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# A Note on Extending Hopf Actions to Rings of Quotients of Module Algebras

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**Abstract.** Given a Hopf algebra $H$ and an $H$-module algebra $A$ it is often important to be able to extend the $H$-action from $A$ to a localisation of $A$. In this paper we discuss sufficient conditions to extend Hopf actions as well as to characterise those localisations where the action can be extended. Our particular interest lies in the maximal ring of quotients of $A$.

## 1. Introduction

Let $k$ be a commutative ring and $H$ a $k$-Hopf algebra. We are interested in studying under which conditions the $H$-action on an $H$-module algebra $A$ can be extended to a localisation of $A$. Sufficient conditions to extend $H$-actions were found by Sidorov [9], Montgomery [6] and Selvan [8]. Given a Gabriel topology $\mathcal{F}$ on $A$ such that $A$ is torsionfree with respect to $\mathcal{F}$, we are going to show that the $H$-action can be extended to a torsion theoretic ring of quotient $Q_{\mathcal{F}}(A)$ of $A$ if and only if $Q_{\mathcal{F}}(A)$ coincides with the ring of quotients with respect to the Gabriel topology $\mathcal{F}_H$ which has a basis of $H$-stable $\mathcal{F}$-dense left ideals. We compare Sidorov, Montgomery and Selvan's condition with a condition by Louden and show that if the embedding $A \to A\#H$ is compatible in the sense of Louden with a Gabriel topology $\mathcal{F}$ and $A$ is non-singular, then the $H$-action can be extended

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to the maximal ring of quotients of $A$. More precisely we show that the $H$-action can be extended from $A$ to $Q_{max}(A)$ if and only if $Q_{max}(A)$ coincides with the ring of quotients of $A$ with respect to the torsion theory cogenerated by the injective hull of $A$ in $A\#H$-Mod.

Throughout $H$ will denote a $k$-Hopf algebra and $A$ a left $H$-module algebra. The smash product of $A$ and $H$ is denoted by $A\#H$.

### 1.1.

We first need two technical basic lemmas on Hopf algebras and their actions.

**Lemma 1.** *Let $H$ be flat as $k$-module. Then $A\#H$ is a flat left $A$-module and, if $H$ has a bijective antipode, then $A\#H$ is also a flat right $A$-module.*

*Proof.* Let $M \in Mod - A$, then $M \otimes_A A\#H \simeq M \otimes H$ as $k$-modules. Since $- \otimes H$ is exact, also $- \otimes_A A\#H$ is exact. If $H$ has a bijective antipode then $A\#H \simeq H \otimes A$ as right $A$-module (see the proof of [6, 7.2.11] that does not use the fact that $k$ is a field). If $M \in A-Mod$, then $A\#H \otimes_A \simeq H \otimes A \otimes_A M \simeq H \otimes M$ as $k$-modules. As $H \otimes -$ is exact, also $A\#H \otimes_A -$ is exact. $\square$

### 1.2.

The next lemma is known for Hopf algebras over fields, but applies also to arbitrary Hopf algebras over rings.

**Lemma 2.** *Assume that $H$ has a bijective antipode. For any left $A\#H$-modules $M, N$ the following hold:*

(1) $\mathrm{Hom}_{A-}(M,N)$ *is a left $H$-module with $H$-action given by*

$$(h \cdot f) : m \mapsto \sum_{(h)} h_2 \cdot (S^{-1}(h_1) \cdot m)f$$

*for all $h \in H$ and $f \in \mathrm{Hom}_{A-}(M,N)$.*

(2) $\forall g \in \mathrm{Hom}_{A-}(M,N), f \in \mathrm{Hom}_{A-}(N,L)$ *and $h \in H$:*

$$h \cdot (f \circ g) = \sum_{(h)} (h_1 \cdot f) \circ (h_2 \cdot g).$$

(3) $S := \mathrm{End}_{A-}(M)$ *is a left $H$-module algebra with $S^H = \mathrm{End}_{A\#H}(M)$.*

(4) $\mathrm{Biend}_{A-}(M) = \mathrm{End}_{-S}(M)$ *is a left $H$-module algebra whose $H$-action is given by*

$$(h \cdot \varphi) : m \mapsto \sum_{(h)} h_1 \cdot \varphi(S(h_2) \cdot m)$$

*for $\varphi \in \mathrm{Biend}_{A-}(M)$.*

(5) *The natural homomorphism $A \longrightarrow \mathrm{Biend}_{A-}(M)$ with $a \mapsto L_a : [m \mapsto am]$ is a homomorphism of left $H$-module algebras with kernel $\mathrm{Ann}_A(M)$.*

*Proof.* (1) It is a routine check that $(m)(h \cdot f) = \sum_{(h)} h_2 \cdot (S^{-1}(h_1) \cdot m)f$ makes $\mathrm{Hom}_{A-}(M, N)$ a left $H$-module.
(2) Let $h \in H$, $f \in \mathrm{Hom}_{A-}(M, N), g \in \mathrm{Hom}_{A-}(N, L)$ and $m \in M$. Then

$$\begin{aligned}\sum_{(h)} (m)(h_1 \cdot f) \circ (h_2 \cdot g) &= \sum_{(h)} h_4 \cdot [S^{-1}(h_3) \cdot h_2 \cdot (S^{-1}(h_1) \cdot m)f]g \\ &= \sum_{(h)} h_2 \cdot ((S^{-1}(h_1) \cdot m)f)g = (m)(h \cdot (f \circ g)).\end{aligned}$$

(3) follows from (1, 2) and from $(m)[h \cdot id_M] = \sum_{(h)} h_1 \cdot (S(h_2) \cdot m) = \epsilon(h)m$, i.e. $h \cdot id_M = \epsilon(h) id_M$.
(4) Can be shown analogously to (1–3).
(5) We just need to show that the homomorphism $j : A \to \mathrm{Biend}_{A-}(M)$ sending $a \in A$ to the biendomorphism $L_a$ is $H$-linear. Let $a \in A, h \in H, m \in M$. Then

$$(h \cdot a)m = \sum_{(h)} h_1 \cdot (a(S(h_2) \cdot m)) = (h \cdot L_a)(m),$$

thus $j(h \cdot a) = h \cdot j(a)$. □

## 2. Torsion theoretical localisation

We fix the following notation: for any subset $I \subseteq A$ and element $a \in A$ set

$$Ia^{-1} := \{b \in A \mid ba \in I\}.$$

### 2.1.

A filter $\mathcal{F}$ of left ideals of a ring $A$ is called *Gabriel topology*, if the following conditions are fulfilled:

(i) $\forall I \in \mathcal{F}, a \in A : Ia^{-1} \in \mathcal{F}$.
(ii) $\forall J \subseteq A : [\{\exists I \in \mathcal{F}, \forall a \in I : Ja^{-1} \in \mathcal{F}\} \Rightarrow J \in \mathcal{F}]$.

We refer to left ideals $I \in \mathcal{F}$ also as $\mathcal{F}$-dense left ideals.

### 2.2.

There exists a bijection between Gabriel topologies $\mathcal{F}$ and hereditary torsion theories $\tau$, where $\tau$ corresponds to the Gabriel topology

$$\mathcal{F}_\tau := \{I \subseteq A \mid A/I \text{ ist } \tau\text{-torsion}\}$$

and $\mathcal{F}$ corresponds to the torsion theory whose torsion class is

$$\{M \in A - Mod \mid \forall m \in M : \mathrm{Ann}_R(m) \in \mathcal{F}\}.$$

**2.3.**

Let $\mathcal{F}$ be a Gabriel topology of $A$ and $\tau$ the torsion theory corresponding to $\mathcal{F}$. Let $M$ be a left $A$-module and set $\overline{M} := M/\tau(M)$. Let $E(\overline{M})$ be the injective hull of $\overline{M}$ in $A$-Mod. The quotient module $Q_{\mathcal{F}}(M)$ of $M$ is defined as the submodule $\overline{M} \subseteq Q_{\mathcal{F}}(M) \subseteq E(\overline{M})$ such that $Q_{\mathcal{F}}(M)/\overline{M} = \tau(E(\overline{M})/\overline{M})$.

It is well-known that $Q_{\mathcal{F}}(M)$ can be described as the direct limit of the Hom-groups:

$$Q_{\mathcal{F}}(M) \simeq \varinjlim \mathrm{Hom}_A(I, \overline{M})$$

and also as

$$Q_{\mathcal{F}}(M) = \{x \in E(\overline{M}) \mid \exists I \in \mathcal{F} : Ix \subseteq \overline{M}\}$$

(see [10, IX.2.2]).

$Q_{\mathcal{F}}(A)$ is a ring for $M = A$. We are interested to extend the $H$-action of a module algebra $A$ to its quotient ring $Q_{\mathcal{F}}(A)$.

**2.4.**

Let $\mathcal{F}$ be a Gabriel topology of $A$. To extend the $H$-action from $A$ to $Q_{\mathcal{F}}(A)$ it would be nice if $\mathcal{F}$ had a bases of $H$-stable ideals. Let $I$ be a left ideal of $A$, then denote by $I_H$ the largest $H$-stable left ideal of $A$ which contains $I$. Define $\mathcal{F}_H := \{I \in \mathcal{F} \mid I_H \in \mathcal{F}\}$. It is not hard to see that $\mathcal{F}_H$ is again a filter, but in general it is not clear whether $\mathcal{F}_H$ is a Gabriel filter ( $(Ia^{-1})_H$ might be zero for example). In case $H$ is finitely generated as $k$-module we get:

**Lemma 3.** *$\mathcal{F}_H$ is a Gabriel topology provided $H$ is finitely generated as $k$-module.*

*Proof.* The proof is like in [8, Lemma 2.1.7] where the lemma had been stated for Hopf algebras over fields. The base field does not play any role in the proof. We include the proof for the reader's convenience: Let $\{b_1, \ldots, b_n\}$ be a generating set of $H$ as $k$-module. Let $I \in \mathcal{F}_H$ and $a \in A$. We have to show that each $Ia^{-1}$ contains an $H$-stable $\mathcal{F}$-dense left ideal. Since $H$ is finitely generated we have:

$$K := \bigcap_{i=1}^{n} I_H(b_i \cdot a)^{-1} = \bigcap_{h \in H} I_H(h \cdot a)^{-1} \subseteq Ia^{-1}.$$

As a finite intersection of $\mathcal{F}$-dense left ideals: $K \in \mathcal{F}$. Moreover $K(H \cdot a) \subseteq I_H$. Let $g \in H$, $1 \leq i \leq n$ and $k \in K$. Then

$$(g \cdot k)(b_i \cdot a) = \sum_{(g)} g_1 \cdot (k \cdot (S(g_2)b_i \cdot a)) \subseteq I_H,$$

i.e. $g \cdot k \in I_H(b_i \cdot a)^{-1}$ for all $i$. Hence $g \cdot k \in K$ for all $g \in H$, i.e. $K$ is $H$-stable. Thus $K \in \mathcal{F}_H$ and $K \subseteq (Ia^{-1})_H \subseteq Ia^{-1}$.

Now let $J$ be a left ideal of $A$ and $I \in \mathcal{F}_H$ such that $Ja^{-1} \in \mathcal{F}_H$ for all $a \in I$. Without loss of generality we may assume that $I$ is $H$-stable. Then $(Ja^{-1})_H \in \mathcal{F}$ and $K := \bigcap_{i=1}^{n}(J(b_i \cdot a)^{-1})_H \in \mathcal{F}$ for all $a \in I$. Hence $K(H \cdot a) \subseteq J$ and, since all $(J(b_i \cdot a)^{-1})_H$ are $H$-stable, we also have $K$ is $H$-stable. It follows from $H \cdot a \subseteq I$ and $I$ $H$-stable that $KI$ is $H$-stable, i.e. $Ka \subseteq KI \subseteq J_H$. Thus $K \subseteq J_H a^{-1}$ for all $a \in I$. Since $\mathcal{F}$ is a Gabriel topology we have $J_H \in \mathcal{F}$. $\square$

**2.5.**

If $\mathcal{F}_H$ is a Gabriel topology, then $\tau_H(M)$ is $H$-stable for any left $A\#H$-module $M$, because if $m \in \tau_H(M)$ then there exists an $H$-stable left ideal $I$ of $A$ such that $Im = 0$. For any $h \in H$ and $x \in I$ we have

$$x(h \cdot m) = \sum_{(h)} S(h_1) \cdot ((h_2 \cdot x)m) \in H \cdot ((H \cdot I)m) = 0,$$

since $I$ is $H$-stable. Hence $h \cdot m \in \tau_H(M)$ for any $h \in H$. Note that $\tau_H(A)$ is then an $H$-stable twosided ideal of $A$ and $\overline{A} = A/\tau_H(A)$ is a left $H$-module algebra. By construction, $\overline{A} \subseteq Q_{\mathcal{F}_H}(A)$ and as we will see $Q_{\mathcal{F}_H}(A)$ is a left $H$-module algebra.

**2.6.**

We are now in a position to prove our first result.

**Theorem 1.** *Assume that $H$ is finitely generated as $k$-module and has a bijective antipode. The following statements are equivalent for a left $H$-module algebra $A$ with Gabriel topology $\mathcal{F}$.*

(a) *$Q_{\mathcal{F}}(A)$ is a left $H$-module algebra, $\tau(A)$ is $H$-stable and $A/\tau(A)$ is a submodule algebra of $Q_{\mathcal{F}}(A)$,*

(b) *$Q_{\mathcal{F}}(A) = Q_{\mathcal{F}_H}(A)$ and $\tau(A) = \tau_H(A)$,*

*where $\tau$ denotes the torsion radical associated to $\mathcal{F}$.*

*Proof.* (a)$\Rightarrow$(b) Let $\{b_1, \dots, b_n\}$ be a generating set of $H$ as $k$-module. Let $\overline{A} := A/\tau(A)$. By Lemma 3 $\mathcal{F}_H$ is a Gabriel topology and since $\mathcal{F}_H \subseteq \mathcal{F}$ holds we get $Q_{\mathcal{F}_H}(A) \subseteq Q_{\mathcal{F}}(A)$. We will show $Q_{\mathcal{F}}(A) \subseteq Q_{\mathcal{F}_H}(A)$. Let $q \in Q_{\mathcal{F}}(A)$. Denote by $\overline{A}q^{-1} := \{a \in A \mid aq \in \overline{A}\}$. Then for any $b_k$ we have $\overline{A}(b_k \cdot q)^{-1} \in \mathcal{F}$. Set $I := \bigcap_{k=1}^n \overline{A}(b_k \cdot q)^{-1}$. Since for all $h \in H$ with $h = \sum_{k=1}^n r_k b_k$ and for all $x \in I : x(h \cdot q) = \sum_{k=1}^n r_k x(b_k \cdot q) \in \overline{A}$, we have $I(H \cdot q) \subseteq \overline{A}$. Moreover $I$ is $H$-stable, since for all $x \in I, h \in H$ and $1 \leq k \leq n$

$$(h \cdot x)(b_k \cdot q) = \sum_{(h)} h_1 \cdot (x(S(h_2)b_k \cdot q)) \subset H \cdot (I(H \cdot q)) \subset \overline{A},$$

as $\overline{A}$ is an $H$-module by hypothesis. Thus $h \cdot x \in \overline{A}(b_k \cdot q)^{-1}$. Since $k$ was arbitrary, $h \cdot x \in I$, i.e. $I$ is $H$-stable. Consequently $I \in \mathcal{F}_H$. As $Iq \subseteq \overline{A}$, $q \in Q_{\mathcal{F}_H}(A)$.

(b)$\Rightarrow$(a) Without loss of generality we might assume $\tau(A) = \tau_H(A) = 0$. Then one gets:

$$Q_{\mathcal{F}}(A) = Q_{\mathcal{F}_H}(A) \simeq \varinjlim{}_{I \in \mathcal{F}_H} \mathrm{Hom}_{A-}(I, A) \simeq \varinjlim{}_{I_H \in \mathcal{F}_H} \mathrm{Hom}_{A-}(I_H, A).$$

By Lemma 2(1) the $k$-modules $\mathrm{Hom}_{A-}(I_H, A)$ are left $H$-modules. Therefore the direct limit of those $H$-modules is again a left $H$-module. The module algebra property follows from Lemma 2(2). $\square$

**2.7.**

We are going to investigate some sufficient conditions to extend the $H$-action to $Q_{\mathcal{F}}(A)$. Selvan has termed a Gabriel topology $\mathcal{F}$ to be $H$-invariant if $\mathcal{F} = \mathcal{F}_H$ holds. Obviously a Gabriel topology is $H$-invariant if and only if it posses a coinitial subset of $H$-stable left ideals. As Theorem 1 shows, $\mathcal{F}$ being $H$-invariant is a sufficient condition for extending the $H$-action to $Q_{\mathcal{F}}(A)$.

**2.8.**

The Gabriel topology $\mathcal{F}$ turns the additive group of $A$ into a topological group by considering $\mathcal{F}$-dense left ideals as a base of the neighborhoods of 0. Sidorov in [9] and Montgomery in [6] showed that if every element $h$ of the Hopf algebra acts continuously (with respect to the topology induced by $\mathcal{F}$) on the algebra $A$, then one can extend the action to $Q_{\mathcal{F}}(A)$. More precisely, a $k$-endomorphism $f \in \mathrm{End}_R(A)$ is called $\mathcal{F}$-continuous if preimages of $\mathcal{F}$-dense left ideals are $\mathcal{F}$-dense. One says that $H$ acts $\mathcal{F}$-continuously on $A$ if all endomorphisms $L_h$ of $A$ with $L_h(a) := ha$ are $\mathcal{F}$-continuous.

It is straightforward that if $\mathcal{F}$ is an $H$-invariant Gabriel topology then $H$ acts $\mathcal{F}$-continuously since for any $h \in H$ and $I \in \mathcal{F}$, $I_H \in \mathcal{F}$ and $hI_H \subseteq I$, i.e. $I_H \subseteq L_h^{-1}(I)$. On the other hand, if $H$ is finitely generated as $k$-module and acts $\mathcal{F}$-continuously on $A$, then $\bigcap_{i=1}^n L_{h_i}^{-1}(I) \subseteq I_H$ for any $I \in \mathcal{F}$ and generating set $h_1, \ldots, h_n$.

**2.9.**

Actually Sidorov proved in [9] that any $\mathcal{F}$-continuous local action of a Hopf algebra on $A$ can be extended to a global action on $Q_{\mathcal{F}}(A)$. Sidorov says that $H$ acts locally on $A$ if the action of an element $h \in H$ is defined on some $\mathcal{F}$-dense left ideal.

**2.10.**

Conditions on compatibility of torsion theories and ring extensions had been studied in [2], [5], [4] and [1].

One says that $\mathcal{F}$ is compatible with the embedding $A \hookrightarrow A\#H$, if

$$\tilde{\mathcal{F}} := \{I \subseteq A\#H \mid I \cap (A\#1) \in \mathcal{F}\}$$

is a Gabriel topology on $A\#H$.

Louden calls a Gabriel topology $\tau$ $A\#H$-good in case the associated filter $\mathcal{F}$ is compatible with the embedding $A \hookrightarrow A\#H$ (see [5, p. 517]).

We have the following relation between an $\mathcal{F}$-continuous action and a compatible topology:

**Theorem 2.** *Let $H$ be a $k$-Hopf algebra with bijective antipode. If $H$ acts $\mathcal{F}$-continuously on $A$ then $\mathcal{F}$ and $A \hookrightarrow A\#H$ are compatible.*

*Proof.* Let $\tau$ be the torsion radical associated to $\mathcal{F}$ and let $M \in A\#H$-Mod. Let $m \in \tau(M)$ then there exists $I \in \mathcal{F}$ with $Im = 0$. Let $h \in H$ with $\Delta(h) = \sum_{k=1}^{n} h_k \otimes h'_k$. By hypothesis, for all $1 \leq k \leq n$ there is $I_k \in \mathcal{F}$ with $S^{-1}(h'_k)I_k \subseteq I$. Set $J := I_1 \cap \cdots \cap I_n \in \mathcal{F}$, then for all $x \in J$:

$$x(hm) = \sum_{k=1}^{n} h_k[(S^{-1}(h'_k)x)m] \in \sum_{k=1}^{n} h_k(Im) = 0.$$

Hence $hm \in \tau(M)$, i.e. $\tau(M)$ is $H$-stable and a left $A\#H$-module. By [5, Theorem 2.5] $\mathcal{F}$ and $A \hookrightarrow A\#H$ are compatible. □

**2.11.**

The following is a special case of [5, Theorem 2.7]:

**Theorem 3.** *Let $H$ be flat as $k$-module and let $\mathcal{F}$ be a Gabriel topology on $A$. If $\mathcal{F}$ is compatible with $A \hookrightarrow A\#H$, then $Q_{\mathcal{F}}(A)$ is a left $A\#H$-module.*

*Proof.* Let $B := A\#H$ and let $I$ be a left ideal of $A$. Since $B_A$ is flat, $B \otimes_A I \simeq B \cdot I$ as left $B$-modules. Let $\tau$ be the torsion radical associated to $\mathcal{F}$. By [5, Theorem 2.5], $\tau(A)$ is a left $B$-module. Moreover by the adjunction of $B \otimes_A -$:

$$\mathrm{Hom}_{A-}(I, \overline{A}) \simeq \mathrm{Hom}_{B-}(B \otimes_A I, \overline{A}) \simeq \mathrm{Hom}_{B-}(B \cdot I, \overline{A}),$$

where $\overline{A} := A/\tau(A)$ and $f \in \mathrm{Hom}_{A-}(I, \overline{A})$ is mapped to the $B$-homomorphism $\tilde{f} \in \mathrm{Hom}_{B-}(BI, \overline{A})$ with $bx \mapsto b \cdot (x)f$. For all $\tilde{I} \in \tilde{\mathcal{F}}$ is $I \cap A \in \mathcal{F}$ and hence $B \cdot (I \cap A) \in \tilde{\mathcal{F}}$, i.e. $\{B \cdot I \mid I \in \mathcal{F}\}$ is a basis for $\tilde{\mathcal{F}}$. In particular:

$$Q_{\mathcal{F}}(A) \simeq \varinjlim_{I \in \mathcal{F}} \mathrm{Hom}_{B-}(B \cdot I, \overline{A}) = \varinjlim_{\tilde{I} \in \tilde{\mathcal{F}}} \mathrm{Hom}_{B-}(\tilde{I}, \overline{A})$$

shows that $Q_{\mathcal{F}}(A)$ has a left $B$-module structure. □

**2.12.**

Note that if the Gabriel topology is induced by a torsion theory of a left $A\#H$-module which is injective as left $A$-module and contains $A$, then $Q_{\mathcal{F}}(A)$ is also a left $H$-module algebra. Recall that the torsion theory cogenerated by a injective module $M$ is defined as the torsion class given by

$$\tau := \{X \in A - Mod \mid \mathrm{Hom}_{A-}(X, M) = 0\}.$$

**Lemma 4.** *Assume that $H$ has a bijective antipode. Let $M \in A\#H - Mod$ with ${}_AM$ injective and $A \subseteq M$. Then $Q_{\mathcal{F}}(A)$ is a $H$-module algebra with $A$ as submodule algebra where $\mathcal{F}$ is the Gabriel topology of the torsion theory cogenerated by $M$.*

*Proof.* Since $A \subseteq M$, $M$ is a cyclic module over its endomorphism ring $\mathrm{End}_A(M)$, and by [10, IX.3.3] there exists a ring isomorphism $\Psi : Q_{\mathcal{F}}(A) \simeq \mathrm{Biend}_{A-}(M)$. From Lemma 2(4,5) it follows that $\mathrm{Biend}_{A-}(M)$ is a left $H$-module algebra with $A$ isomorphic to a subalgebra by the mapping $a \mapsto L_a : [x \mapsto ax]$. The isomorphism $\Psi$ induces a left $H$-module algebra structure on $Q_{\mathcal{F}}(A)$. From the proof of [10, IX.3.3] one sees $\Psi(a) = L_a$ for all $a \in A$, i.e. the embedding $A \subseteq Q_{\mathcal{F}}(A)$ is left $H$-linear and $A$ is a submodule algebra of $Q_{\mathcal{F}}(A)$. □

## 3. The maximal ring of quotients

Let $Q_{max}(A)$ be the left maximal ring of quotients of $A$. It is known that $Q_{max}(A)$ is left self-injective and von Neumann regular if $A$ is left non-singular. The question arises as to whether the $H$-action can be extended from $A$ to $Q_{max}(A)$. It has been shown in [3] that for a semiprime Goldie PI module algebra $A$ with central invariants and a semisimple Hopf algebra $H$ the action can be extended to $Q_{max}(A)$ (which in this case is equal to the classical ring of quotients of $A$) if and only if $A\#H$ is semiprime. In this case $Q_{max}(A)$ is the localisation of $A$ by the regular elements of $A^H$. The main theorem of [3] showed that for any commutative reduced module algebra $A$ and semisimple Hopf algebra $H$ over a field of characteristic 0 the smash product $A\#H$ is semiprime. Hence if $A$ is an integral domain for example and $H$ a semisimple Hopf algebra $H$ over a field of characteristic 0 the Hopf action can be extended to the quotient field $Q_{max}(A)$ of $A$ (the extension is trivial since in this case $Q_{max}(A)$ is the field obtained by inverting the non-zero elements of the ring of invariants of $A$).

In a paper by Rumynin [7] it had been claimed that for any semisimple Hopf algebra, the action on a module algebra can be extended to its maximal ring of quotients. Unfortunately there is a gap in the proof of this claim (that had been also confirmed by Rumynin). In the sequel we will follow Rumynin's basic idea and compare the maximal ring of quotients with the quotient ring of $A$ relative to the Gabriel topology induced by the injective hull $E(A)$ of $A$ in $A\#H$-Mod. Let $\mathcal{L}$ denote the Gabriel topology associated to the torsion theory cogenerated by $E(A)$ in $A\#H$-Mod and let $\mathcal{D}$ denote the Gabiel topology associated to the Lambek torsion theory, i.e. the torsion theory cogenerated by the injective hull $\widehat{A}$ of $A$ in $A$-Mod. In what follows let $H$ be a flat $k$-Hopf algebra with bijective antipode.

### 3.1.

From Theorem 3 and Lemma 4 we conclude

**Corollary 1.** *If $\mathcal{D}$ and $A \hookrightarrow A\#H$ are compatible and $A$ is left non-singular, then the $H$-action can be extended from $A$ to $Q_{max}(A)$.*

*Proof.* If $A$ is left non-singular, then the left singular ideal of $A$ is zero and $Q_{max}(A) = E(A)$ is injective as left $A$-module. Hence $A$ is torsionfree with respect to the torsion theory cogenerated by $Q_{max}(A)$. Moreover by Theorem 3, $Q_{max}(A)$ is a left $A\#H$-module. By Lemma 4, $Q_{max}(A)$ is a left $H$-module algebra with submodule algebra $A$. □

This corollary allows us to extend the $H$-action provided the embedding $A \to A\#H$ and the Lambek torsion theory are compatible. In the last part of this paper we are comparing the Lambek torsion theory with the torsion theory cogenerated by the injective hull of $A$ as $A\#H$-module.

### 3.2.

We will need the following fact whose proof can be found for instance in [7, Lemma 2.4]: If $S \subseteq T$ is a ring extension with $T_S$ flat, then every injective left $T$-module is also injective as left $S$-module.

In particular if $A\#H_A$ is flat then $E(A)$ becomes an injective left $A$-module and $\widehat{A}$ will be a direct summand of $E(A)$.

Let $H, A, E(A), \widehat{A}, \mathcal{D}$ and $\mathcal{L}$ be as above.

**Lemma 5.** *$\mathcal{L} \subseteq \mathcal{D}$ and $\mathcal{L}_H = \mathcal{D}_H$ hold.*

*Proof.* By Lemma 1, $A\#H_A$ is flat and hence $E(A)$ is injective as left $A$-module. Since $A \subseteq E(A)$ and $A$ is essential in $\widehat{A}$ we have that $\widehat{A}$ is isomorphic to a direct summand of $E(A)$. Hence $\mathcal{L} \subseteq \mathcal{D}$.

Obviously $\mathcal{L} \subseteq \mathcal{D}$ implies $\mathcal{L}_H \subseteq \mathcal{D}_H$. Let $I \in \mathcal{D}_H, f \in \mathrm{Hom}_{A-}(A/I, E(A))$ and $q = (1)f \in E(A)$. Then $Iq = 0$ and also $I(Hq) = 0$, since for all $h \in H, x \in I$:

$$x(hq) = \sum_{(h)} h_2[(S^{-1}(h_1)x)q] = 0,$$

as $I$ is $H$-stable. Now set $K := A\#Hq$ and let $k := \sum_i a_i(h_i q) \in A \cap K$ with $a_i \in A$ and $h_i \in H$. Since $\mathcal{D}$ is a Gabriel topology there exists an ideal $J \in \mathcal{D}$ with $0 \neq Jk \subseteq I$ and $0 \neq Ja_i \subseteq I$ for all $i$, but that is a contradiction to $Jk = \sum_i Ja_i(h_i q) \subseteq I(Hq) = 0$. Hence $K \cap A = 0$ and as $A$ is essential as $A\#H$-submodule of $E(A)$ we get $K = 0$, i.e. $q = 0$ and $f = 0$. Thus $I \in \mathcal{L}$ and $I \in \mathcal{L}_H$. □

### 3.3.

As a consequence of Lemma 5 and Lemma 4 we conclude:

**Corollary 2.** *$Q_{\mathcal{L}}(A)$ is a left $H$-module algebra with submodule algebra $A$.*

### 3.4.

It follows from the last corollary that if $Q_{max}(A) = Q_{\mathcal{L}}(A)$ then the $H$-action can be extend from $A$ to $Q_{max}(A)$. Our final result shows that those conditions are actually equivalent.

**Theorem 4.** *Let $H$ be a $k$-Hopf algebra with bijective antipode and assume $H$ is finitely generated and flat as $k$-module. If $A$ is left non-singular then the $H$-action can be extended from $A$ to its maximal ring of quotients $Q_{max}(A)$ if and only if $Q_{max}(A) = Q_{\mathcal{L}}(A)$ where $\mathcal{L}$ is the Gabriel topology associated to the torsion theory cogenerated by the injective hull $E(A)$ of $A$ in $A\#H$-Mod.*

*Proof.* Let $\{h_1, \ldots, h_n\}$ be a generating set for $H$. Since $\mathcal{L}_H \subseteq \mathcal{L}$ we get $Q_{\mathcal{L}_H}(A) \subseteq Q_{\mathcal{L}}(A)$. Let $q \in Q_{\mathcal{L}}(A)$. Since $Q_{\mathcal{L}}(A)$ is a left $H$-module algebra by Lemma 4 we have $h_i q \in Q_{\mathcal{L}}(A)$ for all $i$ and by 2.3 there exists $D_i \in \mathcal{L}$ such that

$D_i(h_i q) \subseteq A$. Set $D := \bigcap_{i=1}^n D_i \in \mathcal{L}'$ then $D(Hq) \subseteq A$. Moreover $D$ is $H$-stable, because for all $g \in H$:

$$(gd)(h_i q) = \sum_{(g)} g_1(d(S(g_2)h_i q) \subseteq A \ \ (1 \leq i \leq n).$$

Thus $gd \in D_i$ implies $gd \in D$. This showed that $D$ is $H$-stable and therefore $D \in \mathcal{L}_H$. Since $Dq \subseteq A$ we get $q \in Q_{\mathcal{L}_H}(A)$, i.e. $Q_{\mathcal{L}}(A) = Q_{\mathcal{L}_H}(A)$. By Lemma 5 $\mathcal{D}_H = \mathcal{L}_H$ holds and $Q_{\mathcal{L}}(A) = Q_{\mathcal{D}_H}(A)$ follows.

We have $\mathcal{L} \subseteq \mathcal{D}$ and $Q_{\mathcal{L}}(A) \subseteq Q_{\mathcal{D}}(A) = Q_{max}(A)$. Moreover $A$ is torsionfree with respect to both torsion theories. It follows now by Theorem 1, that the action can be extended to $Q_{max}(A)$ if and only if $Q_{max}(A) = Q_{\mathcal{D}_H}(A) = Q_{\mathcal{L}}(A)$. □

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