - \#(G(\mathfrak{b})_6) = 13$ so the minimal values of $\lambda_{1,5}(\mathcal{N}(\mathfrak{b}))$ and $\lambda_{2,5}(\mathcal{N}(\mathfrak{b}))$ are for $G(\mathfrak{b})_6 = \emptyset$. We have for this only the following possibilities:

$\lambda_{1,5}(\mathcal{N}(\mathfrak{b})) = 11$ and $\lambda_{2,5}(\mathcal{N}(\mathfrak{b})) = 2$; or
$\lambda_{1,5}(\mathcal{N}(\mathfrak{b})) = 12$ and $\lambda_{2,5}(\mathcal{N}(\mathfrak{b})) = 1$; or
$\lambda_{1,5}(\mathcal{N}(\mathfrak{b})) = 13$ and $\lambda_{2,5}(\mathcal{N}(\mathfrak{b})) = 0$, which give ordinately:

$$\begin{aligned}\mathfrak{b}_1 = (&T^5, ZT^4, Z^2T^3, Z^3T^2, YT^4, YZT^3, YZ^2T^2, Y^2T^3, Y^2ZT^2, Y^3T^2, Z^6T, Z^7,\\ &YZ^5T, YZ^6, Y^2Z^4T, Y^2Z^5, Y^3Z^3T, Y^3Z^4, Y^4Z^2T, Y^4Z^3, Y^5ZT, Y^5Z^2, Y^6T,\\ &Y^6Z, Y^7, XZ^5T, XZ^6, XYZ^4T, XYZ^5, XY^2Z^3T, XY^2XZ^4, XY^3Z^2T,\\ &XY^3Z^3, XY^4ZT, XY^4Z^2, XY^5T, XY^5Z, XY^6, X^2Z^4T, X^2Z^5, X^2YZ^3T,\\ &X^2YZ^4, X^2Y^2Z^2T, X^2Y^2Z^3, X^2Y^3ZT, X^2Y^3Z^2, X^2Y^4T, X^2Y^4Z, X^2Y^5) +\\ &X^3(Y,Z,T)^4 + X^4(Y,Z,T)^3 + X^5(Y,Z,T)^2 + X^6(Y,Z,T) + (X^7),\end{aligned}$$

$$\tilde{\mathcal{M}}(\mathfrak{b}_1) = \begin{pmatrix} 1 & 4 & 10 & 20 & 35 & 46 & 59 \\ 1 & 3 & 6 & 10 & 15 & 11 & 13 \\ 1 & 2 & 3 & 4 & 5 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix};$$

$$\begin{aligned}\mathfrak{b}_2 = (&T^5, ZT^4, Z^2T^3, Z^3T^2, Z^4T, YT^4, YZT^3, YZ^2T^2, Y^2T^3, XT^4, Z^7, YZ^6, Y^2Z^5,\\ &Y^3Z^3T, Y^3Z^4, Y^4ZT^2, Y^4Z^2T, Y^4Z^3, Y^5T^2, Y^5ZT, Y^5Z^2, Y^6T, Y^6Z, Y^7,\\ &XZ^6, XY^2Z^3T, XYZ^5, XY^2Z^4, XY^3ZT^2, XY^3Z^2T, XY^3Z^3, XY^4T^2,\\ &XY^4ZT, XY^4Z^2, XY^5T, XY^5Z, XY^6) + X^2(Z^5, YZ^3T, YZ^4, Y^2ZT^2,\\ &Y^2Z^2T, Y^2Z^3, Y^3T^2, Y^3ZT, Y^3Z^2, Y^4T, Y^4Z, Y^5) + X^3[(Y,Z,T)^4 \setminus \{T^4\}] +\end{aligned}$$

$X^4(Y,Z,T)^3 + X^5(Y,Z,T)^2 + X^6(Y,Z,T) + (X^7)$,

$$\tilde{\mathcal{M}}(\mathfrak{b}_2) = \begin{pmatrix} 1 & 4 & 10 & 20 & 35 & 46 & 59 \\ 1 & 3 & 6 & 10 & 15 & 12 & 13 \\ 1 & 2 & 3 & 4 & 5 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix};$$

$\mathfrak{b}_3 = (T^5, ZT^4, Z^2T^3, Z^3T^2, Z^4T, YT^4, YZT^3, YZ^2T^2, YZ^3T, XT^4, Z^7, YZ^6, Y^2Z^5, Y^3Z^4, Y^4T^3, Y^4ZT^2, Y^4Z^2T, Y^4Z^3, Y^5T^2, Y^5ZT, Y^5Z^2, Y^6T, Y^6Z, Y^7, XZ^6, XYZ^5, XY^2Z^4, XY^3T^3, XY^3ZT^2, XY^3Z^2T, XY^3Z^3, XY^4T^2, XY^4ZT, XY^4Z^2, XY^5T, XY^5Z, XY^6) + X^2(Z^5, YZ^4, Y^2T^3, Y^2ZT^2, Y^2Z^2T, Y^2Z^3, Y^3T^2, Y^3ZT, Y^3Z^2, Y^4T, Y^4Z, Y^5) + X^3[(Y,Z,T)^4 \setminus \{T^4\}] + X^4(Y,Z,T)^3 + X^5(Y,Z,T)^2 + X^6(Y,Z,T) + (X^7)$, with

$$\tilde{\mathcal{M}}(\mathfrak{b}_3) = \tilde{\mathcal{M}}(\mathfrak{b}_2);$$

$\mathfrak{b}_4 = (T^5, ZT^4, Z^2T^3, Z^3T^2, Z^4T, Z^5, YT^4, YZT^3, XT^4, XZT^3, Y^3Z^2T^2, Y^3Z^3T, Y^3Z^4, Y^3Z^4, Y^4T^3, Y^4ZT^2, Y^4Z^2T, Y^4Z^3, Y^5T^2, Y^5ZT, Y^5Z^2, Y^6T, Y^6Z, Y^7) + X(Y^2Z^2T^2, Y^2Z^3T, Y^2Z^4, Y^3T^3, Y^3ZT^2, Y^3Z^2T, Y^3Z^3, Y^4T^2, Y^4ZT, Y^4Z^2, Y^5T, Y^5Z, Y^6) + X^2(YZ^2T^2, YZ^3T, YZ^4, Y^2T^3, Y^2ZT^2, Y^2Z^2T, Y^2Z^3, Y^3T^2, Y^3ZT, Y^3Z^2, Y^4T, Y^4Z, Y^5) + X^3[(Y,Z,T)^4 \setminus \{T^4, ZT^3\}] + X^4(Y,Z,T)^3 + X^5(Y,Z,T)^2 + X^6(Y,Z,T) + (X^7)$,

$$\tilde{\mathcal{M}}(\mathfrak{b}_4) = \begin{pmatrix} 1 & 4 & 10 & 20 & 35 & 46 & 59 \\ 1 & 3 & 6 & 10 & 15 & 13 & 13 \\ 1 & 2 & 3 & 4 & 5 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

The above Example 5.3 shows that if $n \geq 4$, and $\mathbf{h}$ is such that $\Delta(\mathbf{h})_j \not\leq 0$, for some $j \geq d+1$,then $\mathcal{B}^n_{\mathbf{h}}/\sim$ does not have minimum element but several minimal ones. Recently, using different methods, C. A. Francisco in [5] proved a similar result for the partial ordering on $\mathcal{B}^n_{\mathbf{h}}$ given by the graded Betti numbers.

We prove now that for each O-sequence $\mathbf{h} = (1, 3, \binom{4}{2}, \dots, h_d, \dots, h_s) \in \mathbb{N}^{*(s+1)}$, the $\prec$ above endows $\mathcal{B}^3_{\mathbf{h}}/\sim$ with a lattice structure.

Lemma 5.4. *Let $\mu_0, \dots, \mu_{s-d} \in \mathbb{N}$ satisfy $\lambda_{1,d+\ell}(\mathcal{N}(\mathcal{L}(\mathbf{h}))) \geq \mu_\ell \geq \lambda_{1,d+\ell}(\mathcal{N}(\pounds(\mathbf{h})))$, as ℓ varies from 0 to $s-d$, then there exists an ideal $\mathfrak{d} \in \mathcal{B}^3_{\mathbf{h}}$ with $\lambda_{1,d+\ell}(\mathcal{N}(\mathfrak{d})) = \mu_\ell$.*

Proof. We argue as in the construction of $\pounds(\mathbf{h})$:

- for all $0 \leq j \leq d-1$: $\Delta_j = \mathbf{T}_j$,
- for all $0 \leq \ell \leq s-d$: $\Delta_{d+\ell} = R_{\mu_\ell, d+\ell} \sqcup \{t_1, \dots, t_{c(\ell)}\}$ with $c(\ell) := h_{d+\ell} - \frac{\mu_\ell(2(d+\ell)-\mu_\ell+3)}{2}$ and $t_1 < \dots < t_{c(\ell)}$ smallest terms of $(\Delta_{d+\ell-1})_{(1)} \setminus R_{\mu_\ell, d+\ell}$,
- for all $r \in \mathbb{N}^*$: $\Delta_{s+r} = \emptyset$, $\Delta := \bigsqcup_{j\in\mathbb{N}} \Delta_j \subset \mathbf{T}$ is a Borel subset which is an order ideal. The wanted $\mathfrak{d} \in \mathcal{B}^3_{\mathbf{h}}$ is the monomial ideal having Δ as sous-éscalier.

Remark 5.5. a) The $(d+\ell)$-degree terms of $G(\mathfrak{d})$ ($\mathfrak{d}$ defined in Lemma 5.4 and $0 \le \ell \le s-d$), are ordinately greater or equal than those of any $\mathfrak{b} \in \mathcal{B}^3_{\mathbf{h}}$ with $\lambda_1(\mathcal{N}(\mathfrak{b})_{d+\ell}) = \mu_\ell$. In fact $G(\mathfrak{d})_d$ consists of the $(\frac{(d+2)(d+1)}{2} - h_d)$ biggest elements of $\mathbf{T}_d \setminus R_{\mu_0,d}$, $G(\mathfrak{d})_{d+\ell}$ consists of the greatest $h_{d+\ell}+\mu_\ell-h_{d+\ell+1}$ elements of $(\Delta_{d+\ell})_{(1)} \setminus R_{\mu_{\ell+1},d+\ell}$ for all $1 \le \ell \le s-d$, and $G(\mathfrak{d})_{s+1} = (\Delta_s)_{(1)}$.
b) Lemma 5.4 allows to determine all possible ses-matrix of ideals in $\mathcal{B}^3_{\mathbf{h}}$. Indeed, the second row in the matrix of Remark 4.12 b) must be of the form:

$$(1 \ \ 2 \ \ 3 \ \ \cdots \ \ d \ \ \mu_0 \ \ \cdots \ \ \mu_{s-d})$$

for all $\mu_0 \ge \cdots \ge \mu_{s-d} \in \mathbb{N}$ such that $a_\ell \le \mu_\ell \le (h_{d+\ell}\{d+\ell\})^{-1}$.

Theorem 5.6. *The poset $\mathcal{B}^3_{\mathbf{h}}/\sim$ has a lattice structure.*

Proof. Given $\bar{\mathfrak{b}}, \bar{\mathfrak{b}'} \in \mathcal{B}^3_{\mathbf{h}}/\sim$, let $\mathfrak{b} \in \bar{\mathfrak{b}}, \mathfrak{b}' \in \bar{\mathfrak{b}'}$, and $\mu_0 \ge \cdots \ge \mu_{s-d}$ (resp. $\mu'_0 \ge \cdots \ge \mu'_{s-d}$) be the $(d-s+1)$-tuple defined, for ℓ ranging from 0 to $s-d$, by: $\mu_\ell := \min\{\lambda_1(\mathcal{N}(\mathfrak{b})_{d+\ell}), \lambda_1(\mathcal{N}(\mathfrak{b}')_{d+\ell})\}$ (resp. $\mu'_\ell := \max\{\lambda_1(\mathcal{N}(\mathfrak{b})_{d+\ell}), \lambda_1(\mathcal{N}(\mathfrak{b}')_{d+\ell})\}$).
We set: $\bar{\mathfrak{b}} \wedge \bar{\mathfrak{b}'} := \bar{\mathfrak{d}}$, $\bar{\mathfrak{b}} \vee \bar{\mathfrak{b}'} := \bar{\mathfrak{d}'}$, with $\mathfrak{d} \in \mathcal{B}^3_{\mathbf{h}}$ (resp. $\mathfrak{d}' \in \mathcal{B}^3_{\mathbf{h}}$), the ideal constructed, as in Lemma 5.4, from the above $(d-s+1)$-tuple $\mu_0, \ldots, \mu_{s-d}$ (resp. $\mu'_0, \ldots, \mu'_{s-d}$).
Note that if $\bar{\mathfrak{b}} \preceq \bar{\mathfrak{b}'}$, then, by construction, $\bar{\mathfrak{b}} \wedge \bar{\mathfrak{b}'} := \bar{\mathfrak{b}}$ and $\bar{\mathfrak{b}} \vee \bar{\mathfrak{b}'} := \bar{\mathfrak{b}'}$.
For all $\mathfrak{b}, \mathfrak{b}' \in \mathcal{B}^3_{\mathbf{h}}$ we have $\bar{\mathfrak{b}} \wedge \bar{\mathfrak{b}'} \preceq \bar{\mathfrak{b}}, \bar{\mathfrak{b}'}$, and $\bar{\mathfrak{b}} \vee \bar{\mathfrak{b}'} \succeq \bar{\mathfrak{b}}, \bar{\mathfrak{b}'}$. Moreover, $\bar{\mathfrak{a}} \preceq \bar{\mathfrak{b}} \wedge \bar{\mathfrak{b}'}$ and $\bar{\mathfrak{a}'} \succeq \bar{\mathfrak{b}} \vee \bar{\mathfrak{b}'}$ for all $\mathfrak{a}, \mathfrak{a}' \in \mathcal{B}_h$ with $\bar{\mathfrak{a}} \preceq \bar{\mathfrak{b}}, \bar{\mathfrak{b}'}$ and $\bar{\mathfrak{a}'} \succeq \bar{\mathfrak{b}}, \bar{\mathfrak{b}'}$. All this proves that we have the claimed lattice structure.

Example 5.7. If $h = (1,3,6,10,15,16,11)$, then $\#\mathcal{B}^3_{\mathbf{h}} = 20$ and $\mathcal{B}_h/\sim$ consists of six classes $\overline{\mathfrak{b}_1} = \overline{\Lambda(h)}, \ldots, \overline{\mathfrak{b}_6} = \overline{\mathcal{L}(h)}$. The poset structure is described in next picture where for all $1 \le i \le 6$, i represents the class $\overline{\mathfrak{b}_i}$ and an oriented arrow from i to j ($i \ne j$) indicates that $\overline{\mathfrak{b}_i} \succ \overline{\mathfrak{b}_j}$

$$\begin{array}{ccccc} 2 & \leftarrow & 4 & \leftarrow & 6 \\ \downarrow & & \downarrow & & \downarrow \\ 1 & \leftarrow & 3 & \leftarrow & 5. \end{array}$$

Moreover,

$$\begin{array}{ll} \overline{\mathfrak{b}_2} \wedge \overline{\mathfrak{b}_3} = \overline{\mathfrak{b}_1} & \overline{\mathfrak{b}_2} \vee \overline{\mathfrak{b}_3} = \overline{\mathfrak{b}_4} \\ \overline{\mathfrak{b}_2} \wedge \overline{\mathfrak{b}_5} = \overline{\mathfrak{b}_1} & \overline{\mathfrak{b}_2} \vee \overline{\mathfrak{b}_5} = \overline{\mathfrak{b}_6} \\ \overline{\mathfrak{b}_4} \wedge \overline{\mathfrak{b}_5} = \overline{\mathfrak{b}_3} & \overline{\mathfrak{b}_4} \vee \overline{\mathfrak{b}_5} = \overline{\mathfrak{b}_6} \end{array}$$

Finally notice that, according to Remark 4.14 c), for all $\mathfrak{a} \in \overline{\mathfrak{b}_2} \cup \overline{\mathfrak{b}_3}$ we have $\beta_0 = 23$, $\beta_1 = 39$, $\beta_2 = 17$. Of course $\beta_{i,j+i}$'s distinguish the elements of $\overline{\mathfrak{b}_2}$ from those of $\overline{\mathfrak{b}_3}$.

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