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Submanifolds in the Variety of Planar Normal Sections

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Abstract. The present paper continues the study of the nature of the variety $X[M]$ of directions of pointwise planar normal sections for the manifold M of complete flags of a compact simple Lie group G_u.
The main results concern submanifolds embedded in RP^{m-1} ($m = \dim M$) which are subsets of $X[M]$. One of them is an open set in the natural topology of $X[M]$ whose dimension is related to that of M and the rank of the Lie group G_u. Others are projective subspaces of "minimal" dimension contained in $X[M]$ for the groups $G_u = SU(n+1)$.

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1. Introduction

In [5] the variety $X[M]$ of directions of pointwise planar normal sections of a natural embedding of an R-space M, was introduced. This is a real algebraic variety in the real projective space RP^{m-1}, where $m = \dim M$ and in some sense it measures the difference between the given manifold and a symmetric R-space. This variety in general has singularities but in the present paper, when M is the manifold of complete flags of a compact simple Lie group G_u, we observe the presence of an open set in the natural topology of $X[M]$ which is a differentiable manifold whose dimension is related to that of M and the rank of the Lie group

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G_u. The presence of this manifold gives some indication of the "size" of $X[M]$ inside the projective space RP^{m-1}.

On the other hand, the existence of certain submanifolds such as projective subspaces in an algebraic variety like $X[M]$ indicates that the variety is rather special. Therefore the knowledge of the presence of these submanifolds seems to be desirable. In a couple of papers [6] and [7] the presence of these submanifolds in $X[M]$ was considered when the R-space M is the manifold of complete flags of a compact simple Lie group G_u. In the first one, a family of maximal projective subspaces in $X[M]$ was described. In the second one, the study of projective subspaces of $X[M]$ was continued and the main result there relates a family of these subspaces, which are of the maximal possible dimension with tangent spaces to some of the inner symmetric spaces corresponding to the group G_u.

Also in [12], related to the study of extrinsic symmetric CR-structures on the manifold of complete flags M, it was observed a strong connection between the holomorphic tangent spaces of these structures and those subspaces of the tangent space to M which give rise to projective subspaces in $X[M]$. This particular fact throws new light on the interest of the study of these subspaces in $X[M]$.

Following this line, in the present paper and for the manifold $M_n = SU(n+1)/T^n$, we obtain information about the minimal possible dimension of those maximal $Ad(T^n)$-invariant vector subspaces in the tangent space defining projective subspaces in $X[M]$. Since we have already determined the maximal possible dimension for these subspaces we have thus a description of the range of the admissible dimensions for them.

This paper is organized as follows. Section 2 contains the notation and basic facts necessary to get acquainted with the theme of the paper. Section 3 contains information on the nature of the polynomials associated to the variety $X[M]$. The main result here is Theorem 3.2 which shows the existence of submanifolds in $X[M]$ when M is the manifold of complete flags of a compact simple Lie group G_u. It is proven that the variety $X[M]$ contains an open set (for the induced topology from RP^{m-1}) which is a differentiable submanifold of dimension $m-n$ embedded in RP^{m-1}, where n is the rank of the Lie group G_u. This set is the projection of the non-singular points in R^m of the function whose coordinate are the polynomials defining the variety $X[M]$.

In Section 4, for the family $M_n = SU(n+1)/T^n$ we study the varieties $X[M_n]$. In the first place we give recursive formulae for the polynomials defining them which allows us to show that, for certain natural embeddings, the set of directions of pointwise planar normal sections of M_n is contained in that of M_{n+1}. In the second one we obtain Theorem 4.4 which is the main result of this section. It shows that a subspace of the tangent space of M_n, which is maximal among the $Ad(T^n)$-invariant subspaces defining projective subspaces in $X[M_n]$, must have dimension greater than or equal to $2n$. Furthermore this results gives a characterization of those of dimension $2n$ relating them with tangent spaces to inner symmetric spaces for the group $SU(n+1)$. We conclude that, if we pose no restriction, when the dimension of the $Ad(T^n)$-invariant subspace defining projective subspaces in $X[M_n]$ is not one of the extreme cases, there is no connection between them and any tangent spaces to a homogeneous manifold of the group $SU(n+1)$.

Also we obtain, as an application, a result about the holomorphic tangent spaces of $SU(n+1)$-invariant minimal almost Hermitian extrinsic symmetric CR-structure on M_n related to [12].

2. Notation and basic facts

Let $f : M \to R^N$ be an isometric embedding. We may identify M with its image by f. Let p be a point in M and X a unit vector in the tangent space $T_p(M)$. If $T_p(M)^{\perp}$ denotes the normal space to M at p, we may consider the affine subspace of R^N defined by $S(p, X) = p + Span\left\{X, T_p(M)^{\perp}\right\}$.

If U is a small enough neighborhood of p in M, then the intersection $U \cap S(p, X)$ can be considered as the image of a C^{∞} regular curve $\gamma(s)$, parametrized by arc-length, such that $\gamma(0) = p$ and $\gamma'(0) = X$. This curve is called the *normal section of M at the point p in the direction of X*.

Following B. Y. Chen, we say that the normal section γ of M at p in the direction of X is *pointwise planar* at p if its first three derivatives $\gamma'(0)$, $\gamma''(0)$ and $\gamma'''(0)$ are linearly dependent, i.e. if $\gamma'(0) \wedge \gamma''(0) \wedge \gamma'''(0) = 0$.

In previous papers we have studied the pointwise planar normal sections of an orbit of an s-representation; i.e. of a natural embedding of an R-space or real flag manifold (the reader is referred to [5, p. 225] and references therein, for basic information concerning R-spaces, canonical connections, etc.). In order to recall one of the results obtained there, that is needed in the present paper, we introduce now some necessary notation.

Let $f : M \to R^N$ be a natural embedding of an R-space and let ∇ denote the Riemannian connection associated to the metric induced from the Euclidean metric. Let ∇^c denote the canonical connection associated to the "usual" reductive decomposition of the Lie algebra of the compact Lie group defining M. Let $D = \nabla - \nabla^c$ denote the difference tensor and let α be the second fundamental form of the embedding f. The indicated result is the following.

Theorem 2.1. [5, p. 226,(2.5)] *If $f : M \to R^N$ is a natural embedding of an R-space and p is a point in M, then the normal section γ with $\gamma(0) = p$ and $\gamma'(0) = X$ is pointwise planar at p if and only if the unit tangent vector X at p satisfies the equation*

$$\alpha(D(X, X), X) = 0. \qquad \square$$

This result allows us to define $X_p[M]$ as the image (for the canonical projection) in the real projective space RP^{m-1} ($m = \dim M$) of the set of those directions that define pointwise planar normal section at p. Since M is an orbit of a group of isometries of the ambient space R^N, it is clear that $X_p[M]$ does not depend on the point p and we may denote it by $X[M]$.

It is known that $X[M]$ is a real algebraic variety of RP^{m-1} defined by homogeneous polynomials of degree 3 (see [5, p. 227, (2.9)]).

In this paper we restrict our attention to manifolds of complete flags which have been the subject of our work for some time and where we have obtained the most interesting results. These are, among complex flag manifolds M, those where $X[M]$ presents the greatest simplicity in geometric terms (i.e. minimal number of defining polynomials). At the same time, these manifolds are sufficiently complicated to yield interesting information about the varieties of planar normal sections. This is so because they are the flag manifolds which stand further apart from the corresponding Hermitian symmetric spaces, if the group under consideration has one (this is not the case for E_8, F_4 and G_2). For a Hermitian symmetric space H, by results of Chen [3] and Ferus [10] (compare [5, p. 224]), it is known that the variety $X[H]$ is the real projective space RP^{h-1} where $h = \dim H$.

We need to introduce the following notation.

Let G be a simply connected, complex, simple Lie group and let $\mathfrak{g}$ be its Lie algebra. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ the root system of $\mathfrak{g}$ relative to $\mathfrak{h}$. We may write

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\gamma \in \Delta^+} (\mathfrak{g}_\gamma \oplus \mathfrak{g}_{-\gamma})$$

where Δ^+ indicates the set of positive roots with respect to some order.

Let us consider in $\mathfrak{g}$ the Borel subalgebra

$$\mathfrak{b} = \mathfrak{h} \oplus \sum_{\gamma \in \Delta^+} \mathfrak{g}_{-\gamma}.$$

Let B be the analytic subgroup of G corresponding to the subalgebra $\mathfrak{b}$. B is closed and its own normalizer in G. The quotient space $M = G/B$ is a complex homogeneous space called the manifold of complete flags of G.

Let $\pi = \{\gamma_1, \ldots, \gamma_n\} \subset \Delta^+$ be a system of simple roots. We may take in $\mathfrak{g}$ a Weyl basis [13, III, 5] $\{X_\beta : \beta \in \Delta\}$ and $\left\{H_{\gamma_j} : \gamma_j \in \pi\right\}$. The following set of vectors provides a basis of a compact real form $\mathfrak{g}_u$ of $\mathfrak{g}$

$$\begin{cases} U_\gamma = \frac{1}{\sqrt{2}}(X_\gamma - X_{-\gamma}) & \gamma \in \Delta^+ \\ U_{-\gamma} = \frac{i}{\sqrt{2}}(X_\gamma + X_{-\gamma}) & \gamma \in \Delta^+ \\ iH_{\gamma_j} & \gamma_j \in \pi. \end{cases} \tag{1}$$

We shall denote by $\mathfrak{H}$ the real vector space generated by $\left\{iH_{\gamma_j} : \gamma_j \in \pi\right\}$ and by $\mathfrak{m}_\gamma$ that of $\{U_\gamma, U_{-\gamma}\}$. Then we may write

$$\mathfrak{g}_u = \mathfrak{H} \oplus \sum_{\gamma \in \Delta^+} \mathfrak{m}_\gamma = \mathfrak{H} \oplus \mathfrak{m}.$$

Let G_u be the analytic subgroup of G corresponding to $\mathfrak{g}_u$. G_u is compact and acts transitively on M which can be written as $M = G_u/(G_u \cap B)$. The subgroup $T = G_u \cap B = \exp \mathfrak{H}$ is a maximal torus in G_u. Then M is a compact, simply connected, complex manifold called manifold of complete flags for the simple Lie group G_u.

Let $E \in \mathfrak{g}_u$ be a regular element [15, p. 48]. We want to consider the orbit of E by the adjoint action of G_u on $\mathfrak{g}_u$, i.e. $Ad\,(G_u)\,E = \{Ad\,(g)\,E : g \in G_u\}$. It is clear that the isotropy subgroup of the point E is precisely $G_u \cap B$ and we have a natural embedding of M in $\mathfrak{g}_u$. We may take in $\mathfrak{g}_u$ the inner product given by the opposite of the Killing form and therefore the induced Riemannian metric on M, by the embedding $f : M \to \mathfrak{g}_u$, is invariant by the action of G_u. Then the tangent space to M at E is $T_E\,(M) = [\mathfrak{g}_u, E] = [\mathfrak{m}, E] = \mathfrak{m}$ and $T_E\,(M)^{\perp} = \mathfrak{H}$ is the normal space there.

3. Polynomials associated to $X\,[M]$

In order to study the variety $X\,[M]$ for M a complete flag manifold for the compact, connected, simple Lie group G_u, we need to consider the polynomials defining it. These polynomials are the components of the second fundamental form with respect to a convenient basis in the normal space.

If α is the second fundamental form of the imbedding $f : M \to \mathfrak{g}_u$ and $X = \sum_{\gamma \in \Delta^+} (x_\gamma U_\gamma + x_{-\gamma} U_{-\gamma}) \in \mathfrak{m}$, we may write

$$\alpha\left([X,E]\,, D\left([X,E]\,,[X,E]\right)\right) = \sum_{1 \le r \le n} p_r i H_{\gamma_r}. \tag{2}$$

In [6, Lemma 3.1] we obtain the following expression

$$\alpha\left([X,E]\,, D\left([X,E]\,,[X,E]\right)\right) = \sum_{\substack{\sigma,\tau,\delta \in \Delta^+ \\ \delta = \sigma+\tau}} d_{(\sigma,\tau,\delta)} i H_\delta + \sum_{\substack{\varepsilon,\rho,\beta \in \Delta^+ \\ |\varepsilon-\rho| = \beta}} b_{(\varepsilon,\rho,\beta)} i H_\beta$$

where

$$\begin{aligned} d_{(\sigma,\tau,\delta)} &= \tfrac{\tau(iE)}{\sqrt{2}} N_{\sigma,\tau} \left\{ (x_\sigma x_\tau - x_{-\sigma} x_{-\tau}) x_\delta + (x_\sigma x_{-\tau} + x_{-\sigma} x_\tau) x_{-\delta} \right\} \\ b_{(\varepsilon,\rho,\beta)} &= \tfrac{\rho(iE)}{\sqrt{2}} N_{\varepsilon,-\rho} \left\{ (x_\varepsilon x_\rho + x_{-\varepsilon} x_{-\rho}) x_\beta + sg_{\rho-\varepsilon} (x_\varepsilon x_{-\rho} - x_{-\varepsilon} x_\rho) x_{-\beta} \right\} \end{aligned}$$

and

$$\begin{array}{lll} |\varepsilon-\rho| = \varepsilon-\rho \;\text{ and }\; sg_{\varepsilon-\rho} = 1 & \text{if } \varepsilon-\rho \in \Delta^+ \\ |\varepsilon-\rho| = \rho-\varepsilon \;\text{ and }\; sg_{\varepsilon-\rho} = -1 & \text{if } \rho-\varepsilon \in \Delta^+. \end{array}$$

In the previous expression, we may cluster the coefficients corresponding to the terms (σ,τ,δ) and (τ,σ,δ) and similarly those of (ε,ρ,β) and (ρ,ε,β). Let us denote by $d_{\{\sigma,\tau\}} = d_{(\sigma,\tau,\delta)} + d_{(\tau,\sigma,\delta)}$ and $b_{\{\varepsilon,\rho\}} = b_{(\varepsilon,\rho,\beta)} + b_{(\rho,\varepsilon,\beta)}$. Since $N_{\sigma,\tau} = -N_{\tau,\sigma}$ and $N_{\varepsilon,-\rho} = N_{\rho,-\varepsilon}$ follows that

$$\begin{aligned} d_{\{\sigma,\tau\}} &= \frac{N_{\sigma,\tau} u}{\sqrt{2}} \left\{ (x_\sigma x_\tau - x_{-\sigma} x_{-\tau}) x_\delta + (x_\sigma x_{-\tau} + x_{-\sigma} x_\tau) x_{-\delta} \right\} \\ b_{\{\varepsilon,\rho\}} &= \frac{N_{\varepsilon,-\rho} v}{\sqrt{2}} \left\{ (x_\varepsilon x_\rho + x_{-\varepsilon} x_{-\rho}) x_\beta + sg_{\rho-\varepsilon} (x_\varepsilon x_{-\rho} - x_{-\varepsilon} x_\rho) x_{-\beta} \right\} \end{aligned}$$

where $u = \tau\,(iE) - \sigma\,(iE)$ and $v = \rho\,(iE) + \varepsilon\,(iE)$.

Set $\Gamma = \{\{\sigma, \tau\} : \sigma, \tau, \sigma + \tau \in \Delta^+\}$ and $\Upsilon = \{\{\varepsilon, \rho\} : \varepsilon, \rho, |\varepsilon - \rho| \in \Delta^+\}$. Then we have

$$\alpha\left([X, E], D\left([X, E], [X, E]\right)\right) = \sum_{\{\sigma,\tau\}\in\Gamma} d_{\{\sigma,\tau\}} iH_{\sigma+\tau} + \sum_{\{\varepsilon,\rho\}\in\Upsilon} b_{\{\varepsilon,\rho\}} iH_{|\varepsilon-\rho|}. \tag{3}$$

To express each root β in terms of the simple roots, it will be useful the following notation

$$\beta = \sum_{1\le j\le n} k_j(\beta)\gamma_j. \tag{4}$$

The coefficient p_r in (2) is obtained as sum of all the coefficients $d_{\{\sigma,\tau\}}$ and $b_{\{\varepsilon,\rho\}}$ in (3), such that $k_r(\sigma + \tau) \neq 0$ and $k_r(\varepsilon - \rho) \neq 0$.

In order to simplify notation, if $\{\sigma, \tau\} \in \Gamma$ we write

$$q_{\{\sigma,\tau\}} = (x_\sigma x_\tau - x_{-\sigma} x_{-\tau}) x_{\sigma+\tau} + (x_\sigma x_{-\tau} + x_{-\sigma} x_\tau) x_{-(\sigma+\tau)}. \tag{5}$$

Since each of the sets Γ and Υ gives rise the other one, we may adequately change the clustering of the terms in (3) to conclude

$$\alpha\left([X, E], D\left([X, E], [X, E]\right)\right) = \sum_{\{\sigma,\tau\}\in\Gamma} \left\{ \frac{N_{\sigma,\tau}}{\sqrt{2}} (\tau\,(iE) - \sigma\,(iE)) q_{\{\sigma,\tau\}} iH_{\sigma+\tau} + \right.$$

$$\left. + \frac{N_{\sigma+\tau,-\sigma}}{\sqrt{2}} (2\sigma\,(iE) + \tau\,(iE)) q_{\{\sigma,\tau\}} iH_\tau + \frac{N_{\sigma+\tau,-\tau}}{\sqrt{2}} (\sigma\,(iE) + 2\tau\,(iE)) q_{\{\sigma,\tau\}} iH_\sigma \right\}$$

$$= \sum_{\{\sigma,\tau\}\in\Gamma} \frac{3}{\sqrt{2}} N_{\sigma,\tau} q_{\{\sigma,\tau\}} \left\{\tau\,(iE)\, iH_\sigma - \sigma\,(iE)\, iH_\tau\right\}$$

$$= \sum_{\{\sigma,\tau\}\in\Gamma} \frac{3}{\sqrt{2}} N_{\sigma,\tau} q_{\{\sigma,\tau\}} \left\{ \sum_{1\le r\le n} (\tau\,(iE)\, k_r(\sigma) - \sigma\,(iE)\, k_r(\tau)) iH_{\gamma_r} \right\}$$

and then

$$\alpha\left([X, E], D\left([X, E], [X, E]\right)\right) \\ = \tfrac{3}{\sqrt{2}} \sum_{1\le r\le n} \{ \sum_{\{\sigma,\tau\}\in\Gamma} N_{\sigma,\tau} \left(\tau\,(iE)\, k_r(\sigma) - \sigma\,(iE)\, k_r(\tau)\right) q_{\{\sigma,\tau\}} \} iH_{\gamma_r}.$$

The preceding development may be summarized in the following lemma.

Lemma 3.1. *For each r, $1 \le r \le n$, the coefficient p_r in (2) is given by*

$$p_r = \frac{3}{\sqrt{2}} \sum_{\{\sigma,\tau\}\in\Gamma} N_{\sigma,\tau} \left(\tau\,(iE)\, k_r(\sigma) - \sigma\,(iE)\, k_r(\tau)\right) q_{\{\sigma,\tau\}} \tag{6}$$

where $q_{\{\sigma,\tau\}}$ is defined by (5). $\square$

This result allows us to obtain information about the family of polynomials defining $X[M]$ as follows:

Theorem 3.1. *The polynomials* p_r *(*$1 \le r \le n$*) given in* (2) *satisfy:*

(i) $\sum_{1\le r\le n} \gamma_r(iE)p_r = 0.$

(ii) *For any* j *such that* $1 \le j \le n$ *the set* $\{p_r : 1 \le r \le n,\ r \ne j\}$ *is* $\mathbb{R}$*-linearly independent.*

Proof. (i) By Lemma 3.1 and (4) we have

$$\begin{aligned}
&\sum_{1\le r\le n} \gamma_r(iE)p_r \\
&= \frac{3}{\sqrt{2}} \sum_{\{\sigma,\tau\}\in\Gamma} N_{\sigma,\tau} \left\{ \sum_{1\le r\le n} \gamma_r(iE)[k_r(\sigma)\tau\,(iE) - k_r(\tau)\sigma\,(iE)] \right\} q_{\{\sigma,\tau\}} \\
&= \frac{3}{\sqrt{2}} \sum_{\{\sigma,\tau\}\in\Gamma} N_{\sigma,\tau} \left\{ \sigma(iE)\tau\,(iE) - \tau\,(iE)\,\sigma\,(iE) \right\} q_{\{\sigma,\tau\}} = 0.
\end{aligned}$$

(ii) Let us fix j such that $1 \le j \le n$ and assume that there are some real numbers c_r such that

$$\sum_{\substack{1\le r\le n\\ r\ne j}} c_r p_r = 0. \tag{7}$$

Since given two polynomials $q_{\{\sigma,\tau\}}$ and $q_{\{\varphi,\psi\}}$ with $\{\sigma,\tau\} \ne \{\varphi,\psi\}$ in Γ their monomials are all different, by Lemma 3.1, we have that (7) implies

$$\sum_{\substack{1\le r\le n\\ r\ne j}} c_r[\tau\,(iE)\,k_r(\sigma) - \sigma\,(iE)\,k_r(\tau)] = 0 \tag{8}$$

for every σ, τ such that $\{\sigma,\tau\} \in \Gamma$.

For $\sigma = \gamma_j$ let $\tau = \gamma_s$ be such that $\sigma + \tau \in \Delta^+$. By (8) we have $-c_s\gamma_j(iE) = 0$ and therefore

$$c_s = 0. \tag{9}$$

If $r \ne j \ne t$ are such that $\gamma_r + \gamma_t \in \Delta^+$, taking $\sigma = \gamma_r$ and $\tau = \gamma_t$ in (8) we may write $c_r\gamma_t(iE) - c_t\gamma_r(iE) = 0$ and then

$$c_r = \frac{\gamma_r(iE)}{\gamma_t(iE)} c_t. \tag{10}$$

By connectedness of the Dinkyn diagrams, if $m \ne j$, there exist positive, distinct integers $j = k_0, k_1, \ldots, k_h = m$ such that $\gamma_{k_l} + \gamma_{k_{l+1}} \in \Delta^+, 0 \le l \le h-1$. By (9) and (10), we have $0 = c_{k_1} = \cdots = c_{k_h} = c_m$. Therefore (ii) follows. □

The following observation with respect to the partial derivatives of the polynomials $q_{\{\sigma,\tau\}}$ will be useful below:

Remark 3.1. For $\{\sigma,\tau\}\in\Gamma$ and $\sigma+\tau=\delta$ it occurs

$$\begin{array}{ll}
\frac{\partial q_{\{\sigma,\tau\}}}{\partial x_\delta}=x_\sigma x_\tau - x_{-\sigma}x_{-\tau} & \frac{\partial q_{\{\sigma,\tau\}}}{\partial x_{-\delta}}=x_\sigma x_{-\tau}+x_{-\sigma}x_\tau \\
\frac{\partial q_{\{\sigma,\tau\}}}{\partial x_\sigma}=x_\delta x_\tau + x_{-\delta}x_{-\tau} & \frac{\partial q_{\{\sigma,\tau\}}}{\partial x_{-\sigma}}=x_{-\delta} x_{\tau}-x_{\delta}x_{-\tau} \\
\frac{\partial q_{\{\sigma,\tau\}}}{\partial x_\tau}=x_\delta x_\sigma + x_{-\delta}x_{-\sigma} & \frac{\partial q_{\{\sigma,\tau\}}}{\partial x_{-\tau}}=x_{-\delta} x_{\sigma}-x_{\delta}x_{-\sigma} \\
\frac{\partial q_{\{\sigma,\tau\}}}{\partial x_{\pm\varepsilon}}=0 \text{ if } \varepsilon\notin\{\sigma,\tau,\delta\}\,. &
\end{array}$$

Then it is easy to see that at the point $X=\sum_{\gamma\in\Delta^+}(x_\gamma U_\gamma+x_{-\gamma}U_{-\gamma})$ of $\mathfrak{m}$, $\left(\frac{\partial q_{\{\sigma,\tau\}}}{\partial x_\delta},\frac{\partial q_{\{\sigma,\tau\}}}{\partial x_{-\delta}}\right)\neq(0,0)$ if and only if $(x_\sigma,x_{-\sigma})\neq(0,0)$ and $(x_\tau,x_{-\tau})\neq(0,0)$.

Theorem 3.2. *The variety $X[M]$ contains an open set which is an embedded submanifold in RP^{m-1} of dimension $m-n$, where $m=\dim M$ and $n=rank\ \mathfrak{g}$.*

Proof. The proof consists of showing that there exists a point X in S^{m-1}, the sphere unit of $\mathfrak{m}$, such that $\alpha\left([X,E],D\left([X,E],[X,E]\right)\right)=0$ and where the Jacobian matrix of the function $P:S^{m-1}\to R^{n-1}$ given by $P=(p_1,p_2,\dots,p_{n-1})$ has rank $n-1$. Then the theorem follows from the Implicit Function Theorem.

Let

$$X=\sum_{j=1}^{n}(x_{\gamma_j}U_{\gamma_j}+x_{-\gamma_j}U_{-\gamma_j})\in\mathfrak{m}$$

be a unit vector such that $(x_{\gamma_j},x_{-\gamma_j})\neq(0,0)$ for each j $(1\leq j\leq n)$, where γ_j are the simple roots.

By Theorem 4.2 in [6, p. 226], $\widetilde{\mathfrak{p}}=\sum_{\gamma_j\in\pi}\mathfrak{m}_{\gamma_j}$ defines a projective subspace in $X[M]$ and therefore it follows that $\alpha\left([X,E],D\left([X,E],[X,E]\right)\right)=0$.

To show that the rank of the Jacobian matrix of P at this point X is $n-1$ we shall consider the partial derivatives of the polynomials p_r with respect to the variables x_δ and $x_{-\delta}$ moving δ over the set of the $n-1$ positive roots of height 2.

When the simple Lie algebra is different from $\mathfrak{d}_n$ and $\mathfrak{e}_j$ $(n\geq 4$ and $j=6,7,8)$, by (6) and Remark 3.1 (using the notation from [13, p. 470]) we get for each r, such that $1\leq r\leq n-1$, at the point X

$$\left(\frac{\partial p_r}{\partial x_\delta},\frac{\partial p_r}{\partial x_{-\delta}}\right)=\begin{cases}(a,b)\neq(0,0) & \text{if } \delta=\gamma_r+\gamma_{r+1}\\ (0,0) & \text{if } \delta=\gamma_s+\gamma_{s+1} \text{ for } r<s\leq n-1.\end{cases}\tag{11}$$

and therefore the rank of the Jacobian matrix of P at this point X is $n-1$. For the other Lie algebras we get analogous expressions to (11) and so the result follows. ☐

Remark 3.2. Let us notice that the open set obtained in the proof of the previous theorem is the projection of the whole set of non-singular points of P that give rise to pointwise planar normal sections.

4. On $X[SU(n+1)/T^n]$

The present section is devoted to study of the variety $X\,[M_n]$ of directions of pointwise planar normal sections of a natural embedding the manifold of complete flags $M_n = SU(n+1)/T^n$, where $n \geq 2$ and T^n is a maximal torus in $SU(n+1)$, for which we shall indicate some results that we feel are interesting.

We shall add to our previous notation, an extra subindex n which is the rank of the Lie algebra $\mathfrak{sl}(n+1,\mathbb{C})$. Let us recall that $\mathfrak{su}(n+1) = \mathfrak{H}_n \oplus \sum_{\gamma\in\Delta_n^+} \mathfrak{m}_{\gamma;n} = \mathfrak{H}_n \oplus \mathfrak{m}_n$.

We use for the Lie algebra $\mathfrak{a}_n$ the notation given in [13, III, 8] and so $e_i(H)$ $(1 \leq i \leq n)$ are the diagonal elements of $H \in \mathfrak{h}_n$. It is known that, for $\beta = e_r - e_s$,

$$X_{\beta;n} = (2n+2)^{-\frac{1}{2}}\, E_{r,s;n+1}$$

where $E_{r,s;n+1}$ denotes the $(n+1)\times(n+1)$ matrix with entry 1 where the r-th row and the s-th column meet, all other entries being 0.

Therefore, if $\gamma_j \in \pi_n$ and $\beta,\gamma \in \Delta_n$ we have

$$\begin{aligned} H_{\gamma_j;n} &= [X_{\gamma_j;n}, X_{-\gamma_j;n}] = (2n+2)^{-1}\,(E_{j,j;n+1} - E_{j+1,j+1;n+1}) \\ N_{\beta,\gamma;n} &= \begin{cases} \pm\,(2n+2)^{-\frac{1}{2}} & \text{if } \beta+\gamma\in\Delta_n \\ 0 & \text{if } \beta+\gamma\notin\Delta_n \end{cases}. \end{aligned}$$

We consider the inclusion of the algebra $\mathfrak{a}_{n-1}$ into the $\mathfrak{a}_n$ given by the natural inclusion of $\pi_{n-1} = \{\gamma_1,\ldots,\gamma_{n-1}\}$ into $\pi_n = \{\gamma_1,\ldots,\gamma_n\}$. Then for $\beta,\gamma \in \Delta_{n-1} \subset \Delta_n$ it follows that

$$N_{\beta,\gamma;n} = \left(\frac{n}{n+1}\right)^{\frac{1}{2}} N_{\beta,\gamma;n-1}.$$

For each n we consider in the Cartan subalgebra $\mathfrak{h}_n \subset \mathfrak{sl}(n+1,\mathbb{C})$ the Wolf basis [16]

$$\{v_l\}_l \ \text{ defined by } \ \gamma_j(v_l) = \delta_{jl},\ 1 \leq j,l \leq n. \tag{12}$$

Set iE_n the point in $\mathfrak{sl}(n+1,\mathbb{C})$ defined by

$$iE_n = \sum_{1\leq l\leq n} 2^l v_l.$$

It follows that:

(a) For $\beta,\gamma \in \Delta_n^+$, $\beta \neq \gamma$, $\beta(iE_n)$ and $\gamma(iE_n)$ are different positive real numbers (see for instance [12]).

(b) For $\gamma_j \in \pi_m$,

$$\gamma_j(iE_n) = \begin{cases} 2^j \text{ if } j \leq n \\ 0 \ \text{ if } j > n \end{cases}. \tag{13}$$

(c) $E_n \in \mathfrak{H}_n$, it is a regular element and then the orbit $Ad(SU(n+1))E_n$ gives rise to a natural imbedding f_n of the manifold M_n into the Lie algebra of $\mathfrak{su}(n+1)$ defined by $f_n(gT^n) = Ad(g)E_n$.

If $X = \sum_{\gamma\in\Delta_n^+}(x_\gamma U_\gamma + x_{-\gamma}U_{-\gamma}) \in \mathfrak{m}_n$ and α_n is the second fundamental form of the imbedding f_n we may write (2) as

$$\alpha_n\left([X,E_n], D([X,E_n],[X,E_n])\right) = \sum_{1\le r\le n} p_{r;n} i H_{\gamma_r;n}. \tag{14}$$

In order to obtain, for $n \ge 2$, a recursive formula of the polynomials $p_{r;n}$ ($1\le r\le n$) we write

$$\sigma_{j,k} = \sum_{j\le s\le k}\gamma_s \tag{15}$$

and we shall use indistinctly $\sigma_{j,j}$ or γ_j to indicate the simple root γ_j. Then

$$\Delta_n^+ = \{\sigma_{j,k} : 1\le j\le k\le n\} \text{ and } \quad \Gamma = \{\{\sigma_{j,k},\sigma_{k+1,l}\} : 1\le j\le k< l\le n\}.$$

It easy to verify that, for $1\le j\le k<l\le n$, $N_{\sigma_{j,k},\sigma_{k+1,l};n} = \frac{1}{\sqrt{2n+2}}$.

For the algebra $\mathfrak{sl}(n+1,\mathbb{C})$, if σ, $\tau\in\Delta_n^+$ and $k_r(\sigma+\tau)\ne 0$ only one of the numbers $k_r(\sigma)$ or $k_r(\tau)$ is 1 and the other one is 0. By Lemma 3.1 we have

$$\begin{aligned} &p_{r;n} = \tfrac{3}{2\sqrt{n+1}} \sum_{(j,k,l)\in I_{r;n}} c_{j,k,l}(r) q_{\{\sigma_{j,k},\sigma_{k+1,l}\}} \\ &\text{where } c_{j,k,l}(r) = \begin{cases} 2^{l+1}-2^{k+1} & \text{if } j\le r\le k \\ 2^j - 2^{k+1} & \text{if } k<r\le l \end{cases} \\ &\text{and } I_{r;n} = \{(j,k,l) : j\le k<l\le n \text{ and } j\le r\le l\}. \end{aligned} \tag{16}$$

The previous development allows us to obtain recursively the polynomials $p_{r;n}$ as follows.

Proposition 4.1. *Keeping the previous notation, the polynomials $p_{r;n}$, $n\ge 2$ of (14) for the manifold $M_n = SU(n+1)/T^n$ are*

$$p_{1;2} = 2\sqrt{3}\, q_{\{\gamma_1,\gamma_2\}} \quad , \quad p_{2;2} = -\sqrt{3}\, q_{\{\gamma_1,\gamma_2\}}$$

and given $p_{r;n-1}$, for $1\le r\le n-1$,

$$\begin{aligned} p_{r;n} = \sqrt{\tfrac{n}{n+1}}\, p_{r;n-1} + \tfrac{3}{2\sqrt{n+1}} \Bigg\{ &\sum_{1\le j\le r\le k<n} (2^{n+1}-2^{k+1}) q_{\{\sigma_{j,k},\sigma_{k+1,n}\}} + \\ &+ \sum_{1\le j\le k<r\le n} (2^j - 2^{k+1})\, q_{\{\sigma_{j,k},\sigma_{k+1,n}\}} \Bigg\} \end{aligned}$$

and

$$p_{n;n} = \frac{3}{2\sqrt{n+1}} \sum_{1\le j\le k<n} (2^j - 2^{k+1})\, q_{\{\sigma_{j,k},\sigma_{k+1,n}\}}.$$

Proof. The proof follows from (16) and the fact that in order to get $p_{r;n}$ from $p_{r;n-1}$ ($1\le r\le n-1$), it is just necessary to consider the terms of (16) coming from the triples (j,k,n) with $1\le j\le k<n$ and $j\le r\le n$. □

Corollary 4.1. $X[M_{n-1}] \subset X[M_n]$.

Proof. It follows from the previous proposition because $q_{\{\sigma_{j,k},\sigma_{k+1,n}\}}(X) = 0$ for $X = \sum_{\gamma\in\Delta^+_{n-1}} (x_\gamma U_\gamma + x_{-\gamma} U_{-\gamma}) \in \mathfrak{m}_{n-1}$. □

It is well known that the group $SU(n+1)$ gives rise to a family of irreducible symmetric spaces of type I [13, p. 518] and among them, there are those which are inner; i.e. the spaces in which the symmetry at each point belongs to the group $SU(n+1)$. They are of the form $SU(n+1)/K$ where K is a subgroup of maximal rank in $SU(n+1)$. By conjugating K if necessary, we may assume that K contains T^n.

It is known that, among irreducible symmetric spaces G/K of a simple group G, just for the inner ones there exists a simple root γ^* such that the tangent space $T_{[K]}(G/K) = \mathfrak{p}^*$ satisfies

$$\mathfrak{p}^* = \sum_{\gamma\in\Delta^*} \mathfrak{m}_\gamma \quad \text{with } \Delta^* = \left\{\gamma \in \Delta^+ : k_{\gamma^*}(\gamma) = 1\right\}$$

and also that $\mathfrak{p}^*$ gives rise to a projective subspace in $X[G/T]$ which is maximal if and only if $\pi_2(G/K) \neq 0$ (see [6, Remark 4.1, Theorem 4.4] and [12, p. 408–9]).

This fact motivates the study of the subspaces of $\mathfrak{m}_n = T_{[T^n]} SU(n+1)/T^n$ of the form $\widetilde{\mathfrak{p}} = \sum_{\gamma\in\widetilde{\Delta}} \mathfrak{m}_{\gamma;n}$ with $\widetilde{\Delta} \subset \Delta^+_n$, with the goal of getting information about the existence of projective subspaces in $X[M_n]$. These subspaces are exactly those invariant by the natural action of the torus T^n on $\mathfrak{m}_n$ (see for instance [12, p. 407, Th5]).

Following the notation in [7, p. 418] we say that a set $\widetilde{\Delta}$ in Δ^+_n is a *presymmetric set* if it satisfies the following property

$$\varepsilon, \rho \in \widetilde{\Delta} \quad \Rightarrow \quad \varepsilon + \rho \notin \widetilde{\Delta}.$$

We know the following results.

Theorem 4.2. [6, p. 216, (4.2)] *Set* $\widetilde{\mathfrak{p}} = \sum_{\gamma\in\widetilde{\Delta}} \mathfrak{m}_{\gamma;n}$ *with* $\widetilde{\Delta} \subset \Delta^+_n$. *Then*

$$RP(\widetilde{\mathfrak{p}}) \subset X[M_n] \Longleftrightarrow \widetilde{\Delta} \text{ is a presymmetric set.} \qquad \square$$

Set

$$K_n = \begin{cases} S\left(U\left(\frac{n}{2}+1\right) \times U\left(\frac{n}{2}\right)\right) & \text{if } n \text{ is even,} \\ S\left(U\left(\frac{n+1}{2}\right) \times U\left(\frac{n+1}{2}\right)\right) & \text{if } n \text{ is odd;} \end{cases}$$

$$d_n = \begin{cases} \frac{n(n+2)}{2} & \text{if } n \text{ is even,} \\ \frac{(n+1)^2}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Theorem 4.3. [7, p. 416, (1.1), (1.2)] *Let* $\widetilde{\mathfrak{p}} = \sum_{\gamma\in\widetilde{\Delta}} \mathfrak{m}_{\gamma;n}$ *be such that* $\widetilde{\Delta} \subset \Delta^+_n$ *is a presymmetric set. Then*

(i) $\dim \widetilde{\mathfrak{p}} \leq d_n$.

(ii) *If* $\dim \widetilde{\mathfrak{p}} = d_n$ *then* $\widetilde{\mathfrak{p}}$ *is the tangent space to the inner symmetric space* $SU(n+1)/K_n$ *at a fixed point of the action of the torus* T^n. □

These two theorems are written in the indicated references for the manifold of complete flags of any compact simple Lie group. Here we are considering the case of the group $SU(n+1)$ and in this case, the irreducible inner symmetric space of maximal dimension associated is $SU(n+1)/K_n$.

Theorem 4.3 characterizes those subspaces $\widetilde{\mathfrak{p}}$ of $\mathfrak{m}_n$ which are $Ad\,(T^n)$-invariant and define projective subspaces of maximal dimension in $X\,[M_n]$. Making a deeper analysis in this direction we study now those subspaces $\widetilde{\mathfrak{p}}$ that are $Ad\,(T^n)$-invariant and define projective subspaces in $X\,[M_n]$ but in some "interesting" sense are of "minimal" dimension. It is easy to see that these subspaces are those $\widetilde{\mathfrak{p}}$ which are of minimal dimension not properly contained in any other $Ad\,(T^n)$-invariant subspace defining projective subspaces in $X\,[M_n]$. For these subspaces we have obtained the following

Theorem 4.4. *Let $n \geq 2$ and $M_n = SU(n+1)/T^n$ be embedded in $\mathfrak{su}\,(n+1)$ as orbit of any regular element E. Let $\widetilde{\mathfrak{p}} = \sum_{\beta\in\widetilde{\Delta}} \mathfrak{m}_{\beta;n}$ ($\widetilde{\Delta} \subset \Delta_n^+$) be a subspace of $\mathfrak{m}_n = T_E\,(M_n) \subset \mathfrak{su}\,(n+1)$ which is maximal among the subspaces $Ad\,(T^n)$-invariant of $\mathfrak{m}_n$ defining projective subspaces in $X\,[M_n]$. Then*

(i) $\dim \widetilde{\mathfrak{p}} \geq 2n$.

(ii) *If $\dim \widetilde{\mathfrak{p}} = 2n$ then $\widetilde{\mathfrak{p}}$ is the tangent space to the projective space $CP^n = SU(n+1)/S\,(U\,(n) \times U\,(1))$ at a point $E_1 = \widetilde{\sigma}(iv_n)$ where v_n is given by (12) and $\widetilde{\sigma}$ is an element in W_{n+1}, the Weyl group of the pair $(SU(n+1)\,,\, T^n)$.*

Proof. Let us recall that we are using for the Lie algebra $\mathfrak{sl}\,(n+1, C)$ the notation given in [13, III, 8]. Thus, if $\beta \in \Delta_n^+$ then there exist i, j ($1 \leq i < j \leq n+1$) such that $\beta = e_i - e_j$.

It is well known that if $\widetilde{\eta} \in W_{n+1}$ then

$$\widetilde{\eta}\,(e_i - e_j) = e_{\eta(i)} - e_{\eta(j)} \tag{17}$$

where η is a permutation of the set $\{1, \ldots, n+1\}$, that is $\eta \in \mathfrak{S}_{n+1}$. Conversely if $\eta \in \mathfrak{S}_{n+1}$ the equality (17) defines an element $\widetilde{\eta}$ in W_{n+1}.

First we shall prove (ii). Let $\widetilde{\Delta}$ be the set $\{\beta_l = e_{i_l} - e_{j_l} : 1 \leq l \leq n\}$. We consider two cases.

(ii,a): Let us assume that there exists k such that $k \in \{i_l, j_l\}$ for all l such that $1 \leq l \leq n$. Then by taking $\eta \in \mathfrak{S}_{n+1}$ such that $\eta\,(k) = n+1$ and the corresponding $\widetilde{\eta}$ defined in (17), we obtain the following subset of Δ_n^+

$$\widetilde{\Delta}_1 = \{|\widetilde{\eta}\,(\beta_l)| : 1 \leq l \leq n\} = \{e_1 - e_{n+1}, \ldots, e_n - e_{n+1}\}\,.$$

Then

$$\sum_{\beta\in\widetilde{\Delta}_1} \mathfrak{m}_{\beta;n} = T_{iv_n}\,(SU(n+1)/S\,(U\,(n) \times U\,(1)))$$

and therefore $\widetilde{\mathfrak{p}}$ is the tangent space at $E_1 = \widetilde{\sigma}(iv_n)$ where $\widetilde{\sigma} = \widetilde{\eta}^{-1}$.

(ii,b): Let us assume now that for every k $(1 \leq k \leq n+1)$ there is at least one set $\{i_l, j_l\}$ $(1 \leq l \leq n)$ such that $k \notin \{i_l, j_l\}$. In this case we shall arrive to a contradiction by showing that $\widetilde{\Delta}$ is not a maximal presymmetric set.

Let us associate to each r $(1 \leq r \leq n+1)$, the number r_o defined by

$$r_o = \#\{l : 1 \leq l \leq n \text{ and } r \in \{i_l, j_l\}\}.$$

Then, by our assumption, $\max\{r_o : 1 \leq r \leq n+1\} < n$. Let s be such that the corresponding $s_o = \max\{r_o : 1 \leq r \leq n+1\}$.

Let us choose $\eta \in \mathfrak{S}_{n+1}$ such that $\eta(s) = n+1$ and

$$\{e_1 - e_{n+1}, \ldots, e_{s_o} - e_{n+1}\} \subset \left\{|\widetilde{\eta}(\beta)| : \beta \in \widetilde{\Delta}\right\} = \widetilde{\Delta}_1.$$

Then

$$e_i - e_{n+1} \notin \widetilde{\Delta}_1 \text{ for } i > s_o. \tag{18}$$

We shall see that $\widetilde{\Delta}$ is not a maximal presymmetric set by showing that $\widetilde{\Delta}_1$ is not maximal presymmetric. To that end we shall consider two possibilities.

(ii,b,1). There exists a j, $s_o + 1 \leq j \leq n$ such that for every i, $1 \leq i \leq s_o$, $e_i - e_j \notin \widetilde{\Delta}_1$. In this case, we define

$$\widetilde{\Delta}_2 = \widetilde{\Delta}_1 \cup \{e_j - e_{n+1}\},$$

and we shall prove that $\widetilde{\Delta}_2$ is a presymmetric set.

Let ε be the added root $e_j - e_{n+1}$. If there are two positive roots γ and δ such that $\gamma + \delta = \varepsilon$, they will be of the form $\gamma = e_j - e_k$ and $\delta = e_k - e_{n+1}$ with $s_o < j < k$. Then by (18) the root δ cannot belong to $\widetilde{\Delta}_1$. On the other hand, if there are two roots γ and δ in $\widetilde{\Delta}_1$ such that $\delta = \gamma + \varepsilon$, then $\gamma = e_i - e_j$ with $1 \leq i < j$ and $i \leq s_o$. By the assumption in (ii,b,1), this root γ cannot belong to $\widetilde{\Delta}_1$ which shows that $\widetilde{\Delta}_2$ is a presymmetric set. Then $\widetilde{\Delta}_1$ is not maximal.

(ii,b,2) Let us assume now that for every j, $s_o + 1 \leq j \leq n$, there exists i, $1 \leq i \leq s_o$, such that the $e_i - e_j \in \widetilde{\Delta}_1$.

Since the cardinal of the set $\{j : s_o + 1 \leq j \leq n\}$ is $n - s_o$, for each j in this set, there exists a unique i $(1 \leq i \leq s_o)$ such that $e_i - e_j \in \widetilde{\Delta}_1$.

Let us consider the set of positive roots

$$R(s_o) = \{e_i - e_j : 1 \leq i \leq s_o, s_o + 1 \leq j \leq n+1\}$$

then

$$\sum_{\beta \in R(s_o)} \mathfrak{m}_{\beta;n}$$

is the tangent space to the symmetric space $SU(n+1)/S(U(s_o) \times U(t_o))$ at some point $(t_o = n + 1 - s_o)$. Therefore, by [6, Proposition (4.1)], $R(s_o)$ is a presymmetric set.

Since $\widetilde{\Delta}_1 \subsetneq R(s_o)$ we conclude that $\widetilde{\Delta}_1$ is not maximal.

To prove (i) we assume that $\dim \widetilde{\mathfrak{p}} < 2n$, i.e. $\widetilde{\Delta} = \{\beta_1, \dots, \beta_t\}$ with $t < n$. We shall see that in this case the subspace $\widetilde{\mathfrak{p}}$ is not maximal. By proceeding as in (ii) above, it is easy to see that only case (ii,b,1) can occur because $\#\widetilde{\Delta} < n$. This leads us into proving that $\widetilde{\Delta}$ is not maximal which contradicts our hypothesis. □

This result allows us to complete [12, Theorem 8] for the groups $SU(n)$, keeping the notation of that paper, as follows.

Corollary 4.2. *Let $\mathfrak{w}$ be maximal among the holomorphic tangent spaces at the base point of $SU(n+1)$-invariant minimal almost Hermitian extrinsic symmetric CR-structure on $M_n = SU(n+1)/T^n$. Then*

(i) $\dim_R \mathfrak{w} \geq 2n$.

(ii) *If* $\dim_R \mathfrak{w} = 2n$ *then* $\mathfrak{w}$ *is the tangent space to the projective space* $CP^n = SU(n+1)/S\,(U\,(n) \times U\,(1))$ *at some point.*

Proof. It follows from the previous theorem by noticing that the proof of [12, Theorem 8] reduces essentially to show that $\mathfrak{w}$ is an $Ad(T^n)$-invariant subspace of $\mathfrak{m}$ and gives rise to a projective subspace in $X\,[M_n]$. □

Due to the fact that the converse statement of (ii) in Theorem 4.4 is obviously true, this theorem gives us a geometric characterization of the subspaces $\widetilde{\mathfrak{p}}$ of $\mathfrak{m}_n$ which are of dimension $2n$ and maximal among the subspaces $Ad\,(T^n)$-invariant of $\mathfrak{m}_n$ defining projective subspaces in $X\,[M_n]$.

Joining Theorems 4.3 and 4.4, the subspaces $\widetilde{\mathfrak{p}}$ of $\mathfrak{m}_n$ which are maximal among the subspaces $Ad\,(T^n)$-invariant of $\mathfrak{m}_n$ defining projective subspaces in $X\,[M_n]$, satisfy

$$2n \leq \dim \widetilde{\mathfrak{p}} \leq d_n$$

and also, when $\dim \widetilde{\mathfrak{p}}$ is one of the two ends of the above inequality, the subspace $\widetilde{\mathfrak{p}}$ is tangent to the inner symmetric space of minimal and maximal dimension associated to the group $SU(n+1)$.

When the subspace $\widetilde{\mathfrak{p}}$ is such that $2n < \dim \widetilde{\mathfrak{p}} < d_n$, if we pose no restriction on n and $\dim \widetilde{\mathfrak{p}}$, we cannot assure that $\widetilde{\mathfrak{p}}$ is tangent to some inner symmetric space of the group $SU(n+1)$. Furthermore we cannot assure that $\widetilde{\mathfrak{p}}$ is tangent to a homogeneous manifold $SU(n+1)/K$ with $T^n \subset K$, as the following examples show.

Example 4.1. For $n = 6$ we consider the following two subspaces $\widetilde{\mathfrak{p}}$ of $\mathfrak{m}_6$ which are maximal among the subspaces $Ad\,(T^6)$-invariant of $\mathfrak{m}_6$ defining projective subspaces in $X\,[M_6]$.

(i) $\widetilde{\mathfrak{p}} = \sum_{\beta \in \widetilde{\Delta}} \mathfrak{m}_\beta$ with $\widetilde{\Delta} = \{e_1 - e_5, e_1 - e_7, e_2 - e_6, e_2 - e_7, e_3 - e_6, e_3 - e_7, e_4 - e_5, e_4 - e_7, e_5 - e_6\}$.

Since $\dim \widetilde{\mathfrak{p}} = 18$ and the inner symmetric spaces associated to the group $SU(7)$ are of dimension $12, 20$ and 24, $\widetilde{\mathfrak{p}}$ is not tangent to an inner symmetric space of the group $SU(7)$. Furthermore, $\widetilde{\mathfrak{p}}$ is not tangent to a homogeneous space of the group $SU(7)$ because the orthogonal space $\widetilde{\mathfrak{p}}^\perp$ is not a subalgebra (note that $e_1 - e_2$, $e_2 - e_5 \notin \widetilde{\Delta}$ and $e_1 - e_5 \in \widetilde{\Delta}$).

(ii) $\widetilde{\mathfrak{p}} = \sum_{\beta\in\widetilde{\Delta}} \mathfrak{m}_\beta$ with $\widetilde{\Delta} = \{e_1 - e_5, e_1 - e_7, e_2 - e_4, e_2 - e_6, e_2 - e_7, e_3 - e_4, e_3 - e_6, e_3 - e_7, e_4 - e_5, e_5 - e_6\}$.

Even when the dimension of $\widetilde{\mathfrak{p}}$ coincides now with the dimension of one of the symmetric spaces of $SU(7)$, it is not tangent to a homogeneous space because $\widetilde{\mathfrak{p}}^\perp$, as above, is not a subalgebra.

References

[1] Bredon, G.: *Introduction to compact transformation groups*. Pure and Applied Mathematics **46**, Academic Press, New York-London 1972. Zbl 0246.57017

[2] Burstall, F. E.; Rawnsley, J. H.: *Twistor theory for Riemannian Symmetric Spaces*. Lecture Notes in Mathematics **1424**. Springer-Verlag, Berlin etc. 1990. Zbl 0699.53059

[3] Chen, B.-Y.: *Differential geometry of submanifolds with planar normal sections*. Ann. Mat. Pura Appl., IV. Ser. **130** (1982), 59–66. Zbl 0486.53004

[4] Chen, B.-Y.; Verheyen, P.: *Sous-variétés dont les sections normales sont des geodesiques*. (French) C. R. Acad. Sci., Paris, Sér. I **293** (1981), 611–613. Zbl 0489.53008

[5] Dal Lago, W.; García, A.; Sánchez, C.: *Planar normal sections on the natural imbedding of a flag manifold.* Geom. Dedicata **53**(2) (1994), 223–235. Zbl 0839.53034

[6] Dal Lago, W.; García, A.; Sánchez, C.: *Maximal projective subspaces in the variety of planar normal sections of a flag manifold.* Geom. Dedicata **75**(3) (1999), 219–233. Zbl 0964.53033

[7] Dal Lago, W.; García, A.; Sánchez, C.: *Projective subspaces in the variety of normal sections and tangent spaces to a symmetric space.* J. Lie Theory **8**(2) (1998), 415–428. Zbl 0908.53029

[8] Deprez, J.; Verheyen, P.: *Immersions with circular normal sections and normal sections of product immersions.* Geom. Dedicata **20** (1986), 335–344. Zbl 0591.53048

[9] Ferus, D.: *Immersionen mit paralleler zweiter Fundamentalform: Beispiele und Nicht-Beispiele*. (German) Manuscr. Math. **12** (1974), 153–162. Zbl 0274.53058

[10] Ferus, D.: *Symmetric submanifolds of Euclidean space.* Math. Ann. **247** (1980), 81–93. Zbl 0446.53041

[11] Freudenthal, H.; de Vries, H.: *Linear Lie groups.* Pure and Applied Mathematics **35**. Academic Press, New York-London 1969. Zbl 0377.22001

[12] García, A.; Sánchez, C.: *On extrinsic symmetric CR-structures on the manifolds of complete flags.* Beitr. Algebra Geom. **45**(2) (2004), 401–414. Zbl pre02100230

[13] Helgason, S.: *Differential geometry, Lie groups, and symmetric spaces.* Pure and Applied Mathematics **80**. Academic Press, New York-San Francisco-London 1978. Zbl 0451.53038

[14] Harris, J.: *Algebraic geometry. A first course.* Graduate Texts in Mathematics **133**. Springer-Verlag, Berlin etc. 1992. Zbl 0779.14001

[15] Humphreys, J. E.: *Introduction to Lie algebras and representation theory.* Graduate Texts in Mathematics **9**. Springer-Verlag, New York-Heidelberg-Berlin 1972. Zbl 0254.17004

[16] Wolf, J. A.; Gray, A.: *Homogeneous spaces defined by Lie group automorphisms. I.* J. Differ. Geom. **2** (1968), 77–114. Zbl 0169.24103

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