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On Groups Satisfying $|G'| > [G : Z(G)]^{1/2}$

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Abstract. We study conditions under which the commutator subgroup G' of a finite group G satisfies the inequality $|G'| > [G : Z(G)]^{1/2}$.
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1. Introduction

All groups in this paper are finite. We use the standard notation $Z(G)$, $\Phi(G)$ for the centre and the Frattini subgroup of G. We study conditions under which the commutator subgroup G' of a group G satisfies $|G'| > [G : Z(G)]^{1/2}$. As is shown in Example A3 of [2] (see also [1]), the inequality $|G'| > [G : Z(G)]^{1/2}$ does not hold for all non-abelian groups, and not even for all solvable non-abelian groups. The groups in that example have order $2p^\alpha$, where p is an odd prime, and their Sylow p-subgroup is metabelian. Hence, a natural question is whether the inequality above holds when all the Sylow subgroups of G are abelian. We prove the following theorem, which gives an affirmative answer to that question.

Theorem A. *Let G be a non-abelian group with all Sylow subgroups abelian. Then $|G'| > [G : Z(G)]^{1/2}$.*

Our next result, which extends Corollary B1 in [2], provides another type of condition ensuring $|G'| > [G : Z(G)]^{1/2}$.

Theorem B. *Let G be a non-abelian group such that G/G' is cyclic. Then $|G'| > [G : Z(G)]^{1/2}$.*

As has been already mentioned, Theorem A can not be extended to groups having a non-abelian Sylow subgroup. However, assuming solvability and $\Phi(G) = 1$, the following was obtained in [2], Corollary A1:

Theorem 1. *Let $G \neq 1$ be a solvable group such that $Z(G) = \Phi(G) = 1$. Then $|G'| > |G|^{1/3}$.*

It was conjectured[1] in [2] that the inequality of Theorem 1 may be strengthened to $|G'| > |G|^{1/2}$. Our Theorems A and B provide this stronger inequality for certain families of groups, without the assumptions that G is solvable and $\Phi(G) = 1$ (for another type of results on "large" commutator subgroups, see [5]).

We note that the family of Frobenius groups of order $p(p-1)$, where p is a prime, shows that the inequality in Theorems A and B can not be strengthened.

2. Proofs

Lemma 1.

(i) *Let G be a group and let $N \trianglelefteq G$ satisfy $N \cap G' = 1$ (notice that N must be central in G). Then $Z(G/N) = Z(G)/N$.*

(ii) *Let G be a group with all Sylow subgroups abelian, and let $N \leq Z(G)$. Then $Z(G/N) = Z(G)/N$. In particular, $Z(G/Z(G)) = 1$.*

Proof. (i) Let $gN \in Z(G/N)$. Then $[g, x] \in G' \cap N = 1$ for all $x \in G$. Hence $g \in Z(G)$ and the result follows.

(ii) This follows by (i), since a group with all Sylow subgroups abelian satisfies $Z(G) \cap G' = 1$ ([7], Ex. 549(i)). □

We are ready now for the proofs of Theorems A and B.

Proof of Theorem A. We prove by induction on $|G|$. Suppose first that $G > Z(G) > 1$, and notice that $G' \cap Z(G) = 1$ ([7], Ex. 549(i)) and $Z(G/Z(G)) = 1$ by Lemma 1. Thus, by induction applied on the group $G/Z(G)$, we have $|G'| = [G'Z(G) : Z(G)] = |(G/Z(G))'| > |G/Z(G)|^{1/2}$ as required. Hence, we suppose from now on that $Z(G) = 1$. We must show that $|G'| > |G|^{1/2}$.

Assume first that $F := Fit(G) > 1$ (where $Fit(G)$ denotes the Fitting subgroup of G). For any prime divisor p of $|G|$ denote $O_p = O_p(G)$ and $A_p = G/C_G(O_p)$. Then A_p is a p'-group since the Sylow p-subgroups of G are abelian. Considering the action of A_p on the abelian group O_p we have, by a well-known result on coprime actions ([6], 8.4.2), the decomposition $O_p = [G, O_p] \times (O_p \cap Z(G))$. Since $Z(G) = 1$ we obtain by these decompositions that

$$F \leq G'. \tag{1}$$

Put $K/F = Z(G/F)$, then the group K is nilpotent by abelian, and it follows by [2], Theorem D, that $|F| = |Fit(K)| > |K|^{1/2}$. Thus, the proof is completed

[1] In the meantime, this conjecture was proved in [3]

in this case by (1) if $K = G$. We may assume then that $G > K$. Note that $Z(G/K) = 1$ by Lemma 1, since G/K is isomorphic to $(G/F)/(Z(G/F))$. Hence by induction $|G'K/K| = |(G/K)'| > |G/K|^{1/2}$, thus

$$[G' : G' \cap K] > |G/K|^{1/2}. \tag{2}$$

By (1) we have $|G' \cap K| \geq |F| > |K|^{1/2}$. Taking this in account together with (2) we obtain $|G'| > |G|^{1/2}$ as required.

It remains to consider the case $F = Fit(G) = 1$. Suppose, by contradiction, that the claim does not hold. Then $|G'| \leq |G|^{1/2}$. By [7], Ex. 622, there exists a nilpotent subgroup $B < G$ such that $G = G'B$. As B is a nilpotent group with all Sylow subgroups abelian, B is abelian. Hence, by a result of Zenkov ([8], Theorem 1), there exists $g \in G$ such that $B \cap B^g \leq Fit(G)$. Now $BB^g \neq G$ since a group is not a product of two conjugates of a proper subgroup. Since $|B| \geq |G|^{1/2}$, it follows that $B \cap B^g > 1$, implying $Fit(G) > 1$, which is the desired contradiction. □

Proof of Theorem B. The case $Z(G) = 1$ was proved in [2], Corollary B1. We assume then that $G > Z(G) > 1$ and apply induction on $|G|$.

Case (i). $G' \cap Z(G) = 1$. Then $Z(G/Z(G)) = 1$ by Lemma 1, and $(G/Z(G))/(G/Z(G))'$ is cyclic. Thus by induction $|G'| = |G'Z(G)/Z(G)| = |(G/Z(G))'| > |G/Z(G)|^{1/2}$, and the proof is completed.

Case (ii). $N := G' \cap Z(G) > 1$. Then $(G/N)/(G/N)'$ is cyclic. Hence, if G/N is abelian then it is also cyclic, and so $G/Z(G)$ is cyclic. This implies ([7], Ex. 125) that G is abelian, a contradiction. Thus G/N is non-abelian and by induction

$$|G'/N| = |(G/N)'| > [(G/N) : Z(G/N)]^{1/2}. \tag{3}$$

Put $L/N = Z(G/N)$, then $L \geq Z(G)$, and by (3) we have $[G' : N] > [G : L]^{1/2}$, implying $|G'| > |G|^{1/2}/[L : N]^{1/2}$. The proof will be completed by showing $[L : N] \leq |Z(G)|$. We should only consider the case $L > Z(G)$.

Choose $g \in L$. Since $[G, L] \leq N \leq Z(G)$, the function $x \mapsto [g, x]$ is a homomorphism from G to N. We denote this homomorphism by θ_g. Given $g, h \in L$, we note that $\theta_g \neq \theta_h$ if and only if $gZ(G) \neq hZ(G)$. Thus $|\{\theta_g \mid g \in L\}| = [L : Z(G)]$. For any $g \in L$, the centralizer $C_G(g)$ is a normal subgroup of G, since it is equal to $C_G(\langle g\rangle Z(G))$ and $\langle g\rangle Z(G) \trianglelefteq G$. By [4], Corollary 2.1, since G/G' is cyclic, the group G is generated by one of its conjugacy classes. Hence G is not a union of proper normal subgroups. Consequently, there exists $u \in G$ which does not centralize any element of $L - Z(G)$. Thus for any $g, h \in L$ with $gZ(G) \neq hZ(G)$ we have $[g, u] \neq [h, u]$, i.e. $\theta_g(u) \neq \theta_h(u)$. Since $\theta_k(u) \in N$ for all $k \in L$, we obtain $[L : Z(G)] \leq |N|$, implying $[L : N] \leq |Z(G)|$ as required. □

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