Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 47 (2006), No. 1, 29-51.

# On the Depth and Regularity of the Symmetric Algebra

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**Abstract.** Let $(R, \mathfrak{m})$ be a standard graded $K$-algebra whose defining ideal is componentwise linear. Using Gröbner basis techniques, bounds for the depth and the regularity of the symmetric algebra $\mathrm{Sym}(\mathfrak{m})$ are given.

MSC 2000: 13H10, 13P10, 13D02

## Introduction

Let $R$ be a standard graded $K$-algebra with graded maximal ideal $\mathfrak{m}=(x_1,\ldots,x_n)$. The algebra $R$ can be written as $S/I$ where $I \subset \mathfrak{m}^2$ is a graded ideal in the polynomial ring $S = K[x_1,\ldots,x_n]$. In this paper we want to study the depth and the regularity of the symmetric algebra $\mathrm{Sym}(\mathfrak{m})$ of the ideal $\mathfrak{m}$. Depth and regularity have been extensively investigated [10] for the Rees algebra of $\mathfrak{m}$, while for $\mathrm{Sym}(\mathfrak{m})$ only partial results and estimates for the depth are known, see [9].

As a technique to study the symmetric algebra we use Gröbner bases: let $<$ be any term order on $S$, and let $R^* = S/\operatorname{in}(I)$ and $\mathfrak{m}^*$ the graded maximal ideal of $R^*$, where $\operatorname{in}(I)$ denotes the initial ideal of $I$. In Section 1 we compare $\mathrm{Sym}(\mathfrak{m})$ and $\mathrm{Sym}(\mathfrak{m}^*)$. Denote by $\mathfrak{n}$ the graded maximal ideal of $S$. For an element $f \in S$ and a graded ideal $L \subset S$ we set

$$v_L(f) = \max\{j \colon f \in \mathfrak{n}^j L\},$$

and show that

$$\operatorname{reg}\operatorname{Sym}(\mathfrak{m}) \leq \operatorname{reg}\operatorname{Sym}(\mathfrak{m}^*) \quad \text{and} \quad \operatorname{depth}\operatorname{Sym}(\mathfrak{m}^*) \leq \operatorname{depth}\operatorname{Sym}(\mathfrak{m}),$$

if

$$v_{\operatorname{in}(I)}(\operatorname{in}(f)) \geq v_I(f) \quad \text{for all} \quad f \in I.$$

Provided $K$ is a field of characteristic 0, we show in Proposition 1.8 that this last condition is satisfied for the reverse lexicographical term order in generic coordinates, if $I$ is componentwise linear in the sense of [7].

Thus in order to obtain upper bounds for the regularity and lower bounds for the depth of the symmetric algebra of the graded maximal ideal of a standard graded algebra whose defining ideal is componentwise linear, it suffices to study standard graded $K$-algebras with monomial relations. For such algebras we use the theory of $s$-sequence which was introduced in [8]. Recall that $\operatorname{Sym}(\mathfrak{m})$ can be written as $P/J$ where $P = R[y_1, \ldots, y_n]$ and $J \subset P$ is generated by the polynomials $g = \sum_i^n a_i y_i$ with $\sum_i^n a_i x_i = 0$. The sequence $x_1, \ldots, x_n$ is said to be an $s$-sequence if for some term order $<$ on the monomials in $y_1, \ldots, y_n$ which is induced by $y_1 < y_2 < \cdots < y_n$, the initial ideal of $J$ is generated by terms which are linear in the $y_i$.

For the computation of $\operatorname{in}(J)$ we cannot use the standard techniques of Gröbner basis theory because our base ring $R$ is not a field. To overcome this problem we show in Section 2 that in case $I$ is a monomial ideal, $\operatorname{in}(J)$ can be computed as follows: write $\operatorname{Sym}(\mathfrak{m})$ as $S[y_1, \ldots, y_n]/(I, J_0)$, and determine the initial ideal of $(I, J_0)$ with respect to a suitable term order which extends the given term order in the $y_i$, and is induced by $x_1 < x_2 < \cdots < x_n < y_1 < y_2 < \cdots < y_n$. This initial ideal is of the form $(I, L_0)$, and $\operatorname{in}(J)$ is the image of $L_0$ modulo $I$.

With this method we characterize in Theorem 2.2 those monomial ideals for which $x_1, \ldots, x_n$ is an $s$-sequence in $R$. These ideals include the stable ideals.

We apply these results in Section 3 to compute the depth and the regularity of the symmetric algebra $\operatorname{Sym}(\mathfrak{m})$ in case $I$ is strongly stable in the reverse order.

Let $u$ be a monomial. We denote by $m(u)$ the smallest integer $i$ for which $x_i$ divides $u$. The main results are Theorem 3.7:

$$\operatorname{reg} R \leq \operatorname{reg}\operatorname{Sym}(\mathfrak{m}) \leq \operatorname{reg} R + 1, \text{ and } \operatorname{reg}\operatorname{Sym}(\mathfrak{m}) = \operatorname{reg} R \Longleftrightarrow \max\{m(u)\} \leq 2,$$

where the maximum is taken over all Borel generators $u$ of $I$ of maximal degree, and Theorem 3.9:

$$\operatorname{depth}\operatorname{Sym}(\mathfrak{m}) = 0, \text{ if } \operatorname{depth} R = 0,$$

and

$$\operatorname{depth}\operatorname{Sym}(\mathfrak{m}) = \operatorname{depth} R + 1, \text{ if } \operatorname{depth} R > 0.$$

Assuming $\operatorname{char} K = 0$, the generic initial ideal $\operatorname{Gin}(I)$ of $I$ with respect to the reverse lexicographical induced by $x_1 < x_1 < \cdots < x_n$ is strongly stable in the reverse order. Thus if the defining ideal of the standard graded algebra $R$ is componentwise linear the results of Section 1 and Section 3 imply:

$$\operatorname{reg}\operatorname{Sym}(\mathfrak{m}) \leq \operatorname{reg} R + 1, \quad \text{and} \quad \operatorname{depth}\operatorname{Sym}(\mathfrak{m}) \geq \operatorname{depth} R + 1 \quad \text{if} \quad \operatorname{depth} R > 0.$$

## 1. Symmetric algebras and initial ideals

In this section we recall some basic facts about $s$-sequences, and discuss the symmetric algebra of an initial ideal.

Let $R$ be a Noetherian ring and $M$ an $R$-module generated by $f_1, \dots, f_n$. Then $M$ has a presentation

$$R^m \to R^n \longrightarrow M \longrightarrow 0$$

with relation matrix and $A = (a_{ij})_{\substack{i=1,\dots,m \\ j=1,\dots,n}}$.

The symmetric algebra $\mathrm{Sym}(M)$ has the presentation

$$R[y_1, \dots, y_n]/J,$$

where $J = (g_1, \dots, g_m)$ and $g_i = \sum_{j=1}^{n} a_{ij} y_j$ with $i = 1, \dots, m$.

We consider $P = R[y_1, \dots, y_n]$ a graded $R$-algebra assigning to each variable $y_i$ the degree 1 and to the elements of $R$ the degree 0. Then $J$ is a graded ideal, and $\mathrm{Sym}(M)$ a graded $R$-algebra.

Let $<$ a monomial order induced by $y_1 < \cdots < y_n$. For $f \in P$, $f = \sum_\alpha a_\alpha y^\alpha$ we put $\mathrm{in}(f) = a_\alpha y^\alpha$ where $y^\alpha$ is the largest monomial with respect to the given order such that $a_\alpha \neq 0$. We call $\mathrm{in}(f)$ the initial term of $f$. Note that in contrast to ordinary Gröbner basis theory the base ring of our polynomial ring $P$ is not a field. Nevertheless we may define the ideal

$$\mathrm{in}(J) = (\mathrm{in}(f) : f \in J).$$

This ideal is generated by terms which are monomials in $y_1, \dots, y_n$ with coefficients in $R$, and is finitely generated since $P$ is Noetherian.

For $i = 1, \dots, n$ we set $M_i = \sum_{j=1}^{i} R f_j$, and let $I_i = M_{i-1} :_R f_i = \{a \in R \colon a f_i \in M_{i-1}\}$. We also set $I_0 = 0$. Note that $I_i$ is the annihilator of the cyclic module $M_i/M_{i-1} \cong R/I_i$.

It is clear that

$$(I_1 y_1, \dots, I_n y_n) \subseteq \mathrm{in}(J),$$

and the two ideals coincide in degree 1.

**Definition 1.1.** *The generators $f_1, \dots, f_n$ of $M$ are called an $s$-sequence (with respect to $<$), if*

$$(I_1 y_1, \dots, I_n y_n) = \mathrm{in}(J).$$

*If in addition $I_1 \subset I_2 \subset \cdots \subset I_n$, then $f_1, \dots, f_n$ is called a strong $s$-sequence.*

Since $\mathrm{Sym}(\mathfrak{m}) = P/J$ may be viewed as the general fiber of a 1-parameter flat family whose special fiber is $P/\mathrm{in}(J)$, invariants of $\mathrm{Sym}(\mathfrak{m}) = S/J$ compared with the corresponding invariants of $P/\mathrm{in}(J)$ can only be better. Thus, for example, if $x_1, \dots, x_n$ is a strong $s$-sequence one has

$$\begin{aligned} \operatorname{depth} \mathrm{Sym}(\mathfrak{m}) &\geq \operatorname{depth} R[y_1, \dots, y_n]/(I_1 y_1, \dots, I_n y_n) \\ &\geq \min\{\operatorname{depth} R/I_i + i \colon i = 0, 1, \dots, n\}, \end{aligned}$$

see [8, Proposition 2.5].

In the same spirit we may do the following comparison: let $I \subset S$ be a graded ideal, and $\mathfrak{m}$ the graded maximal ideal of $R = S/I$. Let furthermore $<$ be a term order on $S$, $\operatorname{in}(I)$ the initial ideal of $I$, $R^* = S/\operatorname{in}(I)$, and $\mathfrak{m}^*$ the graded maximal ideal of $R^*$. How are the invariants of $\operatorname{Sym}(\mathfrak{m})$ and $\operatorname{Sym}(\mathfrak{m}^*)$ related to each other? At least for the depth there seems not to be no obvious relationship as the following examples demonstrate.

**Example 1.2.** Let $<$ be the lexicographical order induced by $y_3 > y_2 > y_1 > x_3 > x_2 > x_1$, and let $I = (x_1x_3 - x_2^2, x_1x_2 - x_1^2)$. Then $\operatorname{in}(I) = (x_1x_2, x_1x_3, x_2^3)$,

$$\operatorname{depth}\operatorname{Sym}(\mathfrak{m}) = 1, \quad \text{and} \quad \operatorname{depth}\operatorname{Sym}(\mathfrak{m}^*) = 2.$$

On the other hand, let $I=(x_2x_3-x_1^2, x_2x_3-x_3^2)$. Then $\operatorname{in}(I)=(x_3^2, x_2x_3, x_1^2x_3, x_1^2x_2^2)$,

$$\operatorname{depth}\operatorname{Sym}(\mathfrak{m}) = 1, \quad \text{and} \quad \operatorname{depth}\operatorname{Sym}(\mathfrak{m}^*) = 0.$$

These examples show that we need some extra hypotheses. Let $L \subset S$ be any graded ideal and $f \in L$, $f \neq 0$. We set

$$v_L(f) = \max\{j : f \in \mathfrak{n}^j L\}.$$

For systematic reasons we set $v_L(f) = \infty$, if $f = 0$.

Let $R(\mathfrak{n}) = \bigoplus_j \mathfrak{n}^j t^j$ be the Rees ring of the graded maximal ideal $\mathfrak{n}$ of $S$. Then

$$R(\mathfrak{n}) = S[x_1t, \dots, x_nt] \subset S[t].$$

The function $v_L : L \to \mathbb{Z}$ has the following interpretation:

**Lemma 1.3.** *Let $f_1, \dots, f_m$ be a homogeneous system of generators of $L$, and consider the ideal $C = (L, Lt) = (f_1, \dots, f_m, f_1t, \dots, f_mt)$ in $R(\mathfrak{n})$. Let $f \in L$ be a homogeneous element and $a \in \mathbb{Z}$, $a \geq 0$. Then*

$$ft^a \in C \iff a \leq v_L(f) + 1.$$

*Proof.* Suppose that $a \leq v_L(f) + 1$. We may assume that $f \neq 0$ and $a > 0$. Otherwise it is trivial that $ft^a \in C$. Let $j = v_L(f)$. Then $f \in \mathfrak{n}^j L$. Hence $f = \sum_{i=1}^m g_i f_i$ with all $g_i$ homogeneous of degree $\geq j$. For $a \in \mathbb{Z}$ with $0 < a \leq j+1$ we write $ft^a = \sum_{i=1,\dots,m} g_i t^{a-1} f_i t$. Note that if $g \in S$ is homogeneous of degree $k$, then $gt^a \in R(\mathfrak{n})$ if and only if $a \leq k$. Therefore $g_i t^{a-1} \in R(\mathfrak{n})$ for all $i$, and hence $ft^a \in C$.

Conversely suppose that $ft^a \in C$. We note that $R(\mathfrak{n})$ is bigraded, $C$ is a bigraded ideal, and $ft^a$ is bihomogeneous, if we assign the following bidegrees to the generators of $R(\mathfrak{n})$:

$$\deg x_i = (1,0) \quad \text{and} \quad \deg x_it = (0,1) \quad \text{for all} \quad i = 1, \dots, n.$$

Thus we can write $ft^a$ as a linear combination

$$ft^a = \sum_{i=1}^{n} g_i t^a f_i + \sum_{i=1}^{n} h_i t^{a-1} f_i t,$$

of the generators of $C$ with bihomogeneous coefficients $g_i t^a, h_i t^{a-1} \in R(\mathfrak{n})$. It follows that $\deg g_i \geq a$ and $\deg h_i \geq a-1$. Here we use the convention that the zero-polynomial has degree $\infty$. Assuming that $a > j+1$, we have $f = \sum_{i=1}^{n}(g_i + h_i) f_i$ with $g_i + h_i \in \mathfrak{n}^{j+1}$, a contradiction. □

With the introduced notation we have

**Theorem 1.4.** *Suppose that*

$$v_{\mathrm{in}(I)}(\mathrm{in}(f)) \geq v_I(f) \quad \textit{for all} \quad f \in I.$$

*Then*

$$\operatorname{reg} \operatorname{Sym}(\mathfrak{m}) \leq \operatorname{reg} \operatorname{Sym}(\mathfrak{m}^*), \quad \textit{and} \quad \operatorname{depth} \operatorname{Sym}(\mathfrak{m}^*) \leq \operatorname{depth} \operatorname{Sym}(\mathfrak{m}).$$

This theorem is again a consequence of the fact that under the given hypotheses, $\operatorname{Sym}(\mathfrak{m})$ may be viewed as the general fiber of a 1-parameter flat family whose special fiber is $\operatorname{Sym}(\mathfrak{m}^*)$. Indeed, write

$$\operatorname{Sym}(\mathfrak{m}) = R[y_1, \ldots, y_n]/J \quad \text{and} \quad \operatorname{Sym}(\mathfrak{m}^*) = R^*[y_1, \ldots, y_n]/J^*.$$

Let $f_1, \ldots, f_m$ be a set of generators of $I$. Write $f_i = \sum_{j=1}^{n} f_{ij} x_j$ for $i = 1, \ldots, m$, and set

$$J_0 = (\{\sum_{j=1}^{n} f_{ij} y_j\}_{i=1,\ldots,m} \cup \{x_i y_j - x_j y_i\}_{1 \leq i < j \leq n}).$$

Similarly, we define $J_0^*$. Then $J = J_0 \bmod I$, and $J^* = J_0^* \bmod \mathrm{in}(I)$. Hence

$$\operatorname{Sym}(\mathfrak{m}) = S[y_1, \ldots, y_n]/(I, J_0) \quad \text{and} \quad \operatorname{Sym}(\mathfrak{m}^*) = S[y_1, \ldots, y_n]/(\mathrm{in}(I), J_0^*).$$

Let $(\mathcal{M}, <)$ be the totally ordered set of monomials of $S = K[x_1, \ldots, x_n]$ where $<$ is the given monomial order. We define a degree-function $d : S[y_1, \ldots, y_n] \to \mathcal{M}$: let $f \in S[y_1, \ldots, y_n]$, $f = \sum_{\nu,\mu} c_{\nu,\mu} x^\nu y^\mu$. Then

$$d(f) = \max\{x^{\nu+\mu} : c_{\nu,\mu} \neq 0\},$$

and we call

$$\mathrm{in}_d(f) = \sum_{\substack{\nu,\mu \\ x^{\nu+\mu} = d(f)}} c_{\nu,\mu} x^\nu y^\mu.$$

the *initial polynomial* of $f$ (with respect to $d$).

This function satisfies the following conditions: for all $f, g \in S[y_1, \ldots, y_n]$ one has

(a) $d(f+g) \leq \max\{d(f), d(g)\}$ and $d(f+g) = \max\{d(f), d(g)\}$ if $d(f) \neq d(g)$;

(b) $d(fg) = d(f)d(g)$.

Let $L \subset S[y_1, \dots, y_n]$ be an ideal. Let $\mathrm{in}_d(L)$ denote the ideal in $S[y_1, \dots, y_n]$ generated by all initial polynomials $\mathrm{in}_d(f)$ with $f \in L$.

Recall the following concept: given a linear function $\omega : \mathbb{Z}^m \to \mathbb{Z}$ we define the *weight* of a term $u = \lambda x^a$ in $K[x_1, \dots, x_m]$ to be $\omega(a)$. Different terms may have the same weight. Nevertheless we may define for any polynomial $f \in K[x_1, \dots, x_m]$ the *initial polynomial* $\mathrm{in}_\omega(f)$ of $f$ with respect to $\omega$ to be the sum of all terms in $f$ which have maximal weight, and we denote by $\omega(f)$ this maximal weight. Finally, if $L \subset K[x_1, \dots, x_m]$ is an ideal one sets

$$\mathrm{in}_\omega(L) = (\{\mathrm{in}_\omega(f) \colon f \in I\}).$$

We shall need the following result:

**Lemma 1.5.** *For any ideal $L \subset S[y_1, \dots, y_n] = K[x_1, \dots, x_n, y_1, \dots, y_n]$ there exists a weight function $\omega \colon \mathbb{Z}^{2n} \to \mathbb{Z}$ such that $\mathrm{in}_d(L) = \mathrm{in}_\omega(L)$.*

*Proof.* Let $f_1, \dots, f_m \in L$. Just as in ordinary Gröbner basis theory one has the following criterion: we consider the relations of $\mathrm{in}_d(f_1), \dots, \mathrm{in}_d(f_m)$, i.e. the $m$-tupels $r = (r_1, \dots, r_m)$ with $r_i \in S[y_1, \dots, y_m]$ such that $\sum_{i=1}^m r_i \,\mathrm{in}_d(f_i) = 0$. Let $\mathcal{R}$ be a generating set of relations of $\mathrm{in}_d(f_1), \dots, \mathrm{in}_d(f_m)$. Then the following conditions are equivalent:

(a) the initial polynomials $\mathrm{in}_d(f_1), \dots, \mathrm{in}_d(f_m)$ generate $\mathrm{in}_d(L)$;

(b) for each $r \in \mathcal{R}$, the polynomial $f = \sum_{i=1}^m r_i f_i$ can be rewritten as

$$g_1 f_1 + g_2 f_2 + \cdots + g_m f_m,$$

such that $d(f) \geq d(g_i f_i)$ for $i = 1, \dots, m$.

The same criterion holds if we replace everywhere $d$ by $\omega$.

Suppose now that $\mathrm{in}_d(f_1), \dots, \mathrm{in}_d(f_m)$ generate $\mathrm{in}_d(L)$, and that we can find a weight function $\omega$ such that

(i) $\mathrm{in}_\omega(f_i) = \mathrm{in}_d(f_i)$ for $i = 1, \dots, m$;

(ii) $\omega(f) \geq \omega(g_i f_i)$ for $i = 1, \dots, m$ for all (the finitely many) equations in (b).

Then the above criterion and (ii) imply that the polynomials $\mathrm{in}_\omega(f_1), \dots, \mathrm{in}_\omega(f_m)$ generate $\mathrm{in}_\omega(L)$. Therefore (i) yields $\mathrm{in}_d(L) = \mathrm{in}_\omega(L)$.

Now we show how we can choose $\omega$ such that (i) and (ii) are satisfied. Given a polynomial $h \in S[y_1, \dots, y_n]$, let $h_i$, $i = 1, 2, \dots$ be the terms in $h$ such that $d(h) > d(h_i)$. Then we define the (finite) set $\mathcal{P}_h = \{(d(h), d(h_i)) \colon i = 1, 2, \dots\}$ of pairs of monomials in $S$.

Now we consider the finite set of pairs of monomials $\bigcup_{i=1}^m \mathcal{P}_{f_i}$ in $S$, and add to this union the sets of pairs $\mathcal{P}_f \cup \bigcup_{i=1}^m \mathcal{P}_{g_i f_i}$ as well as all the pairs $(d(f), d(g_i f_i))$ $(i = 1, \dots, m)$ which correspond to the finitely many relations in $\mathcal{R}$. Altogether this is a finite set of pairs of monomials $(u, v)$ in $S$ with $u > v$ for each pair. Then [5, Proposition 15.16] asserts that there exists a weight function $\omega_0 \colon \mathbb{Z}^n \to \mathbb{Z}$ such that for each of the pairs $(u, v)$ above, we have $\omega_0(u) > \omega_0(v)$.

The weight function $\omega\colon \mathbb{Z}^{2n} \to \mathbb{Z}$ we are looking for is defined as follows: for $(\nu, \mu) \in \mathbb{Z}^{2n}$ with $\nu, \mu \in \mathbb{Z}^n$ we set

$$\omega(\nu, \mu) = \omega_0(\nu + \mu).$$

We note that for a monomial $u = x^\nu y^\mu$ we have $\omega(u) = \omega_0(d(u))$. Thus our choice of $\omega$ guarantees that the conditions (i) and (ii) are satisfied. $\square$

In the proof of the next lemma we need the following notation: for a monomial $u$ in $S$ we set $m(u) = \inf\{i : x_i \text{ divides } u\}$, and $u' = u/x_{m(u)}$. In particular, $u = u'x_m$ where $m = m(u)$.

The following crucial lemma together with the previous lemma will imply Theorem 1.4.

**Lemma 1.6.** $(\mathrm{in}(I), J_0^*) \subset \mathrm{in}_d(I, J_0)$, *and equality holds if*

$$v_{\mathrm{in}(I)}(\mathrm{in}(f)) \geq v_I(f) \quad \textit{for all} \quad f \in I.$$

*Proof.* Let $f_1, \ldots, f_m$ be a Gröbner basis of $I$, and let $u_i = \mathrm{in}(f_i)$ for $i = 1, \ldots, k$. Then

$$(\mathrm{in}(I), J_0^*) = (u_1, \ldots, u_k, u_1' y_{m_1}, \ldots, u_k' y_{m_k}, \{x_i y_j - x_j y_i\}_{1 \leq i < j \leq n})$$

where $m_i = m(u_i)$ for $i = 1, \ldots, k$.

On the other hand, write $f_i = \sum_{j=1}^n f_{ij} x_j$ for $i = 1, \ldots, k$. Then

$$(I, J_0) = (f_1, \ldots, f_k, \sum_{j=1}^n f_{1j} y_j, \ldots, \sum_{j=1}^n f_{kj} y_j, \{x_i y_j - x_j y_i\}_{1 \leq i < j \leq n}).$$

It is clear that $u_i \in \mathrm{in}_d(I, J_0)$ since $\mathrm{in}_d(f_i) = \mathrm{in}_<(f_i) = u_i$. We also have $x_i y_j - x_j y_i \in \mathrm{in}_d(I, J_0)$ for all $i$ and $j$ since $\mathrm{in}_d(x_i y_j - x_j y_i) = x_i y_j - x_j y_i$.

For each $i$ the presentation $f_i = \sum_{j=1}^n f_{ij} x_j$ may be chosen such that each monomial appearing in $f_i$ appears in exactly one of the summands $f_{ij} x_j$. Then if the leading term $u_i$ of $f_i$ appears in the summand $f_{ij} x_j$, then $(u_i/x_j) y_j = \mathrm{in}_d(\sum_{\ell=1}^n f_{i\ell} y_\ell)$. Thus $(u_i/x_j) y_j \in \mathrm{in}_d(I, J_0)$. However since $x_{m_i} y_j - x_j y_{m_i} \in \mathrm{in}_d(I, J_0)$, we also have $u_i' y_{m_i} \in \mathrm{in}_d(I, J_0)$. This shows that

$$(\mathrm{in}(I), J_0^*) \subset \mathrm{in}_d(I, J_0).$$

We suppose now that $v_{\mathrm{in}(I)}(\mathrm{in}(f)) \geq v_I(f)$ for all $f \in I$. Note first that the ideal $B = (\{x_i y_j - x_j y_i\}_{1 \leq i < j \leq n})$ is contained in the ideal $(\mathrm{in}(I), J_0^*)$ as well as in the ideal $\mathrm{in}_d(I, J_0)$. Thus in order to show that $\mathrm{in}_d(I, J_0) \subset (\mathrm{in}(I), J_0^*)$ it suffices to show that $\mathrm{in}_d(I, J_0)/B \subset (\mathrm{in}(I), J_0^*)/B$.

Let $\varphi\colon S[y_1, \ldots, y_n] \to R(\mathfrak{n})$ be the epimorphism given by

$$\varphi(y_i) = x_i t \quad \text{for} \quad i = 1, \ldots, n.$$

It is known and easy to see that $B = \operatorname{Ker}\varphi$. Hence we have $R(\mathfrak{n}) \cong S[y_1, \dots, y_n]/B$. Therefore, if $f_1, \dots, f_m$ is a reduced Gröbner basis of $I$, then

$$(\operatorname{in}(I), J_0^*)/B = \varphi(\operatorname{in}(I), J_0^*) = (\operatorname{in}(f_1), \dots, \operatorname{in}(f_m), \operatorname{in}(f_1)t, \dots, \operatorname{in}(f_m)t),$$

and

$$(I, J_0)/B = \varphi(I, J_0) = (f_1, \dots, f_m, f_1t, \dots f_mt).$$

Now let $f \in (I, J_0)$. We want to prove that $\varphi(\operatorname{in}_d(f)) \in \varphi(\operatorname{in}(I), J_0^*)$. Since $(I, J_0)$ is bigraded, we may assume that $f$ is bihomogeneous of bidegree $(b, a)$. Set $|\alpha| = \sum_i^n \alpha_i$ for $\alpha = (\alpha_1, \dots, \alpha_n)$. Then $f = \sum_{\nu,\mu} c_{\nu\mu} x^\nu y^\mu$ with $|\nu| = b$ and $|\mu| = a$ for all $\nu$ and $\mu$ in the sum, and $\varphi(f) = gt^a$ where $g = \sum_{\nu,\mu} c_{\nu\mu} x^{\nu+\mu}$ belongs to $I$ and is of degree $a + b$. It follows that either $\varphi(\operatorname{in}_d(f)) = 0$ or $\varphi(\operatorname{in}_d(f)) = \operatorname{in}(g)t^a$. In the first case there is nothing to prove. In the second case, note first that by Lemma 1.3, $a \le v_I(g)+1$, since $gt^a = \varphi(f) \in \varphi(I, J_0) = (f_1, \dots, f_m, f_1t, \dots, f_mt)$. Since by assumption $v_{\operatorname{in}(I)}(\operatorname{in}(g)) \ge v_I(g)$, we obtain that $a \le v_{\operatorname{in}(I)}(\operatorname{in}(g)) + 1$. Again applying Lemma 1.3 we conclude that

$$\varphi(\operatorname{in}_d(f)) = \operatorname{in}(g)t^a \in (\operatorname{in}(f_1), \dots, \operatorname{in}(f_m), \operatorname{in}(f_1)t, \dots, \operatorname{in}(f_m)t) = \varphi(\operatorname{in}(I), J_0^*),$$

as desired. □

*Proof.* [Proof of Theorem 1.4] By Lemma 1.5 there exists a weight function $\omega$ such that $\operatorname{in}_d(I, J_0) = \operatorname{in}_\omega(I, J_0)$. Applying [5, Theorem 15.17] we obtain the following inequalities of graded Betti-numbers

$$\beta_{ij}(I, J_0) \le \beta_{ij}(\operatorname{in}_d(I, J_0)) \quad \text{for all} \quad i, j.$$

The assumptions of Theorem 1.4 and Lemma 1.6 imply that $(\operatorname{in}(I), J_0^*) = \operatorname{in}_d(I, J_0)$, and hence $\beta_{ij}(I, J_0) \le \beta_{ij}(\operatorname{in}(I), J_0*)$ for all $i, j$. This yields the desired inequalities for the depth and regularity of the symmetric algebras under consideration. □

The last result of this section describes a case in which the hypotheses of Theorem 1.4 are satisfied. For a given term order $<$ and $f \in S$, we denote by $\operatorname{inm}_<(f)$ (or simply $\operatorname{inm}(f)$) the initial monomial of $f$.

**Proposition 1.7.** *Let $I \subset S$ be a graded ideal, and $<$ a term order. Suppose there exists a minimal system of homogeneous generators $f_1, \dots, f_m$ of $I$ with the property that for each integer t the set of polynomials $\{f_i \colon \deg f_i \le t\}$ is a Gröbner basis of the ideal they generate. Then $v_{\operatorname{in}(I)}(\operatorname{in}(f)) \ge v_I(f)$ for all homogeneous polynomials $f \in I$.*

*Proof.* Let $f_1, \dots, f_m$ be a minimal system of homogeneous generators of $I$ satisfying the conditions as described in the proposition. Let $f \in I$ be a homogeneous polynomial, and let $v_I(f) = j$. Then $f \in \mathfrak{m}^j I$; hence there exist homogeneous polynomials $g_i \in \mathfrak{m}^j$ such that $f = \sum_{i=1}^m g_i f_i$ and $\deg g_i f_i = \deg f$ for all $i$.

Let $u = \min\{\operatorname{inm}(g_i f_i) \colon i = 1, \dots, m\}$. Then $u \le \operatorname{inm}(f)$. Assume $u < \operatorname{inm}(f)$. Let $S = \{i \colon \operatorname{inm}(g_i f_i) = u\}$. Then $\sum_{i \in S} \operatorname{in}(g_i) \operatorname{in}(f_i) = \sum_{i \in S} \operatorname{in}(g_i f_i) =$

0. Let $t = \max\{\deg f_i\colon i \in S\}$, and suppose without loss of generality that $\deg f_i \le t$ for $i \le r$ and $\deg f_i > t$ for $i > r$. Since by assumption $f_1, \dots, f_r$ is a Gröbner basis and $S \subset \{f_1, \dots, f_r\}$, there exist homogeneous polynomials $h_i$ such that

$$\sum_{i\in S} g_i f_i = \sum_{i=1}^{r} h_i f_i \quad \text{with} \quad u < \operatorname{inm}(h_i f_i) \quad \text{and} \quad \deg h_i f_i = \deg f \quad \text{for} \quad i = 1, \dots, r.$$

Replacing $\sum_{i\in S} g_i f_i$ in the sum $\sum_{i=1}^{m} g_i f_i$ by $\sum_{i=1}^{r} h_i f_i$, we can rewrite $f$ as

$$f = \sum_{i=1}^{m} g_i' f_i \quad \text{with} \quad u < \operatorname{inm}(g_i' f_i) \quad \text{for all } i.$$

Note furthermore that $\deg h_i \ge \deg f - t = \deg g_{i_0} \ge j$, where $i_0 \in S$ is the index with $\deg f_{i_0} = t$. Thus we see that all $h_i \in \mathfrak{m}^j$ for $i = 1, \dots, r$, and hence $g_i' \in \mathfrak{m}^j$ for $i = 1, \dots, m$.

After finitely many steps of rewriting $f$ we may assume that $\operatorname{inm}(f) = \operatorname{inm}(g_i f_i)$ for some $i$. Then we conclude that $v_{\operatorname{in}(I)}(\operatorname{in}(f)) \ge j$, since $\operatorname{in}(g_i f_i) = \operatorname{in}(g_i)\operatorname{in}(f_i)$ and $\operatorname{in}(g_i) \in \mathfrak{m}^j$. $\square$

Recall that a graded ideal $I \subset S$ is called *componentwise linear*, if each component $I_j$ of $I$ generates an ideal with linear resolution.

Let $I$ be a componentwise linear ideal. Fix an integer $t$, and let $I_{\le t}$ be the ideal generated by all components $I_j$ with $j \le t$. Then $I_{\le t}$ is again componentwise linear. In fact, $(I_{\le t})_j = I_j$ for $j \le t$, while for $j > t$ one has $(I_{\le t})_j = S_{j-t} I_t$. Thus all components of $I_{\le t}$ generate ideals with linear resolution.

We now assume that $\operatorname{char} K = 0$. Choose generic coordinates $x_1, \dots, x_n$, and let $<$ be the degree reverse lexicographical term order induced by $x_n > x_{n-1} > \cdots > x_1$. Let $f_1, \dots, f_m$ be a minimal homogeneous set of generators of $I$ such that $\operatorname{inm}(f_1) \le \operatorname{inm}(f_2) \le \cdots \le \operatorname{inm}(f_m)$. It follows from [7, Theorem 1.1] that such a minimal system of generators of $I$ is Gröbner basis of $I$. Therefore, since for each integer $t$, the ideal $I_{\le t}$ is componentwise linear it follows that $f_1, \dots, f_{m_1}$ is a Gröbner basis of $I_{\le t}$, where $m_t = \max\{i\colon \deg f_i \le t\}$. Hence we may apply Proposition 1.7 and Theorem 1.4, and obtain

**Corollary 1.8.** *Suppose* $\operatorname{char} K = 0$. *Let* $I \subset S$ *be componentwise linear ideal. Choose generic coordinates* $x_1, \dots, x_n$, *and let* $<$ *be the degree reverse lexicographical term order induced by* $x_1 < x_2 < \cdots < x_n$. *Then*

$$\operatorname{reg}\operatorname{Sym}(\mathfrak{m}) \le \operatorname{reg}\operatorname{Sym}(\mathfrak{m}^*), \quad \textit{and} \quad \operatorname{depth}\operatorname{Sym}(\mathfrak{m}^*) \le \operatorname{depth}\operatorname{Sym}(\mathfrak{m}).$$

## 2. Algebras with monomial relations whose maximal ideal is generated by an $s$-sequence

Let $K$ be a field, $S = K[x_1, \dots, x_n]$ be a polynomial ring, and $I \subset S$ a monomial ideal. We denote by $G(I) = \{u_1, \dots, u_r\}$ the unique minimal monomial set of generators of $I$, and by $\mathfrak{m}$ the graded maximal ideal of $R = S/I$.

For a monomial $u$ in $S$ we set $m(u) = \inf\{i : x_i|u\}$ and $u' = u/x_{m(u)}$. Then in particular, $u_i = u' x_{m_i}$ where $m_i = m(u_i)$.

Let $J_0 \subset S[y_1, \dots, y_n]$ be the ideal which is generated by

$$G_0 = \{u_i' y_{m_i} : i = 1, \dots, r\} \cup \{x_i y_j - x_j y_i : 1 \le i < j \le n\},$$

and let $J \subset P = R[y_1, \dots, y_n]$ be the ideal which is generated by the residue classes of the elements in $G_0$ modulo $I$.

Then we have

$$\mathrm{Sym}_R(\mathfrak{m}) \cong P/J \cong S[y_1, \dots, y_n]/(I, J_0).$$

We fix a term order $<$ on $P$ induced by $y_1 < y_2 < \cdots < y_n$ for which we want to compute $\mathrm{in}_<(J)$. To this end we extend the given order on the monomials in the variables $y_i$ to a term order $\prec$ order satisfying $x_1 < x_2 < \cdots < x_n < y_1 < y_2 < \cdots < y_n$. Since $I$ is a monomial ideal it follows that

$$\mathrm{in}_\prec(I, J_0) = (I, L'),$$

for some monomial ideal $L' \subset S[y_1, \dots, y_n]$. Let $L \subset P$ be the image of $L'$. Then

**Lemma 2.1.** $\mathrm{in}_<(J) = L$.

*Proof.* For a graded module $M$ we denote by $H_M(t) = \sum_i \dim_K M_i t^i$ the Hilbert series of $M$. We claim that

(1) $L \subset \mathrm{in}(J)$;

(2) $H_{P/J}(t) = H_{P/\mathrm{in}(J)}(t)$.

By (1) and (2) it follows that $L = \mathrm{in}(J)$ if and only if $H_{P/L}(t) = H_{P/J}(t)$. In order to prove this equality of Hilbert series we use Macaulay's theorem (see [3, Corollary 4.2.4]) and the isomorphism $P/J \cong S[y_1, \dots, y_n]/(I, J_0)$, and get

$$\begin{aligned} H_{P/J}(t) &= H_{S[y_1,\dots,y_n]/(I,J_0)}(t) = H_{S[y_1,\dots,y_n]/\mathrm{in}(I,J_0)}(t) \\ &= H_{S[y_1,\dots,y_n]/(I,L')}(t) = H_{P/L}(t), \end{aligned}$$

as desired.

Proof of (1): We view $S[y_1, \dots, y_n]$ as a $\mathbb{Z}^n$-graded $K$-algebra by setting for $i = 1, \dots, n$, $\deg x_i = \deg y_i = (0, \dots, 0, 1, 0, \dots, 0)$ where the entry 1 is at the $i$-th position. Notice that $(I, J_0)$ is multi-homogeneous.

Let $g \in (I, J_0)$ be a multi-homogeneous element. We will show that if $\overline{\mathrm{in}(g)} \ne 0$, then $\overline{\mathrm{in}(g)} = \mathrm{in}(\overline{g})$ where $\bar{f}$ denotes the residue class modulo $I$ of an element $f \in S[y_1, \dots, y_n]$. From this observation assertion (1) will follow.

Let $g = \sum_a v_a y^a$ where the sum is taken over all $a \in \mathbb{N}^n$ and where the coefficients $v_a$ are monomials in the variables $x_1, \dots, x_n$, with all but finitely many $v_a$ are zero.

Let $\mathrm{in}(g) = v_{a_0} y^{a_0}$, and assume that $v_{a_0} \notin I$. Then $\overline{\mathrm{in}(g)} \ne 0$, and $\overline{g} = \sum_a \overline{v}_a y^a$. Suppose $\overline{\mathrm{in}(g)} \ne \mathrm{in}(\overline{g})$. Then there exists $a_1$ such that $\overline{v}_{a_1} y^{a_1} > \overline{v}_{a_0} y^{a_0}$.

Then this means that $y^{a_1} > y^{a_0}$. Since $v_{a_0}y^{a_0}$ and $v_{a_1}y^{a_1}$ have the same multi-degree and since $y_j > x_i$ for all $i$ and $j$, it follows that $v_{a_1}y^{a_1} > v_{a_0}y^{a_0}$, a contradiction.

Proof of (2): We show that for each multi-degree $a$ the multi-graded components $J_a$ and $\mathrm{in}(J)_a$ have the same $K$-dimension.

If $g \in R[y_1, \dots, y_n]$ with $\mathrm{in}(g) = uy^c$ with $u \in R$, then we let $\mathrm{inm}(g) = y^c$ be the initial term of this $g$.

Now let $g_1, \dots, g_s$ a $K$-basis of $J_a$ where the $g_j$ are multi-graded with $\deg g_i = a$. We may assume that $\mathrm{inm}(g_1) \geq \mathrm{inm}(g_2) \geq \cdots \geq \mathrm{inm}(g_s)$. We claim that we can modify this $K$-basis such that $\mathrm{inm}(g_1) > \mathrm{inm}(g_2) > \cdots > \mathrm{inm}(g_s)$. In fact, suppose that for $g_1, \dots, g_m$, $(m \leq s)$ we have $\mathrm{inm}(g_1) = \mathrm{inm}(g_2) = \cdots = \mathrm{inm}(g_m)$. Then since the $g_i$ multi-homogeneous all of same degree $a$ this implies that there exist $\lambda_i \in K$, $\lambda_i \neq 0$, such that $\mathrm{in}(g_i) = \lambda_i \,\mathrm{in}(g_1) = \mathrm{in}(\lambda_i g_1)$ for $i = 1, \dots, m$. We replace $g_1, \dots, g_s$ by $g'_1, \dots, g'_s$ where $g'_1 = g_1, g'_i = g_i - \lambda_i g_1$, for $i = 2, \dots, m$ and $g'_i = g_i$ for $i = m+1, \dots, s$. Then $g'_1, \dots, g'_s$ is again a basis of $J_a$ and $\mathrm{inm}(g'_1) > \mathrm{inm}(g'_i)$ for all $i$.

After renumbering we may assume that

$$\mathrm{inm}(g'_1) > \mathrm{inm}(g'_2) \geq \mathrm{inm}(g'_3) \geq \cdots \geq \mathrm{inm}(g'_s).$$

Applying the same argument to $g'_2, \dots, g'_s$ and using induction on $s$, the claim follows.

In particular, the initial terms $\mathrm{in}(g_1), \dots, \mathrm{in}(g_s)$ are linearly independent over $K$. Therefore we conclude that $\dim_K \mathrm{in}(J)_a \geq \dim_K J_a$. The opposite inequality is proved similarly. $\square$

Now we are ready to prove the main result of this section.

**Theorem 2.2.** *Let $K$ be a field, $S = K[x_1, \dots, x_n]$ the polynomial ring, $I \subset S$ a monomial ideal with $G(I) = \{u_1, \dots, u_r\}$, and $R = S/I$. Then for any term order $<$ induced by $x_1 < x_2 < \cdots < x_n < y_1 < y_2 < \cdots < y_n$, the following conditions are equivalent:*

(a) *$G = \{u_1, \dots, u_r\} \cup \{u'_1y_1, \dots, u'_ry_r\} \cup \{x_iy_j - x_jy_i \colon 1 \leq i < j \leq n\}$ is a Gröbner basis of $(I, J_0)$.*

(b) *For all $u \in G(I)$ and all $j > m(u')$ either*

  (i) *$u'x_j/x_{m(u')} \in I$, or*

  (ii) *there exists $v \in G(I)$ such that either*

  $m(v) = m(u)$, *or*

  $m(v) = m(u')$ *and $v'$ divides $u'x_j/x_{m(u')}$.*

*If the equivalent conditions hold, then the elements $x_1, \dots, x_n$ form an $s$-sequence in $R$.*

*Proof.* Note that $G$ is a Gröbner basis of $(I, J_0)$ if and only if all $S$-pairs of $G$ reduce to zero. The only $S$-pairs of $G$ which do not trivially reduce to zero are the $S$-pairs $S(u'y_{m(u)}, x_iy_j - x_jy_i)$ with $u \in G(I)$, $i < j$ and $x_i$ divides $u'$. In that case we have

$$S(u'y_{m(u)}, x_iy_j - x_jy_i) = \frac{u'x_j}{x_i} y_{m(u)}y_i.$$

If $i > m(u')$, then

$$\frac{u'x_j}{x_{m(u')}} y_{m(u)}y_{m(u')} = \frac{u'x_j}{x_i} y_{m(u)}y_i - \frac{u'x_j}{x_ix_{m(u')}} y_{m(u)}(x_{m(u')}y_i - x_iy_{m(u')}).$$

Therefore it suffices to see in which cases $(u'x_j/x_{m(u')})y_{m(u)}y_{m(u')}$ reduces to zero. Since all integers $k$ for which $x_k$ divides $u'x_j/x_{m(u')}$ are $\geq m(u')$, the relations $x_iy_j - x_jy_i$ with $i < j$ can not be used for further reductions.

Therefore it follows that $(u'x_j/x_{m(u')})y_{m(u)}y_{m(u')}$ reduces to zero if and only either $u'x_j/x_{m(u')} \in I$ or $v'y_{m(v)}$ divides $(u'x_j/x_{m(u')})y_{m(u)}y_{m(u')}$ for some $v \in G(I)$. This is exactly condition (b). □

**Corollary 2.3.** *Suppose condition* (b) *of Theorem* 2.2 *is satisfied. Then*

(a) $\mathrm{in}(I, J_0) = (u_1, \ldots, u_r, u'_1y_{m_1}, \ldots, u'_ry_{m_r}, \{x_iy_j\}_{1\leq i<j\leq n})$;

(b) $\mathrm{in}(J)$ *is generated by the residue classes modulo* $I$ *of the set of monomials*

$$\{u'_1y_{m_1}, \ldots, u'_ry_{m_r}\} \cup \{x_iy_j\}_{1\leq i<j\leq n}.$$

*In particular, the annihilator ideals of* $x_1, \ldots, x_n$ *are*

$$I_j = [(x_1, \ldots, x_{j-1}) + L_j] \bmod I$$

*with* $L_j = (\{u' \colon u \in G(I)$ *and* $m(u) = j\})$ *for* $j = 1, \ldots, n$.

As a first application of Theorem 2.2 we have

**Proposition 2.4.** *Let* $I$ *be a monomial ideal generated in degree* 2*. Then following conditions are equivalent:*

(a) $x_1, \ldots, x_n$ *is an s-sequence in R;*

(b) *for all monomials* $x_ix_j \in I$ *with* $i \leq j$ *and for all* $k > i$ *either* $x_ix_k \in I$ *or* $x_jx_k \in I$.

*If the equivalent conditions hold, then* $\mathrm{Sym}(\mathfrak{m})$ *is a Koszul algebra.*

*Proof.* It is obvious that for a monomial ideal which is generated in degree 2 the condition (b) in this proposition is equivalent to condition (b) in Theorem 2.2. Therefore we have the equivalence of (a) and (b).

If the equivalent conditions hold, then the Gröbner basis of the defining ideal $J$ of $\mathrm{Sym}(\mathfrak{m})$ is generated by quadratic forms. It is well known that this implies that $\mathrm{Sym}(\mathfrak{m})$ is Koszul. □

## 3. Algebras defined by stable monomial ideals

As in the previous section we let $I \subset S = K[x_1, \ldots, x_n]$ be a monomial ideal, and denote by $R$ the standard graded $K$-algebra $S/I$, and by $\mathfrak{m}$ the graded maximal ideal of $R$. Without loss of generality we may assume that $I \subset \mathfrak{m}^2$.

Let $I$ be a monomial ideal. We say that $I$ is *stable (resp. strongly stable) in the reverse order* if for all monomials $u \in I$, one has that $u'x_j \in I$ for all $j > m(u)$ (resp. $(u/x_k)x_j \in I$ for all $j > k$ and all $k$ such that $x_k$ divides $u$).

Note that if we renumber the variables so that $x_i$ becomes $x_{n-i+1}$ for $i = 1, \ldots, n$, then an ideal which is stable in the reverse order becomes an ideal which is stable in the usual sense.

**Proposition 3.1.** *$x_1, \ldots, x_n$ is a strong s-sequence in $R$ if $I$ is a stable ideal in the reverse order.*

*Proof.* Let $u \in G(I)$. Since $I$ is stable in the reverse order, we have that $u'x_j \in I$ for all $j > m(u)$. Therefore there exists $v \in G(I)$, and a monomial $w$ such that $u'x_j = vw$. Since $x_{m(u')}$ divides $u'$, it follows that $x_{m(u')}$ divides $vw$. In case, $x_{m(u')}$ divides $w$, one has $u'x_j/x_{m(u')} \in I$, and condition (b)(i) of Theorem 2.2 is satisfied. On the other hand, if $x_{m(u')}$ divides $v$, then $v/x_{m(u')} = v'$ and $u'x_j/x_{m(u')} = v'w$, and condition (b)(ii) of Theorem 2.2 is satisfied, therefore $x_1, \ldots, x_n$ is an $s$-sequence.

To conclude the proof we have to show $I_1 \subset I_2 \subset \cdots \subset I_n$. By Corollary 2.3,

$$I_j = [(x_1, \ldots, x_{j-1}) + L_j] \bmod I.$$

Let $v \in (x_1, \ldots, x_{j-1}) + L_j$. If $x_i$ divides $v$ for $i = 1, \ldots, j$, then $v \in (x_1, \ldots, x_j) + L_{j+1}$. Therefore we assume that $v \in L_j$, and that $i > j$ whenever $x_i$ divides $v$. Then $v = u'w$ with $u \in G(I)$, $m(u) = j$ and $m(u') > j$. Since $I$ is stable, $u'x_{j+1} \in I$, and since $m(u') > j$, $m(u'x_{j+1}) = j + 1$. Thus $u'x_{j+1} = gh$ where $g \in G(I)$, $h$ is a monomial and $m(gh) = j+1$. If $m(g) = j+1$, then $g' \in L_{j+1}$, and hence $v = g'hw$ belongs to $L_{j+1}$. Otherwise, $m(h) = j+1$, and then $v = gh'w \in I$. Therefore, $v \bmod I = 0 \in I_{j+1}$. □

**Examples 3.2.** (a) Let $I = (x_1, \ldots, x_n)^d$. Then $I$ is stable in the reverse order, and so by Proposition 3.1 the sequence $x_1, \ldots, x_n$ is an $s$-sequence in $R$. Using Corollary 2.3 we see that the annihilator ideals of $x_1, \ldots, x_n$ are

$$I_j = (x_1, \ldots, x_{j-1}, (x_{j+1}, \ldots, x_n)^{d-1}) \bmod I.$$

In particular, $x_1, \ldots, x_n$ is a strong $s$-sequence. Using the formulas [8, Proposition 2.4] we get

$$\dim \mathrm{Sym}(\mathfrak{m}) = n \quad \text{and} \quad e(\mathrm{Sym}(\mathfrak{m})) = d - 1,$$

where $e(\mathrm{Sym}(\mathfrak{m}))$ denotes the multiplicity. Note that by the Huneke-Rossi formula [11] one has $\dim \mathrm{Sym}(\mathfrak{m}) = n$ in general.

(b) The ideal $I = (x_1x_2, x_1x_3, x_2x_4, x_3x_4)$ satisfies condition (b) of Theorem 2.2, but is not stable in the reverse order.

For the proof of the next result we need a few lemmata on stable ideals and basic facts on graded resolutions.

**Lemma 3.3.** *Suppose that $I$ is strongly stable in the reverse order. Then for $j = 1, \dots, n$, the ideals $(I, L_j)$ with $L_j = (u')_{u \in G(I), m(u)=j}$ as defined in Corollary 2.3 are strongly stable in the reverse order.*

*Proof.* Let $v \in (I, L_j)$ be a monomial, and suppose that $x_i$ divides $v$. For all $k > i$ we want to show that $(v/x_i)x_k \in (I, L_j)$. Since $I$ is strongly stable in the reverse order, we may assume that $v \in L_j$. Then there exists a monomial $u \in G(I)$ with $m(u) = j$ such that $v = u'w$ for some monomial $w$.

If $x_i | w$, then $(v/x_i)x_k = u'(w/x_i)x_k \in L_j$. If $x_i | u'$, then $x_i | u$ and hence $(u/x_i)x_k \in I$, since $I$ is strongly stable in the reverse order.
Moreover, $m((u/x_i)x_k) = j$. Hence $(u/x_i)x_k = gh$ where $g \in G(I)$ and $h$ is a monomial, such that either $m(g) = j$ or $m(h) = j$. If $m(h) = j$, then $(v/x_i)x_k = gh'w \in I$, and if $m(g) = j$, then $(v/x_i)x_k = g'hw \in L_j$. □

**Lemma 3.4.** *Let $I$ and $J$ be graded ideals in $S$ such that $\operatorname{Tor}_1^S(S/I, S/J) = 0$. Then $\operatorname{reg}(I + J) \geq \operatorname{reg} I$, and equality holds if and only if $J$ is generated by linear forms.*

*Proof.* Let $\mathbb{F}$ be the graded minimal free resolution of $S/I$, and $\mathbb{G}$ the graded minimal free resolution of $S/J$. Then $\operatorname{Tor}_1^S(S/I, S/J) = 0$ implies that $\mathbb{F} \otimes \mathbb{G}$ is the graded minimal free resolution of $S/(I + J)$.

Let $P = \sum_{i,j} \beta_{i,i+j}(S/I)x^i y^{i+j}$ be the graded Poincaré series of $S/I$ and $Q = \sum_{i,j} \beta_{i,i+j}(S/J)x^i y^{i+j}$ the graded Poincaré series of $S/J$. Then $PQ$ is the graded Poincaré series of $S/(I + J)$. One has $\operatorname{reg} S/I = \deg_y P$, where $\deg_y P$ denotes the $y$-degree of $P$. Similarly, $\operatorname{reg} S/J = \deg_y Q$ and $\operatorname{reg} S/(I + J) = \deg_y PQ = \deg_y P + \deg_y Q$. It follows that $\operatorname{reg} S/I = \operatorname{reg} S/J$ if and only $\deg_y Q = 0$, and this is the case if and only if $J$ is generated by linear forms. □

For a graded ideal $I \subset S$ and an integer $j$ we denote by $I_{\geq j}$ the ideal generated by all homogeneous elements $f \in I$ with $\deg f \geq j$.

**Lemma 3.5.** *Let $I \subset S$ be a graded ideal. Then the natural map*

$$\operatorname{Tor}_i(I_{\geq j}, K)_{i+j} \to \operatorname{Tor}_i(I, K)_{i+j}$$

*is surjective for all $i$ and $j$.*

*Proof.* The short exact sequence

$$0 \longrightarrow I_{\geq j} \longrightarrow I \longrightarrow I/I_{\geq j} \longrightarrow 0$$

induces the long exact sequence

$$\operatorname{Tor}_i(I_{\geq j}, K)_{i+j} \longrightarrow \operatorname{Tor}_i(I, K)_{i+j} \longrightarrow \operatorname{Tor}_i(I/I_{\geq j}, K)_{i+j}.$$

Note that $(I/I_{\geq j})_k = 0$ for $k \geq j$. Let $K(\mathbf{x}; I/I_{\geq j})$ be the Koszul complex of the sequence $\mathbf{x} = x_1, \dots, x_n$ with values in $I/I_{\geq j}$. Then $K_i(\mathbf{x}; I/I_{\geq j})_{i+j} = 0$. Now since $\operatorname{Tor}_i(I/I_{\geq j}, K)_{i+j} \cong H_i(\mathbf{x}; I/I_{\geq j})_{i+j}$, we conclude that $H_i(\mathbf{x}; I/I_{\geq j})_{i+j} = 0$, as desired. □

**Lemma 3.6.** *Let $I \subset J \subset S$ be graded ideals, and $f \in S$ a linear form. Suppose that $f$ is a non-zerodivisor on $S/I$ and on $S/J$ and that for some $j$,*

$$\mathrm{Tor}_i(I, K)_{i+j} \to \mathrm{Tor}_i(J, K)_{i+j}$$

*is surjective for all $i$. Then*

$$\mathrm{Tor}_i((I, f), K)_{i+j} \to \mathrm{Tor}_i((J, f), K)_{i+j}$$

*is surjective for all $i$.*

*Proof.* Let $\mathbb{F}$ be the graded minimal free resolution of $S/I$, and $\mathbb{G}$ the graded minimal free resolution of $S/J$. Let $\alpha\colon \mathbb{F} \to \mathbb{G}$ be the complex homomorphism induced by $I \subset J$. We denote by $\alpha_{i,j}$ the $j$th graded component of $\alpha_i$. Then the map $\mathrm{Tor}_i(I, K)_{i+j} \to \mathrm{Tor}_i(J, K)_{i+j}$ can be identified with $\bar{\alpha}_{i,i+j}\colon (F_i/\mathfrak{m}F_i)_{i+j} \to (G_i/\mathfrak{m}G_i)_{i+j}$, where $\bar{\alpha}_{i,i+j}$ denotes the $i+j$th graded component of $\bar{\alpha}_i = \alpha_i \otimes S/\mathfrak{m}$.

The resolution of $S/(I, f)$ is given by $\mathbb{F} \otimes \mathbb{H}$ where $\mathbb{H}$ is the complex

$$0 \longrightarrow S(-1) \xrightarrow{\ f\ } S \longrightarrow 0.$$

Similarly, the resolution of $S/(J, f)$ is given by $\mathbb{G}\otimes\mathbb{H}$. Hence the inclusion $(I, f) \subset (J, f)$ can be lifted by the complex homomorphism $\alpha \otimes \mathrm{id}$. Thus for all $i$ and $j$ we have

$$(\alpha \otimes \mathrm{id})_{i,i+j} = \alpha_{i,i+j} \oplus \alpha_{i-1,i+j-1}\colon (F_i)_{i+j} \oplus (F_{i-1})_{i+j-1} \longrightarrow (G_i)_{i+j} \oplus (G_{i-1})_{i+j-1},$$

which induces the homomorphisms

$$(\overline{\alpha \otimes \mathrm{id}})_{i,i+j}\colon (F_i/\mathfrak{m}F_i)_{i+j}\oplus(F_{i-1}/\mathfrak{m}F_{i-1})_{i+j-1}\to(G_i/\mathfrak{m}G_i)_{i+j}\oplus G_{i-1}/\mathfrak{m}G_{i-1})_{i+j-1}.$$

Since $(\overline{\alpha \otimes \mathrm{id}})_{i,i+j} = \bar{\alpha}_{i,i+j} \oplus \bar{\alpha}_{i-1,i+j-1}$ and since $\bar{\alpha}_{i,i+j}$ is surjective for all $i$, it follows that $(\overline{\alpha \otimes \mathrm{id}})_{i,i+j}$ is surjective, as desired. □

**Theorem 3.7.** *Let $R = S/I$ where $I$ is a strongly stable ideal in the reverse order, let $u_1, \dots, u_r$ be the Borel generators of $I$ and $d = \max\{\deg(u_i)\colon i = 1, \dots, r\}$. Then*

(a) $\operatorname{reg} R \le \operatorname{reg} \operatorname{Sym}_R(\mathfrak{m}) \le \operatorname{reg} R + 1$;

(b) $\operatorname{reg} R = \operatorname{reg} \operatorname{Sym}_R(\mathfrak{m}) \Longleftrightarrow \max\{m(u_i) : \deg(u_i) = d\} \le 2$.

*Proof.* (a) By the Eliahou-Kervaire resolution of $I$ (see [6]) the regularity of $I$ equals $d$ since $I$ is stable in the reverse order. Hence it amounts to show that $d \le \operatorname{reg}(I, J_0) \le d + 1$.

Since the highest degree of a generators of $(I, J_0)$ is $d$ it follows that $d \le \operatorname{reg}(I, J_0)$. In order to prove the upper inequality, it suffices to show that $\operatorname{reg} \operatorname{in}(I, J_0) \le d + 1$ since $\operatorname{reg}(I, J_0) \le \operatorname{reg} \operatorname{in}(I, J_0)$.

For $j = 1, \dots, n$ we consider the ideal

$$K_j = (I, I_1' y_1, I_2' y_2, \dots, I_j' y_j),$$

where $I'_j = (x_1, \ldots, x_{j-1}) + (I, L_j)$, and we set $K_0 = I$. Recall from Corollary 2.3 that the ideals $I_j = I'_j \bmod I$ are the annihilator ideals of $x_1, \ldots, x_n$.

We will show by induction on $j$ that $\operatorname{reg} K_j \leq d+1$. This implies the upper bound, since by Corollary 2.3 we have $\operatorname{in}(I, J_0) = K_n$.

Since $K_0 = I$ is strongly stable in the reverse order, we have $\operatorname{reg} K_0 = d$.

Now let $j > 0$ and assume that $\operatorname{reg} K_{j-1} \leq d+1$. We have

$$K_j = (K_{j-1}, I'_j y_j) = (K_{j-1}, I'_j) \cap (K_{j-1}, y_j),$$

and $I'_1 \subset I'_2 \subset \cdots \subset I'_j$, since $I_1 \subset I_2 \subset \cdots \subset I_j$ by Proposition 3.1. It follows that $(K_{j-1}, I'_j) = I'_j$. Hence we obtain the exact sequence

$$0 \longrightarrow K_j \longrightarrow I'_j \oplus (K_{j-1}, y_j) \longrightarrow (I'_j, y_j) \longrightarrow 0.$$

This together with Lemma 3.4 implies that

$$\begin{aligned} \operatorname{reg} K_j &\leq \max\{\operatorname{reg} I'_j, \operatorname{reg}(K_{j-1}, y_j), \operatorname{reg}(I'_j, y_j) + 1\} \\ &= \max\{\operatorname{reg} K_{j-1}, \operatorname{reg} I'_j + 1\}. \end{aligned}$$

By induction hypothesis $\operatorname{reg} K_{j-1} \leq d+1$. Hence it remains to show that $\operatorname{reg} I'_j \leq d$.

For a monomial ideal $H$ we denote by $H^{\geq j}$ the ideal generated by all monomials $u \in H$ with $m(u) \geq j$. Then we have $I'_j = (x_1, \ldots, x_{j-1}) + (I, L_j)^{\geq j}$. Therefore, by Lemma 3.4, $\operatorname{reg} I'_j = \operatorname{reg}(I, L_j)^{\geq j}$. In Lemma 3.3 it is shown that $(I, L_j)$ is strongly stable in the reverse order. It is clear that then also $(I, L_j)^{\geq j}$ is strongly stable in the reverse order, and that the highest degree of a Borel generator of $(I, L_j)^{\geq j}$ is $\leq d$. This implies that $\operatorname{reg}(I, L_j)^{\geq j} \leq d$, as desired.

(b) Let $m = \max\{m(u_i) : \deg(u_i) = d\}$, and assume $m \leq 2$. Since $d \leq \operatorname{reg}(I, J_0) \leq \operatorname{reg} \operatorname{in}(I, J_0)$, and since $\operatorname{in}(I, J_0) = K_n$ it suffices to prove that $\operatorname{reg}(K_n) = d$. In fact, we show by induction on $j$ that $\operatorname{reg} K_j \leq d$ for $j = 0, \ldots, n$. We first consider the case $m = 1$. The induction begin is trivial because $K_0 = I$. The assumption $m = 1$ implies that $(I, L_j)^{\geq j}$ is generated in degree $\leq d-1$ for all $j$, and this implies $\operatorname{reg}(I, L_j)^{\geq j} \leq d-1$ for all $j$. Arguing as in the proof of (a) it follows that $\operatorname{reg} K_j \leq d$ for all $j$.

Now assume that $m = 2$. Again we show by induction on $j$ that $\operatorname{reg} K_j \leq d$. For $j = 0$ the assertion is trivial. We must also consider the case $j = 1$. Since $K_1 = (I, I'_1 y_1) = (I, L_1) \cap (I, y_1)$ we obtain the exact sequence

$$0 \longrightarrow K_1 \longrightarrow (I, L_1) \oplus (I, y_1) \longrightarrow (I, L_1, y_1) \longrightarrow 0.$$

For all $j$ this yields the long exact sequence

$$\begin{aligned} &\longrightarrow \operatorname{Tor}_{i+1}((I, L_1), K)_j \oplus \operatorname{Tor}_{i+1}((I, y_1), K)_j \longrightarrow \operatorname{Tor}_{i+1}((I, L_1, y_1), K)_j \\ &\longrightarrow \operatorname{Tor}_i(K_1, K)_j \longrightarrow \operatorname{Tor}_i((I, L_1), K)_j \oplus \operatorname{Tor}_i((I, y_1), K)_j. \end{aligned}$$

We need to show that $\operatorname{Tor}_i(K_1, K)_j = 0$ for $j > d+i$. Since $(I, L_1)$ and $I$ are strongly stable ideals in the reverse order, generated in degree $\leq d$, it follows that

$\mathrm{Tor}_i((I,L_1),K)_j \oplus \mathrm{Tor}_i((I,y_1),K)_j = 0$ for $j > d+i$, and $\mathrm{Tor}_{i+1}((I,L_1,y_1),K)_j = 0$ for $j > d+i+1$. Hence we see that $\mathrm{Tor}_i(K_1,K)_j = 0$ for $j > d+i+1$. It remains to show that $\mathrm{Tor}_i(K_1,K)_{d+i+1} = 0$. In fact this is the case since

$$\mathrm{Tor}_{i+1}((I,y_1),K)_{d+i+1} \to \mathrm{Tor}_{i+1}((I,L_1,y_1),K)_{d+i+1}$$

is surjective. Indeed, we have $(I,L_1)_{\geq d} = I$. Thus the surjectivity follows from Lemma 3.5 and Lemma 3.6.

Now let $m \geq 3$. We will show that the first syzygy module of $J$ has a generator of degree $d+3$. This will imply that $\operatorname{reg}\mathrm{Sym}(\mathfrak{m}) \geq \operatorname{reg} R+1$. Then together with (a) the conclusion follows.

Let $u$ be one of the Borel generators with $m(u) \geq 3$, $\deg(u) = d$. We consider the subideal $J'$ of $J$ generated by $u$, $u'y_m$ and $x_1y_2-x_2y_1, x_1y_m-x_my_1, x_2y_m-x_my_2$. It is easy to see that this is the ideal of 4-Pfaffians of the skew-symmetric matrix

$$A = \begin{pmatrix} 0 & 0 & y_m & -y_2 & y_1 \\ 0 & 0 & -x_m & x_2 & -x_1 \\ -y_m & x_m & 0 & 0 & 0 \\ y_2 & -x_2 & 0 & 0 & u' \\ -y_1 & x_1 & 0 & -u' & 0 \end{pmatrix}.$$

Hence by the Buchsbaum-Eisenbud structure theorem ([3, Theorem 3.4.1]), $J'$ is Gorenstein ideal of height 3 whose graded resolution $\mathbb{G}$ is

$$0 \to S(-d-3) \to S^3(-d-1) \oplus S^2(-3) \to S^2(-d) \oplus S^3(-2) \to S \to S/J' \to 0.$$

We construct graded free resolution $\mathbb{F}$ of $J$ such that the inclusion $J' \subset J$ induces a complex homomorphism $\alpha\colon \mathbb{G} \to \mathbb{F}$ for which $\mathbb{G}/\mathfrak{m}\mathbb{G} \to \mathbb{F}/\mathfrak{m}\mathbb{F}$ is injective.

Let $G$ be the Gröbner basis of $J$ as described in Theorem 2.2. Furthermore, let $F_1$ be the free module with basis $e_f$, $f \in G$ and $F_1 \to J$ the epimorphism which sends $e_f$ to $f$. Since the generators of $J'$ are part of the set $G$ of generators of $J$, the map $G_1/\mathfrak{m}G_1 \to F_1/\mathfrak{m}F_1$ is injective.

Next we determine a generating set of relations of $J$. Since $G$ is a Gröbner basis of $J$, we obtain a generating set of relations of $G$ by lifting the minimal set of relations of $\mathrm{in}(G)$. We obtain the following set of relations:

(1) $x_ie_u - x_{m(u)}e_{u'x_i}$ where $u \in G(I)$ and $i > m(u)$;

(2) $x_ie_{u'y_{m(u)}} - x_{m(u')}e_{(u'/x_{m(u')})y_{m(u)}x_i}$ where $u \in G(I)$ and $i > m(u')$;

(3) $y_{m(u)}e_u - x_{m(u)}e_{u'y_{m(u)}}$ where $u \in G(I)$;

(4) $y_ie_{u'y_{m(u)}} - y_{m(u)}e_{u'y_i}$ where $u \in G(I)$ and $m(u) < i \leq m(u')$;

(5) $y_je_u + y_ie_{(u/x_i)x_j} - (u/x_i)e_{f_{ij}}$ where $u \in G(I)$, $x_i$ divides $u$, $j > i$ and $f_{ij} = x_iy_j - x_jy_i$;

(6) $y_ie_u - x_ie_{u'y_{m(u)}} + u'e_{f_{im(u)}}$ where $u \in G(I)$ and $i < m(u)$;

(7) $y_je_{u'y_{m(u)}} - y_ie_{(u'/x_i)x_jy_{m(u)}} + (u'/x_i)y_me_{f_{ij}}$ where $u \in G(I)$, $x_i$ divides $u'$ and $j > i$;

(8) $x_ie_{f_{jk}} - x_je_{f_{ik}} + x_ke_{f_{ij}}$ where $1 \leq i < j < k \leq n$;

(9) $y_i e_{f_{jk}} - y_j e_{f_{ik}} + y_k e_{f_{ij}}$ where $1 \leq i < j < k \leq n$.

We let $F_2$ be the free module whose basis elements are mapped to these relations. Observe that the first row in the matrix $A$ is a relation of type (9), the second of type (8), the third of type (3), the fourth and fifth of type (6). This shows that $G_2/\mathfrak{m}G_2 \to F_2/\mathfrak{m}F_2$ is injective.

Next we choose a minimal homogeneous presentation of $\mathrm{Ker}(F_2 \to F_1)$ and obtain the begin

$$F_3 \to F_2 \to F_1 \to J \to 0\,.$$

of a graded free resolution of $J$.

It remains to show that $G_3/\mathfrak{m}G_3 \to F_3/\mathfrak{m}F_3$ is injective. In order to see this we show that the second syzygy $z_2$ of $J$ which is represented by the image of $S(-d-3) \to S^3(-d-1) \oplus S^2(-3)$ cannot be written as a linear combination of second syzygies of degree $d+2$. Indeed, let $r_1, \dots, r_s$ be all the relations listed above and $h_1, \dots, h_s$ be the homogeneous basis of $F_2$ with $\partial(h_i) = r_i$ for $i = 1, \dots, s$. We may assume that $r_1, \dots, r_5$ are the relations corresponding to the rows of the matrix $A$. Then, since $J'$ is a Gorenstein ideal, its resolution is self-dual, so that

$$z_2 = uh_1 + u'y_m h_2 + f_{12}h_3 + f_{1m}h_4 + f_{2m}h_5.$$

In particular we see that the coefficient of $h_3$ is $f_{12} \neq 0$. Hence if $z_2$ is a linear combination of homogeneous second syzygies, at least one of these syzygies has to have a coefficient of $h_3$ which is non-zero modulo all $x_i$ and $y_i$ with $i \geq 3$. We will show that any such syzygy is of degree $\geq d+3$. In fact, let $w = \sum_{i=1}^s a_i h_i$ be a homogeneous syzygy with $a_3 \neq 0$ modulo all $x_i$ and $y_i$ with $i \geq 3$ and $i \neq m(u)$. Let

$$r_i = \cdots + b_i e_u + c_i e_{u' y_m} + \cdots\,.$$

Then $\sum_i a_i b_i = 0$ and $\sum_i a_i c_i = 0$. We denote by $\bar{f}$ the residue class of an element $f$ modulo the elements $x_i$ and $y_i$ with $i \geq 3$ and $i \neq m(u)$ and get $\sum_i \bar{a}_i \bar{b}_i = 0$ and $\sum_i \bar{a}_i \bar{c}_i = 0$. Inspecting our list of relations we see that in each of these sums only three summands remain, so that we get

$$-\bar{a}_3 y_m + \bar{a}_4 y_2 - \bar{a}_5 y_1 = 0 \quad \text{and} \quad \bar{a}_3 x_m - \bar{a}_4 x_2 + \bar{a}_5 x_1 = 0.$$

Therefore, $\bar{a}_3$ must be a multiple of $f_{12}$. This shows that $w$ is a syzygy of degree $\geq d+3$.

These calculations show that

$$F_3 \to F_2 \to F_1 \to J \to 0\,.$$

is the begin of a graded free resolution of $J$ with the following properties:

(i) the presentation $F_1 \to J$ is minimal (since $G$ is a minimal set of generators of $J$);

(ii) the presentation $F_3 \to \mathrm{Ker}(\partial_2)$ is minimal;

(iii) $F_3$ contains a homogeneous basis element $b$ of degree $d+3$ with $\partial_3(b) \in \mathfrak{m}F_2$.

This complex may not be the begin of a *minimal* graded free resolution of $J$. However by (i) and (ii) one can find graded decompositions $F_2 = F_2' \oplus F_2''$, and $F_3 = F_3' \oplus F_3''$ with $\partial_3(F_3') \subset F_2'$ and $\partial_3(F_3'') \subset F_2''$, such that the induced map $\partial_3 \colon F_3'' \to F_2''$ is an isomorphism and the induced complex

$$F_3' \to F_2' \to F_1 \to J \to 0\,.$$

is the begin of a minimal graded free resolution of $J$. We claim that the image of $b$ under the canonical projection $F_3 \to F_3'$, does not belong to $\mathfrak{m}F_3'$. In fact, let $g_1, \dots, g_s$ be a homogeneous basis of $F_3'$, $h_1, \dots, h_t$ a homogeneous basis of $F_3''$, and write $b = \sum_i a_i g_i + \sum_j c_j h_j$. Then the image of $b$ in $F_3'$ is $\sum_i a_i g_i$. Suppose $a_i \in \mathfrak{m}$ for all $i$. Then, since $b$ is a basis element of $F_3$, there exists an index $j$ such that $c_j \notin \mathfrak{m}$. Since the induced map $\partial_3 \colon F_3'' \to F_2''$ is an isomorphism, this implies that $\partial_3(b) \notin \mathfrak{m}F_2$, a contradiction. Thus we conclude that $F_3'$ has a generator of degree $d+3$, as desired. ☐

Next we want to compute depth of the symmetric algebra. We shall need the following variation of [8, Proposition 2.6].

**Proposition 3.8.** *Let $m$ be an integer with $1 \le m \le n$ and let $(0) = I_{m-1} \subset I_m \subset I_{m+1} \subset \cdots \subset I_n \subset R$ be graded ideals in $R$. Then*

$$\operatorname{depth} R[y_1, \dots, y_n]/(I_m y_m, \dots, I_n y_n) \ge \min\{\operatorname{depth} R/I_i + i\colon i = m-1, \dots, n\}.$$

*Proof.* As in [8] we set $R_i = R[y_1, \dots, y_i]$, and $J_i = (I_1 y_1, \dots, I_i y_i)$ with $I_1 = I_2 = \cdots = I_{m-1} = (0)$ and $I_m \subset \cdots \subset I_n$. Proceeding by induction on $j$, we want to prove that

$$\operatorname{depth} R_j/J_j \ge \{\operatorname{depth} R/I_i + i : i = m-1, \dots, j\} \quad \text{for} \quad j \ge m-1.$$

Since $I_{m-1} = (0)$, it follows that $\operatorname{depth} R_{m-1}/J_{m-1} = \operatorname{depth} R_{m-1} = \operatorname{depth} R + m - 1$.

Now, assume that $j > m-1$. We consider the exact sequences

$$0 \to I_j R_j/J_j \to R_j/J_j \to R_j/I_j R_j \to 0,$$

$$0 \to I_j R_{j-1}/J_{j-1} \to R_{j-1}/J_{j-1} \to R_{j-1}/I_j R_{j-1} \to 0.$$

Since $I_j R_j/J_j \cong I_j R_{j-1}/J_{j-1}$, these exact sequences together with the induction hypothesis imply:

$$\begin{aligned} \operatorname{depth} R_j/J_j &\ge \min\{\operatorname{depth} I_j R_j/J_j, \operatorname{depth} R_j/I_j R_j\} \\ &\ge \min\{\operatorname{depth} R_{j-1}/J_{j-1}, \operatorname{depth} R_{j-1}/I_j R_{j-1} + 1, \operatorname{depth} R/I_j + j\} \\ &\ge \min\{\operatorname{depth} R/I_i + i : i = m-1, \dots, j\}, \end{aligned}$$

as required. ☐

**Theorem 3.9.** *Let $R = S/I$ where $I$ is a strongly stable ideal in the reverse order. Then*

$$\operatorname{depth}\operatorname{Sym}(\mathfrak{m}) = \begin{cases} 0, & \textit{if } \operatorname{depth} R = 0, \\ \operatorname{depth} R + 1, & \textit{if } \operatorname{depth} R > 0. \end{cases}$$

*Proof.* It follows from [1, Proposition 2.1] that the homology classes of the cycles $u' e_{m(u)} \wedge e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_{i-1}}$ with $u \in G(I)$ and $m(u) < j_1 < j_2 < \cdots < j_{i-1} \le n$ form a $K$-basis of the Koszul homology $H_i(x_1, \ldots, x_n; R)$. Therefore, if $t = \operatorname{depth} R$ and $\mathbb{F}$ is the graded minimal free resolution of $R$ then

(i) $t + 1 = \min\{m(u) : u \in G(I)\}$;

(ii) the highest degree of a generator of $F_{n-t}$ is $d - 1 + n - t$, where $d = \max\{\deg u \colon m(u) = t + 1\}$.

In particular, if $\operatorname{depth} R = 0$, then there exists a monomial $u \in G(I)$ with $m(u) = 1$. Note that $u' \notin J$. Therefore, $\operatorname{depth}\operatorname{Sym}(\mathfrak{m}) = 0$, since $u' \in J : (x_1, \ldots, x_n, y_1, \ldots, y_n)$. Indeed, because $u = u' x_1 \in G(I)$, we have $u' x_i \in I \subset J$ for all $i = 1, \ldots, n$. By the definition of the $J$ with each $u' x_i \in I$ we also have $u' y_i \in J$, as desired.

Now let $\operatorname{depth} R = t > 0$. We first show that $\operatorname{depth}\operatorname{Sym}(\mathfrak{m}) \ge t + 1$. Since

$$\operatorname{depth} R[y_1, \ldots, y_n]/J \ge R[y_1, \ldots, y_n]/\operatorname{in}(J) = R[y_1, \ldots, y_n]/(I_1 y_1, \ldots, I_n y_n),$$

it suffices to show that $\operatorname{depth} R[y_1, \ldots, y_n]/(I_1 y_1, \ldots, I_n y_n) \ge t + 1$. We observe that $I_1 = 0$ because $t > 0$. Therefore we may apply Proposition 3.8, and obtain that

$$\operatorname{depth} R[y_1, \ldots, y_n]/(I_2 y_2, \ldots, I_n y_n) \ge \min\{\operatorname{depth} R/I_i + i \colon i = 1, \ldots, n\}.$$

Obviously, $\operatorname{depth} R/I_i + i \ge t + 1$ for $i \ge t + 1$. Therefore it suffices to show that $\operatorname{depth} R/I_i > t - i$ for $i = 1, \ldots, t$. For such $i$ we have $L_i = (0)$, so that

$$R/I_i \cong K[x_i, \ldots, x_n]/I^{\ge i}.$$

Since $G(I) = G(I^{\ge i})$ for $i < t+1$, formula (i) implies that $\operatorname{depth} R/I_i = (t+1) - i > t - i$, as desired.

It remains to show that $\operatorname{depth}\operatorname{Sym}(\mathfrak{m}) \le t + 1$ if $\operatorname{depth} R = t > 0$. We choose $u \in G(I)$ of maximal degree with $m(u) = t + 1$, and let $d = \deg u$. We claim that

$$\operatorname{Tor}_{2n-(t+1)}(K, \operatorname{Sym}(\mathfrak{m}))_{2n-(t+1)+(d-1)} \ne 0,$$

which of course implies the desired inequality.

Suppose first that $t = 1$. We construct a non-zero homology class

$$[z] \in H_{2n-2}(\mathbf{x}, \mathbf{y}; \operatorname{Sym}(\mathfrak{m}))$$

in the Koszul homology of $\operatorname{Sym}(\mathfrak{m})$ with respect to the sequence $x_1, \ldots, x_n, y_1, \ldots, y_n$ with $\deg[z] = 2n - 2 + (d - 1)$.

For $K_1(\mathbf{x}, \mathbf{y}; \operatorname{Sym}(\mathfrak{m}))$ we choose the basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ with $\partial(e_i) = x_i$ and $\partial(f_i) = y_i$ for $i = 1, \ldots, n$, and let $z = u' e_2 \wedge \cdots \wedge e_n \wedge f_2 \wedge \cdots \wedge f_n$. Then

$\deg z = 2n - 2 + (d-1)$ and $z$ is a cycle. In fact, since $I$ is strongly stable in the reverse order we have $u'x_i \in I \subset J$ for all $i \geq 2$. On the other hand, if $v \in I$ is a monomial and $x_i$ divides $v$, then $(v/x_i)y_i \in J$. Thus we see that $u'y_i \in J$ for all $i \geq 2$.

We claim that $z$ is not a boundary. In fact, suppose $z = \partial(b)$. Then

$$\begin{aligned} b &= \sum_{i=1}^{n} c_i e_1 \wedge \cdots \wedge e_{i-1} \wedge e_{i+1} \wedge \cdots \wedge e_n \wedge f_1 \wedge \cdots \wedge f_n \\ &+ \sum_{i=1}^{n} d_i e_1 \wedge \cdots \wedge e_n \wedge f_1 \wedge \cdots \wedge f_{i-1} \wedge f_{i+1} \wedge \cdots \wedge f_n, \quad \text{and} \\ z = \partial(b) &= (d_1 x_1 + (-1)^n c_1 y_1) e_2 \wedge \cdots \wedge e_n \wedge f_2 \wedge \cdots \wedge f_n + \cdots . \end{aligned}$$

Thus we must have that $u' = d_1x_1 + (-1)^n c_1 y_1$. As in Lemma 2.1 we use the fact that $J$ is a multigraded ideal with $\deg x_i = \deg y_i = \varepsilon_i$ for $i = 1, \ldots, n$ where $\varepsilon_i$ denotes the $i$-th vector in the canonical basis of $\mathbb{Z}^n$. Since $z$ is multihomogeneous we may assume that $b$ is multihomogeneous of the same multidegree as $z$. In particular, $d_1x_1 + (-1)^n c_1 y_1$ is multihomogeneous. Let $a = (a_1, \ldots, a_n)$ be the multidegree of $d_1x_1 + (-1)^n c_1y_1$; then $a_1 \neq 0$. This is a contradiction since $u'$ has a multidegree whose first component is zero.

Now suppose that $\operatorname{depth} R = t > 1$. Then $x_1 - y_1$ is a non-zerodivisor on $\operatorname{Sym}(\mathfrak{m})$. Indeed suppose that $f(x_1 - y_1) \in (I, J_0)$. Then with respect to a term order as in Theorem 2.2 we have

$$-\operatorname{in}(f)y_1 = \operatorname{in}(f(x_1 - y_1)) \in \operatorname{in}(I, J_0).$$

Since no monomial in $G(\operatorname{in}(I, J_0))$ contains the factor $y_1$ it follows that $\operatorname{in}(f) \in \operatorname{in}(I, J_0)$, that is, $\operatorname{in}(f) = u \operatorname{in}(f_i)$, with $u$ a monomial in $S[y_1, \ldots, y_n]$ and $f_i$ a generator of $J$. Let $\tilde{f} = f - uf_i$ then $\tilde{f}(x_1 - y_1) = f(x_1 - y_1) - uf_i(x_1 - y_1) \in J$, with $\operatorname{in}(\tilde{f}) < \operatorname{in}(f)$. Induction by $\operatorname{in}(f)$ concludes the proof.

Let $\overline{\mathfrak{m}}$ be the graded maximal ideal of $\bar{R} = K[x_2, \ldots, x_n]/I$ and consider its symmetric algebra $\operatorname{Sym}_{\overline{R}}(\overline{\mathfrak{m}})$ whose defining ideal is

$$I + (L_i y_i \colon i = t+1, \ldots, n) + (x_i y_j - x_j y_i \colon 2 \leq i < j \leq n).$$

Let $\overline{\operatorname{Sym}(\mathfrak{m})} = \operatorname{Sym}(\mathfrak{m})/(x_1 - y_1)$. Its defining ideal is

$$I + (L_i y_i \colon i = t+1, \ldots, n) + (x_1(x_i - y_i) \colon i = 2, \ldots, n) + (x_i y_j - x_j y_i : 2 \leq i < j \leq n).$$

From this it follows that $\operatorname{Sym}(\overline{\mathfrak{m}}) \cong \overline{\operatorname{Sym}(\mathfrak{m})}/(x_1)$. Let $T = S[y_1, \ldots, y_n]$. We have the exact sequence of $T$-modules

$$0 \longrightarrow (x_1) \longrightarrow \overline{\operatorname{Sym}(\mathfrak{m})} \longrightarrow \operatorname{Sym}(\overline{\mathfrak{m}}) \longrightarrow 0. \tag{1}$$

The principal ideal $(x_1)$ is isomorphic to

$$\overline{\operatorname{Sym}(\mathfrak{m})}/\operatorname{Ann}(x_1) \cong R[y_1] \cong T/(I, y_2, \ldots, y_n),$$

and hence its depth is $t+1$.

Considering the long exact sequence of Tor induced by sequence (1), we get the exact sequence

$$\begin{aligned} \operatorname{Tor}_{2n-t}(K, \overline{\operatorname{Sym}(\mathfrak{m})})_{2n-t+d-1} &\to \operatorname{Tor}_{2n-t}(K, \operatorname{Sym}(\overline{\mathfrak{m}}))_{2n-t+d-1} \\ &\xrightarrow{\varphi} \operatorname{Tor}_{2n-(t+1)}(K, R[y_1])_{2n-(t+1)+d} \to \cdots . \end{aligned} \tag{2}$$

Let

$$\mathbb{F} : 0 \to F_{n-t} \to \cdots \to F_i \to \cdots \to F_1 \to F_0$$

be the minimal free resolution of $T/I$, then the minimal free resolution of $R[y_1] \cong T/(I, y_2, \ldots, y_n)$ is the tensor product of $\mathbb{F}$ with the Koszul complex $K(y_2, \ldots, y_n; T)$, and by (ii) the highest degree of a generator of $F_{n-t} \otimes K_{n-1}$ is

$$(d+n-t-1)+(n-1) = 2n-(t+1)+(d-1).$$

Therefore, $\operatorname{Tor}_{2n-(t+1)}(K, R[y_1])_{2n-(t+1)+d} = 0$, and so sequence (2) and our induction hypothesis imply that $\operatorname{Tor}_{2n-t}(K, \overline{\operatorname{Sym}(\mathfrak{m})})_{2n-t+d-1} \neq 0$.

On the other hand, from the exact sequence

$$0 \longrightarrow \operatorname{Sym}(\mathfrak{m})(-1) \longrightarrow \operatorname{Sym}(\mathfrak{m}) \longrightarrow \overline{\operatorname{Sym}(\mathfrak{m})} \longrightarrow 0 .$$

one deduces easily that

$$\operatorname{Tor}_{2n-(t+1)}(K, \operatorname{Sym}(\mathfrak{m}))_{2n-(t+1)+d-1} \cong \operatorname{Tor}_{2n-t}(K, \overline{\operatorname{Sym}(\mathfrak{m})})_{2n-t+d-1}$$

and this yields the desired conclusion. □

Combining Corollary 1.8 with Theorem 3.7 and Theorem 3.9 of this section we obtain

**Corollary 3.10.** *Assume* $\operatorname{char} K = 0$. *Let* $I \subset S = K[x_1, \ldots, x_n]$ *be a componentwise linear ideal, and* $\mathfrak{m}$ *the graded maximal ideal of* $R = S/I$. *Then*

(a) $\operatorname{reg} \operatorname{Sym}(\mathfrak{m}) \leq \operatorname{reg} R + 1$;

(b) $\operatorname{depth} \operatorname{Sym}(\mathfrak{m}) \geq \operatorname{depth} R + 1$ *if* $\operatorname{depth} R > 0$.

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Received July 30, 2004