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A Broken Circuit Ring

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Abstract. Given a matroid M represented by a linear subspace $L \subset \mathbb{C}^n$ (equivalently by an arrangement of n hyperplanes in L), we define a graded ring $R(L)$ which degenerates to the Stanley-Reisner ring of the broken circuit complex for any choice of ordering of the ground set. In particular, $R(L)$ is Cohen-Macaulay, and may be used to compute the h-vector of the broken circuit complex of M. We give a geometric interpretation of $\operatorname{Spec} R(L)$, as well as a stratification indexed by the flats of M.

1. Introduction

Consider a vector space with basis $\mathbb{C}^n = \mathbb{C}\{e_1, \ldots, e_n\}$, and its dual $(\mathbb{C}^n)^\vee = \mathbb{C}\{x_1, \ldots, x_n\}$. Let $L \subset \mathbb{C}^n$ be a linear subspace of dimension d. We define a matroid $M(L)$ on the ground set $[n] := \{1, \ldots, n\}$ by declaring $I \subset [n]$ to be independent if and only if the composition $\mathbb{C}\{e_i \mid i \in I\} \hookrightarrow (\mathbb{C}^n)^\vee \to L^\vee$ is injective. Recall that a minimal dependent subset $C \subset [n]$ is called a *circuit*; in this case there exist scalars $\{a_c \mid c \in C\}$, unique up to scaling, such that $\sum_C a_c x_c$ vanishes on L. Conversely, the support of every linear form that vanishes on L contains a circuit.

The central object of study in this paper will be the ring $R(L)$ generated by the inverses of the restrictions of the linear functionals $\{x_1, \ldots, x_n\}$ to L. More formally, let

$$\mathbb{C}[x, y] := \mathbb{C}[x_1, y_1, \ldots, x_n, y_n]/\langle x_i y_i - 1\rangle,$$

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and let $\mathbb{C}[x]$ and $\mathbb{C}[y]$ denote the polynomial subrings generated by the x and y variables, respectively. Let $\mathbb{C}[L]$ denote the ring of functions on L, which is a quotient of $\mathbb{C}[x]$ by the ideal generated by the linear forms $\left\{ \sum_C a_c x_c \mid C \text{ a circuit} \right\}$. We now set

$$R(L) := \Big(\mathbb{C}[L] \otimes_{\mathbb{C}[x]} \mathbb{C}[x,y]\Big) \cap \mathbb{C}[y].$$

Geometrically, $\operatorname{Spec} R(L)$ is a subscheme of $\operatorname{Spec} \mathbb{C}[y]$, which we will identify with $(\mathbb{C}^n)^\vee$. Using the isomorphism between $\mathbb{C}^n$ and $(\mathbb{C}^n)^\vee$ provided by the dual bases, $\operatorname{Spec} R(L)$ may be obtained by intersecting L with the torus $(\mathbb{C}^*)^n$, applying the involution $t \mapsto t^{-1}$ on the torus, and taking the closure inside of $\mathbb{C}^n$. If C is any circuit of $M(L)$ with $\sum_{c \in C} a_c x_c$ vanishing on L, then we have the relation

$$f_C := \sum_{c \in C} a_c \prod_{c' \in C \setminus \{c\}} y_{c'} = 0 \quad \text{in} \quad R(L).$$

Our main result (Theorem 4) will be that the elements $\{f_C \mid C \text{ a circuit}\}$ are a universal Gröbner basis for $R(L)$, hence this ring degenerates to the Stanley-Reisner ring of the broken circuit complex of $M(L)$ for any choice of ordering of the ground set $[n]$. It follows that $R(L)$ is a Cohen-Macaulay ring of dimension d, and that the quotient of $R(\mathcal{A})$ by a minimal linear system of parameters has Hilbert series equal to the h-polynomial of the broken circuit complex. In Proposition 7 we identify a natural choice of linear parameters for $R(L)$.

The Hilbert series of $R(L)$ has already been computed by Terao [8], using different methods. The main novelty of our paper lies in our geometric approach, and our interpretation of $R(L)$ as a deformation of another well-known ring. The ring $R(L)$ also appears as a cohomology ring in [5], and as the homogeneous coordinate ring of a projective variety in [3, 3.1].

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2. The broken circuit complex

Choose an ordering w of $[n]$. We define a *broken circuit of* $M(L)$ *with respect to* w to be a set of the form $C \setminus \{c\}$, where C is a circuit of $M(L)$ and c the w-minimal element of C. We define the *broken circuit complex* $\mathrm{bc}_w(L)$ to the simplicial complex on the ground set $[n]$ whose faces are those subsets of $[n]$ that do not contain any broken circuit. Note that all of the singletons will be faces of $\mathrm{bc}_w(L)$ if and only if $M(L)$ has no parallel pairs, and the empty set will be a face if and only if $M(L)$ has no loops. We will not need to assume that either of these conditions holds.

Consider the f-vector $(f_0, \ldots, f_d)$ of $\mathrm{bc}_w(L)$, where f_i is the number of faces of order i. Then f_i is equal to the rank of $H^i(A(L))$, where $A(L) = L \setminus \bigcup_{i=1}^n \{x_i = 0\}$ is the complement of the restriction of the coordinate arrangement from $\mathbb{C}^n$ to L (see for example [4]). In particular, the f-vector of $\mathrm{bc}_w(L)$ is independent of the ordering w. The h-vector $(h_0, \ldots, h_{d-1})$ of $\mathrm{bc}_w(L)$ is defined by the formula $\sum h_i z^i = \sum f_i z^i (1-z)^{d-i}$.

The *Stanley-Reisner ring* $\mathrm{SR}(\Delta)$ of a simplicial complex Δ on the ground set $[n]$ is defined to be the quotient of $\mathbb{C}[e_1, \dots, e_n]$ by the ideal generated by the monomials $\prod_{i \in N} e_i$, where N ranges over the nonfaces of Δ. The complex $\mathrm{bc}_w(L)$ is shellable of dimension $d-1$ [1], which implies that $\operatorname{Spec} \mathrm{SR}(\mathrm{bc}_w(L))$ is Cohen-Macaulay and pure of dimension d. Let $\mathbb{C}[L^\vee]$ denote the ring of functions on $L^\vee = (\mathbb{C}^n)^\vee / L^\perp$, which we may think of as the symmetric algebra on L. The inclusion of L into $\mathbb{C}^n$ induces an inclusion of $\mathbb{C}[L^\vee]$ into $\mathbb{C}[e_1, \dots, e_n]$, which makes $\mathrm{SR}(\mathrm{bc}_w(L))$ into an $\mathbb{C}[L^\vee]$-algebra. Let $\mathrm{SR}_0(\mathrm{bc}_w(L)) = \mathrm{SR}(\mathrm{bc}_w(L)) \otimes_{\mathbb{C}[L^\vee]} \mathbb{C}$, where each linear function on $L^\vee$ acts on $\mathbb{C}$ by 0. The following proposition asserts that L constitutes a linear system of parameters (l.s.o.p.) for $\mathrm{SR}(\mathrm{bc}_w(L))$.

Proposition 1. *The Stanley-Reisner ring* $\mathrm{SR}(\mathrm{bc}_w(L))$ *is a free* $\mathbb{C}[L^\vee]$*-module, and the ring* $\mathrm{SR}_0(\mathrm{bc}_w(L))$ *is zero-dimensional with Hilbert series* $\sum h_i z^i$*.*

Proof. By [6, 5.9], it is enough to prove that $\mathrm{SR}_0(\mathrm{bc}_w(L))$ is a zero-dimensional ring. Let π denote the composition $\operatorname{Spec} \mathrm{SR}(\mathrm{bc}_w(L)) \hookrightarrow (\mathbb{C}^n)^\vee \twoheadrightarrow L^\vee$. The variety $\operatorname{Spec} \mathrm{SR}(\mathrm{bc}_w(L))$ is a union of coordinate subspaces, one for each face of $\mathrm{bc}_w(L)$. Let F be such a face, with vertices $(v_1, \dots, v_{|F|})$. The broken circuit complex is a subcomplex of the matroid complex, hence $(v_1, \dots, v_{|F|})$ is an independent set, which implies that π maps the corresponding coordinate subspace injectively to $L^\vee$. Thus $\pi^{-1}(0) = \operatorname{Spec} \mathrm{SR}_0(\mathrm{bc}_w(L))$ is supported at the origin, and we are done. $\square$

3. A degeneration of $R(L)$

In this section we show that $R(L)$ degenerates flatly to the Stanley-Reisner ring $\mathrm{SR}(\mathrm{bc}_w(L))$ for any choice of w.

Lemma 2. *The spaces* $\operatorname{Spec} R(L)$ *and* $\operatorname{Spec} \mathrm{SR}(\mathrm{bc}_w(L))$ *are both pure d-dimensional homogeneous varieties of degree* $t_{M(L)}(1,0)$*, where* $t_M(w,z)$ *is the Tutte polynomial of* M*.*

Proof. The broken circuit complex is pure of dimension $d-1$, hence $\operatorname{Spec} \mathrm{SR}(\mathrm{bc}_w(L))$ is union of d-dimensional coordinate subspaces of $(\mathbb{C}^n)^\vee$. Its degree is the number of facets of $\mathrm{bc}_w(L)$, which is equal to $\sum h_i = t_{M(L)}(1,0)$ [1].

The variety $\operatorname{Spec} R(L)$ is equal to the closure inside of $(\mathbb{C}^n)^\vee \cong \mathbb{C}^n$ of $L \cap (\mathbb{C}^*)^n$, and is therefore d-dimensional. We will now show that $\deg \operatorname{Spec} R(L)$ obeys the same recurrence as $t_{M(L)}(1,0)$. First, suppose that $i \in [n]$ is a loop of $M(L)$. Then L lies in a coordinate subspace of $\mathbb{C}^n$, $L \cap (\mathbb{C}^*)^n$ is empty, and $\operatorname{Spec} R(L)$ is thus empty and has degree 0. In this case, we also have $t_{M(L)}(1,0) = 0$. Next, suppose that i is a coloop of $M(L)$. Then L is invariant under translation by e_i, and $\operatorname{Spec} R(L)$ is similarly invariant under translation by x_i. Write L/i for the quotient of L by this translation, so that $\operatorname{Spec} R(L) = \operatorname{Spec} R(L/i) \times \mathbb{C}$ and $\deg \operatorname{Spec} R(L) = \deg \operatorname{Spec} R(L/i)$. It is clear that $M(L/i) = M(L)/i$, and indeed $t_M(1,0) = t_{M/i}(1,0)$ when i is a coloop.

Now consider the case where i is neither a loop nor a coloop, hence we have

$$t_{M(L)}(1,0) = t_{M(L)/i}(1,0) + t_{M(L)\setminus i}(1,0).$$

In this case, we may apply the following theorem.

Theorem 3. [2, 2.2] *Let X be a homogeneous irreducible subvariety of $\mathbb{C}^n = H \oplus \ell$, with H a hyperplane and ℓ a line such that X is not invariant under translation in the ℓ direction. Let X_1 be the closure of the projection along ℓ of X to H, and let X_2 be the flat limit in $H \times \mathbb{P}^1$ of $X \cap (H \times \{t\})$ as $t \to \infty$. Then X has a flat degeneration to a scheme supported on $(X_1 \times \{0\}) \cup (X_2 \times \ell)$. In particular, $\deg X \geq \deg X_1 + \deg X_2$, with equality if the projection $X \to X_1$ is generically one to one.*

Let $X = \operatorname{Spec} R(L)$, $\ell = \mathbb{C}x_i$, and $H = \mathbb{C}\{x_j \mid j \neq i\}$. Then in the notation of Theorem 3, we have $X_1 = \operatorname{Spec} R(L \setminus i)$, where $L \setminus i$ is the projection of L onto H, and $X_2 = \operatorname{Spec} R(L/i)$. The projection of $\operatorname{Spec} R(L)$ onto H is one to one because the corresponding projection of L in the x_i direction is one to one. Thus the degree of $\operatorname{Spec} R(L)$ is additive. □

We are now ready to prove our main theorem, which asserts that $R(L)$ degenerates flatly to $\mathrm{SR}(\mathrm{bc}_w(L))$ for any choice of w.

Theorem 4. *The set $\{f_C \mid C$ a circuit of $M(L)\}$ is a universal Gröbner basis for $R(L)$. Given any ordering w of $[n]$, with the induced term order on $\mathbb{C}[y]$, we have $\operatorname{In}_w R(L) = \mathrm{SR}(\mathrm{bc}_w(L))$.*

Proof. Suppose given an ordering w of $[n]$ and a circuit C of $M(L)$. Let c_0 denote the w-minimal element of C, so that $\prod_{c' \in C \setminus \{c_0\}} y_{c'}$ is the leading term of f_C with respect to w. Every monomial of this form vanishes in $\operatorname{In}_w R(L)$, hence we deduce that $\operatorname{Spec} \operatorname{In}_w(R(L))$ is a subscheme of $\operatorname{Spec} \mathrm{SR}(\mathrm{bc}_w(L))$. However, Lemma 2 tells us that these two schemes have the same dimension and degree, and $\operatorname{Spec} \mathrm{SR}(\mathrm{bc}_w(L))$ is reduced. Thus they are equal.

Let R be the quotient ring of $\mathbb{C}[y]$ generated by the polynomials $\{f_C\}$. It is clear that $\operatorname{In}_w \operatorname{Spec}(R(L)) \subseteq \operatorname{In}_w \operatorname{Spec} R \subseteq \operatorname{Spec} \mathrm{SR}(\mathrm{bc}_w(L))$. Since the two ends of this chain are equal, we have $\operatorname{In}_w R = \operatorname{In}_w R(L)$, and thus R and $R(L)$ have the same Hilbert series. As $R(L)$ is a quotient ring of R, $R = R(L)$. □

4. A stratification of Spec $R(L)$

Let I be a subset of $[n]$. The *rank* of I is defined to be the cardinality of the largest independent subset of I. If any strict superset of I has strictly greater rank, then I is called a *flat* of $M(L)$. If I is a flat, let $L_I \subset \mathbb{C}^I$ be the projection of L onto the coordinate subspace $\mathbb{C}^I \subset \mathbb{C}^n$, and let $L^I \subset \mathbb{C}^{I^c}$ be the intersection of L with the complimentary coordinate subspace $\mathbb{C}^{I^c}$. The matroid $M(L_I)$ is called the *localization of $M(L)$ at I*, while $M(L^I)$ is called the *deletion of I from $M(L)$*.

For any $I \subset [n]$, let $U_I = \{y \in (\mathbb{C}^n)^\vee \mid y_i = 0 \iff i \notin I\}$, and let $A_I = \operatorname{Spec} R(L) \cap U_I$.

Proposition 5. *The variety A_I is nonempty if and only if I is a flat of $M(L)$. If nonempty, A_I is isomorphic to $A(L_I) = L_I \setminus \bigcup_{i \in I}\{y_i = 0\}$.*

Proof. First suppose that I is not a flat of $M(L)$. Then there exists some circuit C of $M(L)$ and element $c_0 \in C$ such that $C \cap I = C \setminus \{c_0\}$. On one hand, the polynomial $f_C = \sum_{c \in C} a_c \prod_{c' \in C \setminus \{c\}} y_{c'}$ vanishes on A_I. On the other hand, f_C has a unique nonzero term $\prod_{c \in C \setminus \{c_0\}} y_{c'}$ on U_I, and therefore cannot vanish on this set. Hence A_I must be empty.

Now suppose that I is a flat. If $I = [n]$, then we are simply repeating the observation that $\operatorname{Spec} R(L) \cap (\mathbb{C}^*)^n \cong L \cap (\mathbb{C}^*)^n = A(L)$. In the general case, Theorem 4 tells us that $\operatorname{Spec} R(L)$ is cut out of $(\mathbb{C}^n)^\vee$ by the polynomials f_C, so we need to understand the restrictions of these polynomials to the set U_I. If C is not contained in I, then $C \setminus I$ has size at least 2, and therefore f_C vanishes on U_I. Thus we may restrict our attention to those circuits that are contained in I. Proposition 5 then follows from the fact that the circuits of $M(L_I)$ are precisely the circuits of $M(L)$ that are supported on I. □

Remark 6. The stratification of $\operatorname{Spec} R(L)$ given by Proposition 5 is analogous to the standard stratification of L into pieces isomorphic to $A(L^I)$, again ranging over all flats of $M(L)$.

The identification of e_i with y_i makes $R(L)$ into an algebra over $\mathbb{C}[L^\vee]$. We conclude by showing that, as in Proposition 1, L provides a natural linear system of parameters for $R(L)$.

Proposition 7. *The ring $R(L)$ is a free module over $\mathbb{C}[L^\vee]$. The zero dimensional quotient $R_0(L) := R(L) \otimes_{\mathbb{C}[L^\vee]} \mathbb{C}$ has Hilbert series $\sum h_i z^i$.*

Proof. The fact that $R(L)$ is Cohen-Macaulay follows from Theorem 4, which asserts that it is a deformation of the Cohen-Macaulay ring $\mathrm{SR}(\mathrm{bc}_w(L))$. Furthermore, Theorem 4 tells us that any quotient of $R(L)$ by d generic parameters has the same Hilbert series of $SR_0(bc_w(L))$. Therefore, as in Proposition 1, we let π denote the composition $\operatorname{Spec} R(L) \hookrightarrow (\mathbb{C}^n)^\vee \twoheadrightarrow L^\vee$, and observe that it is enough to show that $\pi^{-1}(0)$ is supported at the origin.

Let $I \subset [n]$ and suppose that $y = (y_1, \ldots, y_n) \in A_I = \operatorname{Spec} R(L) \cap U_I$. By Proposition 5, A_I is obtained from $A(L_I)$ by applying the inversion involution of $(\mathbb{C}^*)^I$, hence there exists $x_I \in A(L_I) \subset L_I$ such that $x_i = y_i^{-1}$ for all $i \in I$. Extend x_I to an element $x \in L$. Then $\langle x, y \rangle = \sum x_i y_i = |I|$, hence if y projects trivially onto $L^\vee$, we must have $I = \emptyset$. □

Remark 8. It is natural to ask the question of whether $R_0(L)$ has a g-element; that is an element $g \in R(L)$ in degree 1 such that the multiplication map $g^{r-2i} : R_0(L)_i \to R_0(L)_{r-i}$ is injective for all $i < r/2$, where r is the top nonzero degree of $R_0(L)$. This property is known to fail for the ring $SR_0(bc_w(L))$ [7, §5], but the inequalities that it would imply for the h-numbers are not known to be either true or false. In fact, the ring $R_0(L)$ fares no better than its degeneration; Swartz's counterexample to the g-theorem for $SR_0(bc_w(L))$ is also a counterexample for $R_0(L)$.

Remark 9. All of the constructions and results in this paper generalize to arbitrary fields with the exception of Proposition 7, which uses in an essential manner the fact that $\mathbb{C}$ has characteristic zero.

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