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Single Elements

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Abstract. In this paper we consider single elements in rings and near-rings. If R is a (near)ring, $x \in R$ is called single if $axb = 0 \Rightarrow ax = 0$ or $xb = 0$. In seeking rings in which each element is single, we are led to consider 0-simple rings, a class which lies between division rings and simple rings.

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Introduction

If R is a (not necessarily unital) ring, an element s is called *single* if whenever $asb = 0$ then $as = 0$ or $sb = 0$. The definition was first given by J. Erdos [2] who used it to obtain results in the representation theory of normed algebras. More recently the idea has been applied by Longstaff and Panaia to certain matrix algebras (see [9] and its bibliography) and they suggest it might be worthy of further study in other contexts. In seeking rings in which every element is single we are led to consider 0-simple rings, a class which lies between division rings and simple rings. In the final section we examine the situation in nearrings and obtain information about minimal N-subgroups of some centralizer nearrings.

1. Single elements in rings

We begin with a slightly more general definition. If I is a one-sided ideal in a ring R an element $x \in R$ will be called *I-single* if $axb \in I \Rightarrow ax \in I$ or $xb \in I$. We abbreviate "0-single" to just "single"[1].

[1] Of course one must be careful to distinguish between statements such as "R has a single idempotent" and "R has an idempotent which is single".

The proof of the following lemma is elementary

Lemma 1. *Let I be a left (resp. right) ideal and let x be I-single. Then*

(i) *rx (resp. xr) is I-single for all $r \in R$;*

(ii) *x is $(I : J)$-single for all right ideals J (resp. $(J : I)$-single for all left ideals J);*

(iii) *if $x^n \in I$, $n \geq 3$, $x^2 \in I$;*

(iv) *if x is a unit, I is completely prime.*

Moreover if I is two-sided, then

(v) *x is I-single iff $\overline{x} = x + I$ is single in $\overline{R} = R/I$.*

(vi) *If I is completely prime, every element of R is I-single.*

(vii) *If R is commutative or anti-commutative, x is I-single iff $(I : x)$ is a prime ideal of R.*

In view of (v) of Lemma 1, we will restrict our attention to 0-single elements in the rest of the paper. Also "single element" shall always mean "non-zero single element".

Example 1. In Z/mZ, k is single iff $\dfrac{m}{\gcd(m,k)}$ is prime. The same is true for any Euclidean domain.

Example 2. (See [2]) In a C^*-algebra A, an element x is single iff the image of x under some faithful representation of A is an operator of rank 1 (hence the name).

Example 3. (See [9]) In the ring of upper triangular $n \times n$ matrices over a field K with at least 3 elements, x is single iff x has rank 1. On the other hand, there are single elements of rank > 1 in other matrix rings. For example

$$x = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad a \neq 0, 1$$

is single in the ring of matrices over K of the form

$$\begin{bmatrix} * & 0 & * & * \\ 0 & * & * & * \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix}.$$

Example 4. There are rings in which every element is single. (See Section 2.) Trivially in a domain D every element is single, i.e., the non-zero single elements are precisely the non-zero-divisors. In contrast the single elements of $D \times D$ are precisely the zero-divisors. More generally if $\{R_i\}$ is a family of rings with zero right and left annihilator then $r = (r_i) \in R = \Pi R_i$ is single iff $\exists j$ such that $r_i = 0$ for all $i \neq j$ and r_j is single in R_j.

Example 5. There are rings with no single elements. Take, for instance, the algebra $C_0(X)$ of all functions vanishing at infinity on a locally compact Hausdorff space with no isolated points ([2]). See also Corollary 1 and Examples 7, 8 and 9 below.

Example 6. The inner automorphisms of the dihedral group D_4 of order 8 additively generate a commutative ring of order 16 ([8]). The single elements are precisely the seven zero-divisors since as seen from the multiplication table in [8] each zero-divisor z has the property that $xz = 0$ or z for all x. Thus if $xzy = 0$ and $xz \neq 0$, then $zy = 0$.

Let S be the set of single elements of R. Suppose $S \neq 0$ and define the *middle annihilator* of S by $m(S) = \{x | sxs = 0 \text{ for all } s \in S\}$. Let Prad R denote the prime radical, and N the set of nilpotent elements.

Proposition 1. *$m(S)$ is an ideal containing Prad R. If N is an ideal then $m(S) \supseteq N$.*

Proof. Clearly $m(S)$ is a subgroup of R. Since S is closed under left and right R-multiplication, $r \in m(S),\ s \in S$ and $x \in R \Rightarrow sxrsx = 0$ and since $s \in S$ $sxrs = 0$ or $sx = 0$; but the latter also implies $sxrs = 0$ so $xr \in m(S)$.

If $0 \neq x \in$ Prad R then $xs \in S \cap$ Prad R for all $s \in S$ so xs is (strongly) nilpotent and single. Therefore $(xs)^2 = 0$ by Lemma 1 (iii). It follows that $sxs = 0$ since $s \in S$.

If N is an ideal and $x \in N$ then $xs \in S \cap N$ and we proceed as above. This completes the proof.

One large class of rings in which N is an ideal is the class of 2-primal rings defined by the property $N =$ Prad R. Equivalently (see eg. [7(a), p. 195]) every minimal prime ideal is completely prime. This class includes rings with the "insertion of factors property" (or I.F.P) by which is meant $ab = 0 \Rightarrow arb = 0$ for all $r \in R$ (see eg. [17, Lemma 1.2 and Theorem 1.5]). In turn, the class of I.F.P. rings contains all completely reflexive rings, defined as those in which $ab = 0 \Rightarrow ba = 0$ (see [10]). Commutative rings and reduced rings are completely reflexive.

Proposition 2.

(a) *If R is completely reflexive, Prad $R = N \subseteq m(S) = m_1(S) = \{y \in R | sys_1 = 0$ for all $s, s_1 \in S\}$.*

(b) *If R is completely reflexive, x is single iff Ann x is completely prime.*

(c) *If R is reduced and $0 \neq A =$ Ann P for a prime ideal P then P is a minimal prime ideal and $P =$ Ann x for all $x \in A$. Hence every element of A is single.*

Proof.

(a) $m_1(S)$ is an ideal in any ring and $m_1(S) \subseteq m(S)$. We show equality holds if R is completely reflexive. If $x \in m(S)$ then $ss_1xss_1 = 0$ for all $s, s_1 \in S$. Since s_1 is single, either $ss_1 = 0$ or $s_1xss_1 = 0$ and since s is single $s_1xs = 0$ or $ss_1 = 0$. But $ss_1 = 0 \Rightarrow s_1s = 0 \Rightarrow s_1xs = 0$ by I.F.P.

(b) If Ann x is completely prime and $axb = 0$ then $xba = 0$, i.e., $ba \in \text{Ann } x$ so $b \in \text{Ann } x$ or $a \in \text{Ann } x$. Therefore $bx = 0$ or $ax = 0$ and x is single. Conversely if x is single and $ab \in \text{Ann } x$ then $xab = 0$ so $bxa = 0$ whence $bx = 0$ or $xa = 0$, i.e., $b \in \text{Ann } x$ or $a \in \text{Ann } x$.

(c) Suppose $0 \neq x \in A = \text{Ann } P$. If P is not minimal, $P \supset Q$ where Q is a minimal prime ideal. If $p \in P \backslash Q$ $xp = 0$ so because Q is completely prime $x \in Q \subset P$. But then $x^2 = 0$, a contradiction. Hence P is minimal and $P \subseteq \text{Ann } x$. But also $x \in A = \text{Ann } P \Rightarrow x \notin P$ (or else $x^2 = 0$) so $\text{Ann } x \subset P$. Therefore $\text{Ann } x = P$ is a minimal prime ideal and the result follows from part (a).

We recall that in a semiprime ring R minimal left ideals have the form Re, $e^2 = e$ and Re is minimal iff eR is a minimal right ideal iff eRe is a division ring (see eg. [7(a)]). Also an idempotent e is primitive if whenever $ef = fe = f = f^2 \neq 0$, then $e = f$. Equivalently e is primitive iff e is not the sum of two orthogonal idempotents iff eRe has no non-trivial idempotents. Clearly if Re is a minimal left ideal, e is primitive.

Theorem 1.

(a) *If e is idempotent and single then eRe is a domain so e is primitive.*

(b) *If M is a minimal left ideal of R such that $\forall s \neq 0$ in M $\exists r \in R$ with $rs \neq 0$ then $M = Rs$ and s is single.*

(c) *If $\text{Ann}^\ell R = 0$, s is single and $Rs \supseteq Re$ where $e^2 = e \neq 0$ then $Rs = Re$.*

(d) *If R is regular and s is single then Rs is a minimal left ideal and conversely every minimal left ideal has the form Rs, s single.*

Proof.

(a) If $ereese = erese = 0$, then since e is single, $ere = 0$ or $ese = 0$.

(b) Let $0 \neq s \in M$. Then $rs \neq 0$ for some r so $0 \neq Rs \subseteq M$ and M minimal $\Rightarrow Rs = M$. Suppose $asb = 0$ with $as \neq 0$. Again $Ras = M$ so $Mb = Rasb = 0$. Hence $sb = 0$.

(c) If $Rs \supseteq Re$ then $e = rs$ and for all $x \in R$ we have $e(x - ex) = 0$, i.e., $rs(x - ex) = 0$. Since s is single and $rs(= e) \neq 0$ we have $s(x - ex) = 0$, i.e., $s - se \in \text{Ann}^\ell x$. Since $\text{Ann}^\ell R = 0$, $s = se$ and so $Rs = Re$.

(d) We use the fact that in a regular ring, e is a primitive idempotent $\Rightarrow Re$ is a minimal left ideal ([7(a), Ex. 21.8]). Now if s is single, since R is regular, $Rs = Re$ for $e^2 = e$ and $e = rs$ is single. Therefore e is primitive and so $Rs = Re$ is a minimal left ideal.

Note that if R is semiprime or if $1 \in R$ then the hypothesis of (b) is true for all minimal left ideals M, and the condition $\text{Ann}^\ell R = 0$ of (c) also holds. More generally, the hypothesis of (b) holds if R is right D-regular ($x \in xR$ for all x) and that of (c) holds if R is left D-regular.

Example 3. (continued) The ring of 4×4 matrices described in Example 3 is not semiprime. The nilpotent element x generates a non-minimal nilpotent right (or left) ideal consisting of single elements. For if $x = \left(\begin{array}{c|c} 0 & A \\ \hline 0 & 0 \end{array}\right)$ where $A = \begin{pmatrix} 1 & 1 \\ 1 & a \end{pmatrix}$, $a \neq 0, 1$ and $r = \left(\begin{array}{c|c} D_1 & B \\ \hline 0 & D_2 \end{array}\right)$ where D_1 and D_2 are 2×2 diagonal matrices and B is an arbitrary 2×2 matrix over K then $xr = \left(\begin{array}{c|c} 0 & AD_2 \\ \hline 0 & 0 \end{array}\right)$. Hence $(xr)^2 = 0$, and if $D_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ then $xrR \subsetneqq xR$ or else $x = xrs \Rightarrow A = AD_2D_4$ for some diagonal matrix D_4. Thus $D_2D_4 = I$ which is impossible.

Corollary 1. *Let R be a von Neumann regular ring. If R has no minimal left ideals, R has no single elements. If R has minimal left ideals, then the socle of R is the set of finite sums of single elements. In that case the primitive idempotents are precisely the single ones.*

Example 7(a). Consider the semiprime ring $R = C(X)$ of continuous real-valued functions on a completely regular topological space X. As usual let $Z(f) = \{x | f(x) = 0\}$. Then ([6]) I is a minimal ideal iff $\exists y$ such that $\forall\, 0 \neq f \in I,\ Z(f) = X - \{y\}$. It follows that R has minimal ideals iff X has isolated points. Moreover R is regular iff X is a P-space [4, Ex. 4J]. If X is discrete, $C(X)$ is a regular ring with minimal ideals, i.e., with single elements. On the other hand, there is a P-space without isolated points [4, Ex. 13P] giving a regular ring $C(X)$ with no single elements.

Example 7(b). Let $R = \mathcal{P}(S)$ be the Boolean ring of subsets of a set S. Then R is regular and the single elements are precisely the singleton subsets (!). If S is infinite then $\mathcal{F} = \{T \subseteq S | T \text{ is finite}\}$ is an ideal and $\overline{R} = R/\mathcal{F}$ is regular (Boolean) with no primitive idempotents ([7b, Ex. 21.12]), hence no single elements.

Example 8. On the other hand there are rings which have no idempotent elements and no single elements. Let K be a field and consider the Zassenhaus Algebra A which is the K-algebra with basis $\{u_a | a \in \mathbb{R},\ 0 < a < 1\}$ and multiplication given by $u_a u_b = \begin{cases} u_{a+b} & \text{if } a + b < 1 \\ 0 & \text{if } a + b \geq 1 \end{cases}$. This is an idempotent nil ring. If $0 \neq x \in A$, let $x = \sum_{i=1}^{n} r_i u_{a_i}$ where $r_i \in K$ and $a_i < a_j$ for $i < j$. Let $b = \frac{1 - a_1}{2}$. Then $u_b x u_b = 0$ but $u_b x \neq 0 \neq x u_b$ since $a_1 + b = a_1 + \frac{1 - a_1}{2} < 1$.

Corollary 2. *If R is completely reducible, every single element s can be written $s = se$ where e is an idempotent which is also single, and R is additively generated by its single elements.*

We note that other rings can be generated by their single elements. For example as noted above in the ring of upper triangular $n \times n$ matrices (which is not semiprime) the single elements are precisely the elements of rank 1 so the standard basis provides a generating set of single elements.

Example 9. Let R be a left primitive ring with a faithful simple left module V. Then $K = \mathrm{End}(_RV)$ is a division ring and R is isomorphic to a dense subring of $E = \mathrm{End}(V_K)$. From [7(a), Ex. 11.17] s has rank 1 iff Rs is a minimal left ideal. Since R is semiprime, $Rs = Re$ where $e^2 = e$ is single by Theorem 1 (b). Hence s is single.

Let R be a biregular ring. That is, for every x in R there is a central idempotent e, such that $(x) = eR$. In such a ring the structure space $\mathcal{M}(R)$ of maximal (= prime) ideals is locally compact and totally disconnected.

Proposition 3. *If s is a single element in a biregular ring there is a unique maximal ideal M for which $s \notin M$.*

Proof. Let $s = er$ where $e^2 = e$ so $es = s = se$. Suppose $\exists P_1 \neq P_2 \in \mathcal{M}$ such that $s \notin P_i$. Since $\mathcal{M}(R)$ is totally disconnected there exist closed sets $C_1 \cap C_2 = \emptyset$ with $C_1 \cup C_2 = \mathcal{M}$, $P_i \in C_i$. Hence by [3, Cor. 1.7] $\exists c_i$ such that $c_1 \in \cap\{Q|Q \in C_1\}$, $c_1 - e \in P_2$, $c_2 \in \cap\{Q|Q \in C_2\}$, $c_2 - e \in P_1$. Then $c_1sc_2 = 0$ since $c_1sc_2 \in \cap\{M|M \in \mathcal{M}\}$. However if $c_1s = 0$ then $(c_1 - e)s = c_1s - es = -s \notin P_2$ but $c_1 - e \in P_2$ so we contradict the hypothesis that s is single.

Viewing $s \in R$ in terms of its sheaf representation $\hat{s} : \mathcal{M} \to \cup R/P$, we see that the term "single" is apt because $\hat{s}$ has a single non-zero image.

Proposition 4. *Let $\phi : R \to S$ be a ring homomorphism.*

(a) *If ϕ is a monomorphism and s is single in R then $\phi(s)$ is single in $\phi(R)$.*

(b) *If R is semiprime and Re is a minimal left ideal of R then $\phi(e)$ is single in $\phi(R)$. If, in addition, R is regular, then if s is single $\phi(s)$ is single in $\phi(R)$.*

Proof.

(a) If $\phi(s) = 0$ it is trivially single. If $\phi(x)\phi(s)\phi(y) = 0$ then $xsy \in \ker\phi = 0$ so $xs = o$ or $sy = 0$ whence $\phi(x)\phi(s) = 0$ or $\phi(s)\phi(y) = 0$.

(b) If $\phi(e) \neq 0$ and $\phi(x)\phi(e)\phi(y) = 0$ then $xey \in K = \ker\phi$. If $xey = 0$, proceed as in part (a). Otherwise, since Rey is a minimal left ideal and $0 \neq xey \in K \cap Rey$, we must have $Rey \subseteq K$. Then $ey = eey \in K$ so $\phi(ey) = 0$, i.e., $\phi(e)\ \phi(y) = 0$ as claimed. If further R is regular and s is single in R, then $Rs = Re$, $e^2 = e$ and $\phi(e)$ is single. Since $s = se$, $\phi(s) = \phi(s)\ \phi(e)$ is also single.

2. Generalized domains and 0-simple rings

Definition. *R will be called a generalized domain if every element of R is single. Clearly a domain is a generalized domain and if $1 \in R$ the converse is true. Let $\mathcal{G}$ denote the class of generalized domains. A ring with trivial multiplication ($R^2 = 0$) is in $\mathcal{G}$.*

Proposition 5. *If R is a generalized domain, then either $Ann^\ell R \neq 0$ or $Ann^r R \neq 0$ or R is prime.*

Proof. Suppose $\mathrm{Ann}^\ell R = 0 = \mathrm{Ann}^r R$. If $xRy = 0$ then for all $r \in R$ either $xr = 0$ or $ry = 0$. Thus $R = \mathrm{Ann}^r x \bigcup \mathrm{Ann}^\ell y$. However a group cannot be the union of two proper subgroups so either $\mathrm{Ann}^r x = R$ or $\mathrm{Ann}^\ell y = R$, i.e., either $x \in \mathrm{Ann}^\ell R = 0$ or $y \in \mathrm{Ann}^r R = 0$.

Theorem 2.
(a) *$\mathcal{G}$ is closed under subrings.*
(b) *$\mathcal{G}$ is closed under ultraproducts.*
(c) *$\mathcal{G}$ is not closed under factor rings nor under formation of prime factor rings.*

Proof.
(a) is trivial.
(b) Let $\{A_i | i \in I\} \subseteq \mathcal{G}$ and let U be an ultrafilter on I. Consider $R = \Pi A_i/U$. If $(a_i)(s_i)(b_i) \equiv 0 \bmod U$ then $J = \{i \in I | a_i s_i b_i = 0\} \in U$. Now $J = \{i \in I | a_i s_i = 0\} \cup \{i \in I | a_i s_i \neq 0$ and $s_k b_i = 0\} = L_1 \dot\cup L_2$ say. Since U is an ultrafilter $L_1 \in U$ or $L_2 \in U$. But $L_2 \subseteq L_3 = \{i \in I | s_i b_i = 0\}$ and U is a filter so if $L_2 \in U$ then $L_3 \in U$. Thus $(a_i)(s_i) \equiv 0 \bmod U$ or $(s_i)(b_i) \equiv 0 \bmod U$ as claimed.
(c) For the first part simply consider Z/mZ where m is not prime (Example 1). Secondly let $A = A_1(k)$ be the Weyl algebra over a field k of characteristic 0. Then A is a domain so $A \in \mathcal{G}$. A can be represented as a dense ring of linear transformations of an infinite dimensional vector space V over k. Then for all n, A contains a subring B whose homomorphic image is the full $n \times n$ matrix ring over k. Then $B \in \mathcal{G}$ by part (a) but its image is in $\mathcal{G}$ iff $n = 1$.

Theorem 3.
(a) *A nil ring R is in $\mathcal{G}$ if and only if $R^2 = 0$.*
(b) *A semi-prime ring R is in $\mathcal{G}$ iff it has no zero-divisors.*
(c) *If $R \in \mathcal{G}$ then $\mathrm{Ann}^\ell R \subseteq \mathrm{Ann}^r R$ or $\mathrm{Ann}^r R \subseteq \mathrm{Ann}^\ell R$.*
(d) *If $R \in \mathcal{G}$ then $R/\mathrm{Ann}^\ell R$ and $R/\mathrm{Ann}^r R \in \mathcal{G}$.*
(e) *Let R be in $\mathcal{G}$, $\mathrm{Ann}^r R \subseteq \mathrm{Ann}^\ell R \neq R$. Then $R/\mathrm{Ann}^\ell R$ is (in $\mathcal{G}$ and) prime.*
(f) *If $\mathrm{Ann}^r R \subseteq \mathrm{Ann}^\ell R$ and $R/\mathrm{Ann}^r R$ has no zero-divisors then $R \in \mathcal{G}$. Hence if $\mathrm{Ann}^r R = \mathrm{Ann}^\ell R$ the converse to part* (e) *is true.*

Proof.
(a) Let R be a nil ring in $\mathcal{G}$. By Lemma 1 (iii) $x^2 = 0$ for all $x \in R$. Thus R is anticommutative. By Lemma 1 (vii), $(0 : x)$ is a prime ideal or R for each x. But R is a prime radical ring (e.g. because it is a nil PI ring) so $(0 : x) = R$ for every x. Thus $R^2 = 0$. The converse is clear.
(b) If R is semi-prime and $R \in \mathcal{G}$, then by Proposition 5 R is prime. Suppose there exist elements a and b with $ab \neq 0$. Since R is prime, there exist f and g so that $afa \neq 0$, $bgb \neq 0$. Then $a(fa + bg)b = afab + abgb = 0$ but $a(fa + bg) = afa \neq 0$ and $(fa + bg)b = bgb \neq 0$ so $fa + bg$ is not a single element.

(c) Suppose the two annihilators of R are incomparable. Then there exist $a, b \in R$ with $aR = 0$, $Rb = 0$ but $ra, bs \neq 0$ for some $r, s \in R$. Then $r(a+b)s = (ra+rb)s = ras = 0$, while $r(a+b) = ra \neq 0$ and $(a+b)s = bs \neq 0$, so $a+b$ is not single. Thus $R \notin \mathcal{G}$.

(d) (for $R/\mathrm{Ann}^{\ell}R$). Let $\overline{x} = x + \mathrm{Ann}^{\ell}R$ for all $x \in R$. If $\overline{abc} = 0$, i.e., $abc \in \mathrm{Ann}^{\ell}R$, then $abcd = 0$ for all $d \in R$. As b is single, either $ab = 0$ or $bcd = 0$ for all d, i.e., $ab = 0$ or $bc \in \mathrm{Ann}^{\ell}R$, i.e., $\overline{ab} = 0$ or $\overline{bc} = 0$. Hence every $\overline{b}$ is single, so $R/\mathrm{Ann}^{\ell}R \in \mathcal{G}$.

(e) Let $B/\mathrm{Ann}^{\ell}R = \mathrm{Ann}^{\ell}(R/\mathrm{Ann}^{\ell}R)$. Then $B \lhd R, (B/\mathrm{Ann}^{\ell}R)^2 = 0$ and $(\mathrm{Ann}^{\ell}R)^2 = 0$ so B is nilpotent and in $\mathcal{G}$ (Theorem 2). By (a) $B^2 = 0$. But then $BRB = (BR)B \subseteq B^2 = 0$. If $a \in R$ and $Ba \neq 0$, let $ba \neq 0$, $b \in B$. Then $baB \subseteq BRB = 0$ so (as a is single) $aB = 0$. Thus $R = \{a : Ba = 0\} \cup \{a : aB = 0\}$. But then $R = \{a : Ba = 0\}$ or $R = \{a : aB = 0\}$ (as R can't be a union of two proper additive subgroups). This means that $BR = 0$ or $RB = 0$, so that $B \subseteq \mathrm{Ann}^{\ell}R \subseteq B$ or $B \subseteq \mathrm{Ann}^{\ell}R \subseteq \mathrm{Ann}^{\ell}R \subseteq B$. In any case, $B = \mathrm{Ann}^{\ell}R$, so $R/\mathrm{Ann}^{\ell}R$ has zero left annihilator.
Let $C/\mathrm{Ann}^{r}R = \mathrm{Ann}^{r}(R/\mathrm{Ann}^{\ell}R)$. Then as with B we have $C \lhd R$, $C \in \mathcal{G}$ and C is nilpotent, so $C^2 = 0$ and $CRC = 0$. Hence $CR = 0$ or $RC = 0$, so $C \subseteq \mathrm{Ann}^{\ell}R \subseteq C$ or $C \subseteq \mathrm{Ann}^{r}R \subseteq \mathrm{Ann}^{\ell}R \subseteq C$. In any case, $C = \mathrm{Ann}^{\ell}R$ so $R/\mathrm{Ann}^{\ell}R$ has zero right annihilator. By (d) $R/\mathrm{Ann}^{\ell}R \in \mathcal{G}$ so by Proposition 5, $R/\mathrm{Ann}^{\ell}R$ is prime.

(f) Suppose $axb = 0$. Then since $\mathrm{Ann}^{r}R$ is a completely prime ideal one of a, x, b is in $\mathrm{Ann}^{r}R \subseteq \mathrm{Ann}^{\ell}R$. Hence $ax = 0$ or $xb = 0$.

Definition. *A non-zero element $y \in R$ will be called a strong generator of R if for all $x \in R$ there exist $a, b \in R$ such that $x = ayb$. R will be called* 0*-simple if every non-zero element is a strong generator.*

The second definition and terminology coincide with that used in semigroups [5]. Thus a strong generator y is a generator in the usual sense that the ideal generated by y is all of R. Clearly any left or right unit in a ring with identity is a strong generator. If $1 \in R$ and R is Dedekind finite ($uv = 1 \Rightarrow vu = 1$) then the only strong generators are the units. However, if R is not Dedekind finite and $uv = 1$, $vu \neq 1$ then u, v and vu are all strong generators and vu is neither a left nor a right unit.

Clearly a division ring is 0-simple and a 0-simple ring is simple. C^*-algebras which are 0-simple are precisely those which are purely infinite [15]. A 0-simple ring is a λ-ring in the sense of [1], i.e., $\forall x \neq 0$, $x \in RxR$. If $1 \in R, R$ is 0-simple iff for all $y \neq 0$ $\exists a, b$ with $1 = ayb$.

Example 10. Let V be a vector space over a division ring K. If V has finite dimension, $E = \mathrm{End}\, V$ is simple but not 0-simple since E is Dedekind finite. On the other hand if V has countably infinite dimension, let I be the ideal of elements of finite rank. I is the unique proper non-trivial ideal of E so $R = E/I$ is simple.

In fact R is 0-simple. Suppose $R = \bigoplus_{1}^{\infty} e_i K$ and $g \in E\backslash I$. Then $V = \ker(g) \oplus U$ where U has a basis $\{u_1, u_2, \ldots\}$. Since $\{g(u_i)\}$ is a linearly independent set we can define $f \in E\backslash I$ by $f(g(u_i)) = e_i\ \forall i$ and $h \in E\backslash I$ by $h(e_i) = u_i$. It follows that $fgh = 1$.

Finally suppose V has dimension α, an infinite cardinal. Then ([7(a), Ex. 3.16]) the ideals of $E = \text{End}\ V$ are 0, E, and $E_\beta = \{T \in E | \text{rank}\ T < \beta\}$ for all infinite cardinals $\beta \leq \alpha$, and moreover [7(b), Ex. 3.16] if rank $S \leq$ rank $T\ \exists\ P, Q \in E$ with $S = PTQ$. Therefore if $\beta < \gamma$ are consecutive infinite cardinals E_γ / E_β is a 0-simple ring without identity.

Lemma 2. *If R is 0-simple and $e^2 = e$, then eRe is 0-simple.*

Proof. Since e is the identity of eRe it suffices to show that for all $ere \neq 0$ there exist ese, ete with $e = eseereete = eserete$. Since ere is a strong generator of R there exist s, t with $e = seret$ and the result follows.

A unital 0-simple ring R which is not a division ring is rich in idempotents. In the first place R is not Dedekind finite so $\exists u, v$ with $uv = 1$ and $e = vu \neq 1$. Then for all i and j, $e_{ij} = v^i(1 - e)u^j$ is an idempotent. Moreover for any non-unit x there exist a, b with $1 = axb$ so $f = xba$ and $g = bax$ are idempotents at least one of which is different from 1.

Theorem 4. *If R is 0-simple and has a primitive idempotent, it is regular and a generalized domain. If R is unital, it is a division ring.*

Proof. The regularity is a consequence of the semigroup structure of R alone [5, Lemma 3.2.7]. Let e be the primitive idempotent. For all $b \in R$ there exist x, y with $b = xey$. Since R is regular, Re is a minimal left ideal and e is single. Therefore b is single. If $1 \in R$ since eRe is a division ring and $\exists a, b \in R$ with $1 = aeb$ we have $e = ae\ be$ and ebe has an inverse ece in eRe. Therefore $ece = aebece = ae$ and $1 = aeb = eceb$ so $e = eceb = 1$. Thus 1 is a primitive idempotent, i.e., R is a division ring.

3. Single elements in nearrings

Recall (see e.g. [14]) that a (right) nearring is a triple $(N, +, \cdot)$ where $(N, +)$ is a (possibly non-abelian) group, $(N, \cdot)$ is a semi-group and $(a + b)c = ac + bc$ for all $a, b, c \in N$. Many of the results of Section 1 will hold in nearrings with suitable modifications such as the replacement of "left ideal" by "N-subgroup" and with attention paid to the presence or absence of zero-symmetry. It is not obvious that all of these appear in the literature, but whenever the proof is a straight-forward adaptation of the ring theoretic proof, it will be omitted. Nearrings provide a natural setting for single elements in that there are large and important classes of nearrings which are functions (on groups) and, as we have seen, the original impetus for single elements came from rings of functions.

Let N be a right nearring with constant subnearring $N_c = \{n \in N | n0 = n\} = \{n \in N | nn' = n \text{ for all } n' \in N\}$. Clearly every constant element is single. The converse may or may not hold:

Example 11. Let N be the nearring on $Z_2 \times Z_2 \times Z_2$ whose multiplication table is given by (see [11, Example 2.6]):

	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	0	1	0	0	0	0
2	0	0	2	2	0	0	2	2
3	0	1	2	3	0	0	2	2
4	4	4	4	4	4	4	4	4
5	4	5	4	5	4	4	4	4
6	4	4	6	6	4	4	6	6
7	4	5	6	7	4	4	6	6

Direct calculation shows that 1, 2, 4 and 5 are all single elements but only 4 is a constant. For example if $s = 5$ then since $5b = 4$ or 5 for all b, $a5b = 0 \Rightarrow a4 = 0$ or $a5 = 0$. But, in fact, $a4 = 0 \Leftrightarrow a5 = 0$ and s is single.

Example 12. In $M(G) = \{f : G \to G\}$ where G is a group, every "single" element is constant. For suppose $f \in M(G)$ where f is single and $f(g_1) = h_1 \neq h_2 = f(g_2)$ for some $g_i \in G$. We can assume $h_2 \neq 0$. Define $\alpha, \beta \in M(G)$ so that $\alpha(h_1) \neq 0$, $\alpha(x) = 0\ \forall x \neq h_1$ and $\beta(x) = g_2\ \forall x$. Then $\alpha f \beta(x) = \alpha f(g_2) = \alpha(h_2) = 0$ but $\alpha f(g_1) = \alpha(h_1) \neq 0$ and $f\beta(x) = f(g_2) = h_2 \neq 0$.

Example 13. Let N be a subnearring of $M_0(G)$. Then any function $f \in N$ which has only one non-trivial element in its range is single. For suppose $\emptyset \neq X \subset G$ and $f(x) \equiv f_{g,X}(x) = g\ \forall x \in X$ and $f(y) = 0\ \forall y \notin X$. Then if $\alpha f \beta = 0$ for some $\alpha, \beta \in N$ there are two cases. If $\beta(g) \subseteq G \backslash X$, $f\beta = 0$. On the other hand if $\exists t \in G$ such that $\beta(t) \in X$, $\alpha f \beta(t) = 0 \Rightarrow \alpha(g) = 0 \Rightarrow \alpha f(x) = 0\ \forall x \in X$ so $\alpha f = 0$.

Example 13(a). If $N = M_0(G)$ the converse to Example 13 holds, that is a single element must have only one non-trivial element in its range. The proof given for Example 12 goes through. Moreover the single elements which are idempotent are those $f_{g,X}$ for which $g \in X$.

Example 13(b). Endomorphism nearrings can have single elements of the type described in Example 13. For instance (see [12]), $E(D_4)$ can be additively generated by 5 such idempotent single elements and $I(D_4)$ has seven such elements (eg. $2id$ is one).

Example 13(c). If V is a faithful N-group where N is zero-symmetric, Scott [16] has defined a pre-image of V as a triple (n, v, S) where $n \in N$, $v \in V^*$, $\emptyset \neq$

$S \subseteq V^*$ and $ns = v\ \forall s \in S,\ ns' = 0$ otherwise. In this case n is a single element by Example 13.

Example 14. If $N = M_A^{\circ}(G)$ is a centralizer nearring where $A \neq \{id\}$ is a group of automorphisms of G then N has single elements with more than one non-trivial element in their range. In particular $\forall g \in G$ define $e_g(x) = x\ \forall x \in Ag$ and $e_g(y) = 0$ otherwise. Then e_g is single and idempotent. For suppose $f, h \in N$ and $fe_gh(y) = 0\ \forall y \in G$. If $h(y) \notin Ag$ for all y then $e_gh(y) = 0$. On the other hand if $h(y) \in Ag$ for some y then $h(y) = \alpha(g)$ for some $\alpha \in A$ and $fe_gh(y) = 0 \Rightarrow fh(y) = 0$. We claim that $fe_g(x) = 0$ for all x. Certainly $fe_g(x) = 0$ if $x \notin Ag$. If $x \in Ag,\ x = \beta(g)$ for some $\beta \in A$ so $x = \beta\alpha^{-1}h(y)$. Then $fe_g(x) = f\beta\alpha^{-1}h(y) = \beta\alpha^{-1}fh(y) = 0$ as claimed.

Lemma 3. *If s is single so is xs for all $x \in N$; if N is zero-symmetric sx is also single for all $x \in N$.*

To extend Theorem 1 to nearrings we need the following definitions:

(1) N is called 3-*semiprime* if $xNx = 0 \Rightarrow x = 0$;

(2) N is 2-*semiprime* if it has no non-zero nilpotent N-subgroups;

(3) N is 1-*semiprime* if it has no non-zero nilpotent left ideals;

(4) N is said to have "DCCN" if it has the descending chain condition on N-subgroups;

(5) an idempotent e is *primitive* if for all idempotents $f \neq 0,\ ef = fe = f \Rightarrow e = f$.

Note that in nearrings this neither implies nor is implied by the statement "e is not the sum of two orthogonal idempotents". As in rings, an idempotent which is single is primitive.

Lemma 4. *e is primitive iff the semigroup eNe has no non-zero idempotent elements except e.*

Proof. If $f = ene$ satisfies $f^2 = f$, then $fe = ef = f$ so $f = e$. Conversely if $ef = fe = f$ then ef is an idempotent since $efef = eff = ef$ and moreover $ef = efe \in eNe$ so $f = e$.

Proposition 6. *Consider the following conditions on $e^2 = e$.*

(a) *Ne is a minimal N-subgroup of N.*

(b) *$(eNe - \{0\}, \cdot)$ is a group.*

(c) *e is primitive.*

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c)*. Also* (b) $\Rightarrow$ (a) *if N is zero-symmetric and* 3*-semiprime, and* (c) $\Rightarrow$ (a) *if N is regular or if N is zero-symmetric and* 2*-semiprime with* DCCN*.*

Proof. (a) $\Rightarrow$ (b): If $ene \neq 0$, then $0 \neq ene = e(ene) \in Nene \subseteq Ne$ so by the minimality of Ne, $Nene = Ne$. Therefore there is an $x \in N$ such that $e = xene$

so $e = exene = (exe)(ene)$. Thus $eNe - \{0\}$ is a semigroup with identity in which each element has a left inverse. Hence it is a group.

(b) $\Rightarrow$ (c) is trivial since a group does not have idempotent elements other than the identity.

(b) $\Rightarrow$ (a) if N is zero-symmetric and 3-semiprime: Suppose $0 \neq I \subseteq Ne$, I an N-subgroup. Let $0 \neq x = ne \in I$. Since N is 3-semiprime, $neNne \neq 0$ so $\exists x$ such that $nexne \neq 0$. Since N is zero-symmetric $exne \neq 0$, so by hypothesis $\exists t$ such that $eteexne = e$. Thus $e \in Nne$ so $Ne \subseteq Nne \subseteq I$ so $I = Ne$ as claimed.

(c) $\Rightarrow$ (a): If N is regular and $0 \neq a \in I \subseteq Ne$ where I is an N-subgroup of N then $Na = Nf$ for some idempotent f so $Nf \subseteq Ne$. Hence $f = ne$ so $fe = f$ and ef is an idempotent in eNe. Since e is primitive $ef = e$ so $e \in Nf$ and $Ne = Nf = Na$ as claimed.

With the alternate hypothesis given, if Ne is not minimal then Ne contains some minimal N-subgroup which by [14, Theorem 3.51(a)] has the form Nf, $f^2 = f$. The proof then proceeds as above.

Remark. The result just quoted ([14, Theorem 3.51(a)]) has DCCN as a hypothesis. However, more generally, we have by a standard proof:

Proposition 7. *If N is zero-symmetric and L is a minimal N-subgroup of N with $L^2 \neq 0$ then $L = Ne$ for some idempotent e.*

Theorem 5. (a) *If L is a minimal N-subgroup of N such that $\forall 0 \neq x \in L\ \exists n \in N$ with $nx \neq 0$ then $L = Nx$ and x is single.*

(b) *If N is regular and $e^2 = e$ is single, then Ne is a minimal N-subgroup of N.*

Proof. (a) The proof given in Theorem 1(b) goes through.

(b) If e is single, eNe has no zero-divisors. As in the proof of Theorem 1(d), this implies eNe has no idempotents other than e. Hence by the previous result Ne is a minimal N-subgroup.

Corollary 3. *If N is regular, s is single iff Ns is a minimal N-subgroup.*

Example 15. The zero-symmetry is necessary for Proposition 5 (c) $\Rightarrow$ (a). Example 11 above is non-zero symmetric and 1 is a primitive idempotent. N is 3-semiprime and so is 2-semiprime, and being finite it has DCCN. However $N1 = \{0, 1, 4, 5\} \supsetneqq N4 = \{0, 4\}$.

Theorem 6. *Let $N = M_A^\circ(G)$.*

(a) *Every minimal left ideal (if any exist) consists of single elements, Moreover if N is regular then*

(b) *minimal N-subgroups exist and consist of single elements*

(c) *every N-subgroup contains a minimal N-subgroup.*

Proof.

(a) Let e_g be the idempotent defined in Example 13. By [13, Theorem 2.1] if L is a minimal left ideal, $L \subseteq Ne_g$ for some g. Since e_g is single, so is every element of L.

(b) By Theorem 5, Ne_g is a minimal N-subgroup. Furthermore if L is any minimal N-subgroup $L^2 \neq 0$ since N is regular so by Proposition 7, $L = Ne$ and e is single by the corollary.

(c) If L is any N-subgroup and $0 \neq f \in L$ then $f(x) \neq 0$ for some $x \in G$. Let $e \equiv e_{f(x)}$. Since $ef(x) = f(x)$, ef is a non-zero single element in L. Thus L contains the N-subgroup Nef which is minimal by the corollary.

Remark. Theorem 6 applies in particular to the regular nearring $M_0(G)$ and, despite the fact that much has been known for some time about the structure of $M_0(G)$ (see e.g. [14, Ch. 7]) Theorem 6(c) appears to be new. This also provides a good example of how nearrings can differ markedly from rings. By [7(a), 3.10] a simple ring with a minimal left ideal is left artinian. In contrast, $M_0(G)$ is a simple nearring in which every N-subgroup contains a minimal N-subgroup but if G is infinite it does not have the descending chain condition on N-subgroups [14, 7.19].

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