Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 47 (2006), No. 1, 121-135.

The Effect of Quadratic Transformations on Degree Functions

R. Debremaeker[†] **V. Van Lierde**[‡]

[†] *Monitoraat Wiskunde*
K.U.Leuven, Celestijnenlaan 200C
3001 Heverlee, Belgium
raymond.debremaeker@wis.kuleuven.be

[‡] *Al Akhawayn University*
P.O. Box 1828 Avenue Hassan II, Ifrane 53000, Morocco

Abstract. We study the effect of a quadratic transformation on the degree function of a 0-dimensional ideal with only one Rees valuation in a 2-dimensional regular local ring with algebraically closed residue field.
A number of important results of Zariski and Lipman about complete ideals in a 2-dimensional regular local ring follow as quick corollaries.
Necessary and sufficient conditions for the regularity of a 2-dimensional normal local domain are proved.

MSC 2000: 13H05, 13H15
Keywords: degree function, quadratic transformation, regular local ring, complete ideals

1. Introduction

The main objective of this paper is to study the effect of a quadratic transformation on the degree function of a 0-dimensional ideal with only one Rees valuation in a 2-dimensional regular local ring.

We begin by introducing the relevant definitions and giving some background.

By a local ring we will always mean a commutative Noetherian ring with a unique maximal ideal. We will also assume throughout that the residue field is infinite.

Let $(R, \mathcal{M})$ be a local domain with quotient field K. With an $\mathcal{M}$-primary ideal I of R, Rees [5] associated an integer-valued function d_I on $\mathcal{M} \setminus \{0\}$ as follows:

$$d_I(x) = e\Big(\frac{I + xR}{xR}\Big)$$

where $e(\frac{I+xR}{xR})$ denotes the multiplicity of $\frac{I+xR}{xR}$. The function d_I is called the *degree function* defined by I.

With every prime divisor v of R, there is associated a non-negative integer $d(I, v)$, with $d(I, v) = 0$ for all except finitely many v, such that

$$d_I(x) = \sum_v d(I, v)v(x) \qquad \forall 0 \neq x \in \mathcal{M}$$

where the sum is over all prime divisors v of R ([5], Theorem 3.2). By a prime divisor v of R we mean a discrete valuation v of K which is non-negative on R and has center $\mathcal{M}$ on R and whose residual transcendence degree is $\dim R - 1$. The set of all prime divisors of R will be denoted by $P(R)$. In case $(R, \mathcal{M})$ is analytically unramified, $d(I, v) \neq 0$ for all $v \in P(R)$ that are Rees valuations of I as defined by Rees in [5], whereas $d(I, v') = 0$ for all other prime divisors v' of R.

More background information on degree functions will be given in Section 2.

Now we turn to the other term in our title, quadratic transformations.

Let $(R, \mathcal{M})$ be a noetherian local domain with field of fractions K. We denote by $B\ell_{\mathcal{M}}R$ the scheme $Proj(\oplus_{n\geq 0}\mathcal{M}^n)$ obtained by blowing up $\mathcal{M}$. For any $x \in \mathcal{M} \setminus \mathcal{M}^2$ and any maximal ideal N in $R[\frac{\mathcal{M}}{x}]$ containing $\mathcal{M}R[\frac{\mathcal{M}}{x}]$, the ring

$$S := R[\frac{\mathcal{M}}{x}]_N$$

is called a *first (or an immediate) quadratic transform* of R.

From now till the end of the introduction we shall assume that the local ring $(R, \mathcal{M})$ is regular. Then the ring S is a 2-dimensional regular local ring birationally dominating R.

Let I be an $\mathcal{M}$-primary ideal in R. The ideal

$$I^S := IS \cdot (IS)^{-1}$$

is called the *transform of I in S*.

There are only finitely many immediate quadratic transforms S of R for which $ord_S(I^S) \neq 0$, where ord_S is the order valuation associated with the maximal ideal of S; these are called the *immediate base points of I*. By the local factorization theorem of Zariski-Abhyankar, we know that any 2-dimensional regular local ring S birationally dominating R can be reached by a finite number of quadratic transformations:

$$R = R_0 < R_1 < R_2 < \cdots < R_n = S.$$

This sequence is unique and is called *the quadratic sequence from R to S*. Let I^{R_i} denote the transform of $I^{R_{i-1}}$ in R_i for $1 \leq i \leq n$. Then I^{R_n} is called the *transform of I in S*, denoted by I^S.

By a *base point S of I* we mean a 2-dimensional regular local ring S birationally dominating R such that $I^S \neq S$.

A 2-dimensional regular local ring S birationally dominating R is a base point of I if and only if S is dominated by a Rees valuation ring of I (see [4], p. 295). Hence a given $\mathcal{M}$-primary ideal I in R has only finitely many base points.

We recall that the Rees valuation rings of an $\mathcal{M}$-primary ideal I of R are defined as follows. Let $\overline{R[It,t^{-1}]}$ be the integral closure of $R[It,t^{-1}]$ in the quotient field $K(t)$, and let $\{P_1,\dots,P_n\}$ be the minimal primes of $(t^{-1})\overline{R[It,t^{-1}]}$. Then $\overline{R[It,t^{-1}]}$ is a Krull domain and each P_i is a height one prime and hence $(\overline{R[It,t^{-1}]})_{P_i}$ is a discrete valuation ring of $K(t)$ for $i=1,\dots,n$.

The *Rees valuation rings of I* are

$$V_i := (\overline{R[It,t^{-1}]})_{P_i} \cap K \qquad i=1,\dots,n.$$

The corresponding discrete valuations $v_1,\dots,v_n$ are called the *Rees valuations of I* and the set of these Rees valuations is denoted by $T(I)$, i.e.

$$T(I) = \{v_1,\dots,v_n\}.$$

The main purpose of this paper is to study, in case of a 2-dimensional regular local ring with algebraically closed residue field, the effect of a quadratic transformation on the numbers $d(I,v)$ in case I is a *one-fibered* $\mathcal{M}$-primary ideal (i.e. I has only one Rees valuation v and hence $T(I)=\{v\}$).

The key result is the following: the transform I^S of I in every base point S of I satisfies the property that $T(I^S)=\{v\}$ and $d(I^S,v)=d(I,v)$. A proof of the result (and of a number of related results) will be given in Section 3.

Besides Göhner's work ([3], Section 2) on normal complete models over R in K, we will use the so-called Length Formula of Hoskin-Deligne.

A short and elementary proof of this formula is presented in [2]. As a consequence of this elementary proof a remarkable short proof of Zariski's Product Theorem (ZPT), i.e. the product of complete ideals is again complete, can be given in a way which is logically independent of the material in this paper.

See e.g. J. K. Verma's manuscript "Zariski-Lipman Theory of complete ideals in 2-dimensional regular local rings" of July 5, 2003.

In what follows the above mentioned property ZPT of a 2-dimensional regular local ring will be used in an essential way.

Further in Section 3 we will show that the following well-known results of a 2-dimensional regular local ring with algebraically closed residue field, follow from our key result:

- Every simple complete $\mathcal{M}$-primary ideal of R has exactly one Rees valuation,
- Zariski's one to one correspondence,
- Zariski's unique factorization theorem,
- Lipman's reciprocity and multiplicity formula.

From the results of Section 3 it follows a.o. that in a 2-dimensional regular local ring with algebraically closed residue field the following holds: for every prime divisor v of R there exists a complete $\mathcal{M}$-primary ideal I in R such that $T(I) = \{v\}$ and $d(I, v) = 1$.

In Section 4 we will give examples of 2-dimensional normal local domains $(R, \mathcal{M})$ showing that this property does no longer hold if R is not regular.

So one can ask to what extent this property is characteristic for the regularity of a 2-dimensional normal local domain. It is proved that the above mentioned property, completed with some natural conditions on the unique maximal ideal $\mathcal{M}$ of R does imply the regularity of R.

2. Background on degree functions

As we have seen in the introduction, the degree function d_I of an $\mathcal{M}$-primary ideal I in a noetherian local domain $(R, \mathcal{M})$ can be written as follows:

$$d_I(x) = \sum_{v \in P(R)} d(I, v) \cdot v(x) \qquad \forall 0 \neq x \in \mathcal{M}.$$

In [6] Rees and Sharp prove that the integers $d(I, v)$ are uniquely determined by the previous condition, i.e. suppose that

$$\sum_{v \in P(R)} d(I, v) v(x) = \sum_{v \in P(R)} d'(I, v) v(x) \qquad \forall 0 \neq x \in \mathcal{M}$$

then $d(I, v) = d'(I, v)$ for every prime divisor v of R. From this uniqueness it follows that for $\mathcal{M}$-primary ideals I and J in a 2-dimensional noetherian local domain $(R, \mathcal{M})$, one has that

$$d(IJ, v) = d(I, v) + d(J, v)$$

for every prime divisor v of R ([6], Lemma 5.1, p. 459). If we make the additional assumption that R is analytically unramified and normal, then this implies that

$$T(IJ) = T(I) \cup T(J).$$

In [6] Theorem 4.3, p. 457, Rees and Sharp show that for an $\mathcal{M}$-primary ideal I in a 2-dimensional local domain $(R, \mathcal{M})$, the multiplicity $e(I)$ of I is given by

$$e(I) = \sum_{v \in P(R)} d(I, v) v(I).$$

For I and J $\mathcal{M}$-primary ideals in a 2-dimensional Cohen-Macaulay local domain $(R, \mathcal{M})$, Rees and Sharp define

$$d_I(J) = \min\{d_I(x) | 0 \neq x \in J\}$$

and they prove ([6], Theorem 5.2, p. 460) that

$$d_I(J) = \sum_{v \in P(R)} d(I, v) \cdot v(J)$$

and

$$d_I(J) = d_J(I) = e_1(I|J)$$

where $e_1(I|J)$ denotes the mixed multiplicity of I and J and is defined by $e(IJ) = e(I) + 2e_1(I|J) + e(J)$ (see a.o. [7], p. 1037).

We end this section with the following result of Rees and Sharp ([6], Corollary 5.3, p. 461).

Let I and J be $\mathcal{M}$-primary ideals in the 2-dimensional $\mathcal{M}$-primary ideals in the 2-dimensional Cohen-Macaulay local domain $(R, \mathcal{M})$. Then the following three statements are equivalent:

(1) $\bar{I} = \bar{J}$ where "$^{-}$" denotes the integral closure.
(2) $d_I(x) = d_J(x) \quad \forall x \in \mathcal{M} \setminus \{0\}$
(3) $d(I, v) = d(J, v) \quad \forall v \in P(R)$

3. Main result

In this section, $(R, \mathcal{M})$ will denote a 2-dimensional regular local ring with algebraically closed residue field and fraction field K. Following Göhner [3, Section 2] we shall regard $B\ell_{\mathcal{M}}R$ as a model over R.

The assumption that the residue field is algebraically closed is not strictly necessary, but the restriction is made for the purpose of simplifying the presentation.

Now we prove the main result of this section.

Proposition 3.1. *Let I be a complete $\mathcal{M}$-primary ideal of R with $T(I) = \{v\}$ and $v \neq ord_R$. Then I has exactly one immediate base point R_1 and if I_1 is the transform of I in R_1, then $T(I_1) = \{v\}$ and $d(I_1, v) = d(I, v)$.*

Proof. Let $(R_1, \mathcal{M}_1)$ be the unique local ring of the complete normal model $B\ell_{\mathcal{M}}R$ dominated by the valuation ring $(V, \mathcal{M}_V)$ of v. Since $(V, \mathcal{M}_V)$ is the unique Rees valuation ring of I, it follows that $(R_1, \mathcal{M}_1)$ is the unique immediate base point of I.

According to Göhner ([3], Proposition 2.9, p. 414) there is an integer $e > 0$ such that the transform of I^e in R_1 has v as its unique Rees valuation. Since R_1 is regular and hence a UFD, we can take $e = 1$ and so $T(I_1) = \{v\}$.

To finish the proof it remains to show that $d(I_1, v) = d(I, v)$. We commence by noting that the unique immediate base point R_1 of I is a local ring of the form

$$R_1 = R[\frac{\mathcal{M}}{x}]_N$$

with N a height 2-prime ideal of $R[\frac{\mathcal{M}}{x}]$ lying over $\mathcal{M}$ and $x \in \mathcal{M} \setminus \mathcal{M}^2$.

Denote $r := ord_R(I)$, then $IR_1 = x^r \cdot I_1$. Since $T(I) = \{v\}$, we have according to Section 2 that $e(I) = d(I, v)v(I)$. Hence $d(I, v) = \frac{e(I)}{v(I)}$. Similarly $T(I_1) = \{v\}$ implies $d(I_1, v) = \frac{e(I_1)}{v(I_1)}$. From the Length formula of Hoskin-Deligne it follows

$$e(I) = \sum_s ord_s(I^s)^2$$

where the sum is over all the base points s of I. Thus

$$e(I) = r^2 + e(I_1).$$

Since $IR_1 = x^r \cdot I_1$ we have

$$v(I) = r \cdot v(\mathcal{M}) + v(I_1)$$

and hence

$$d(I, v) = \frac{e(I_1) + r^2}{v(I_1) + r \cdot v(\mathcal{M})} = \frac{d(I_1, v) + \frac{r^2}{v(I_1)}}{1 + r \cdot \frac{v(\mathcal{M})}{v(I_1)}}.$$

It follows that

$$d(I, v)\Big(1 + r \cdot \frac{v(\mathcal{M})}{v(I_1)}\Big) = d(I_1, v) + \frac{r^2}{v(I_1)}.$$

Since $d_I(\mathcal{M}) = d_{\mathcal{M}}(I)$ and $e(\mathcal{M}) = 1$, it follows that

$$d(I, v) \cdot v(\mathcal{M}) = r$$

and this together with the previous relation yields $d(I, v) = d(I_1, v)$. □

Conversely, let us start with a complete $\mathcal{M}_1$-primary ideal I_1 in an immediate quadratic transform $(R_1, \mathcal{M}_1)$ of $(R, \mathcal{M})$. Then R_1 is of the form $R_1 = R[\frac{\mathcal{M}}{x}]_N$, with $x \in \mathcal{M} \setminus \mathcal{M}^2$ and N a maximal ideal in $R[\frac{\mathcal{M}}{x}]$ lying over $\mathcal{M}$.

We now recall the definition of the *inverse transform* of I_1 in R.

Let a be the smallest positive integer so that $x^a I_1$ is *extended* from R, i.e. there exists an ideal J of R such that $x^a I_1 = JR_1$. Then $I := x^a I_1 \cap R$ is called the *inverse transform* of I_1 in R. It is clear that

$$x^a I_1 = IR_1$$

and

$$IR_1 \cap R = I$$

in other words, I is contracted from R_1. Note also that

$$a = ord_R(I).$$

Since I_1 is $NR[\frac{\mathcal{M}}{x}]_N$-primary, there is exactly one N-primary ideal in $R[\frac{\mathcal{M}}{x}]$, say I', such that

$$I_1 = I'_N.$$

Lemma 3.2. *I' is the transform of I in $R[\frac{\mathcal{M}}{x}]$ i.e. $IR[\frac{\mathcal{M}}{x}] = x^a \cdot I'$.*

Proof. Let b be the smallest positive integer such that $x^b I'$ is *extended* from R i.e. $x^b I' = (x^b I' \cap R)R[\frac{\mathcal{M}}{x}]$.

This implies that $x^b I_1$ is *extended* from R and hence $b \geq a$.

Since $I = x^a I' \cap R$, it is sufficient to prove that $b = a$. Suppose $b > a$, then

$$(x^b I' \cap R)R[\mathcal{M}x]_N = \mathcal{M}^{b-a} \cdot IR[\frac{\mathcal{M}}{x}]_N$$

and contraction to R implies

$$x^b I' \cap R = \mathcal{M}^{b-a} I$$

because $x^b I' \cap R$ as well as $\mathcal{M}^{b-a} I$ is contracted from $R[\frac{\mathcal{M}}{x}]_N$. Extension to $R[\frac{\mathcal{M}}{x}]$ yields

$$x^b I' = x^{b-a} I R[\frac{\mathcal{M}}{x}]$$

and this implies

$$x^a I' = I R[\frac{\mathcal{M}}{x}].$$

Thus $x^a I'$ is extended from R, so by the choice of b one has $a \geq b$, a contradiction with $b > a$. $\square$

Corollary 3.3. *The inverse transform I of I_1 has R_1 as its unique immediate base point.*

Proof. As the residue field of R is infinite, we may suppose without loss of generality that the element $x \in \mathcal{M} \setminus \mathcal{M}^2$ is chosen in such a way that all immediate base points of I are localizations of $R[\frac{\mathcal{M}}{x}]$. This together with Lemma 3.2 proves the assertion. $\square$

So far I_1 was a complete $\mathcal{M}_1$-primary ideal of R_1. Now we assume additionally that I_1 has only one Rees valuation v, i.e. $T(I_1) = \{v\}$. Then we can prove the following converse of Proposition 3.1.

Proposition 3.4. *The inverse transform I of I_1 in R is a complete $\mathcal{M}$-primary ideal such that $T(I) = \{v\}$ and $d(I, v) = d(I_1, v)$.*

Proof. Since $I = x^a I_1 \cap R$, it is clear that I is a complete $\mathcal{M}$-primary ideal in R.

Next we prove that $T(I) = \{v\}$. Because of Corollary 3.3, we know that I has exactly one base point among all the immediate quadratic transforms of R, namely R_1. This implies that the blow-up $B\ell_{I\mathcal{M}}R$ of R at $I\mathcal{M}$ is obtained from $B\ell_{\mathcal{M}}R$ by blowing up R_1 at I_1 while leaving unaltered all the other local rings of $B\ell_{\mathcal{M}}R$.

It follows that

$$T(I\mathcal{M}) = T(I) \cup T(\mathcal{M}) = \{ord_R, v\}$$

where ord_R denotes the $\mathcal{M}$-adic order valuation of R. So it remains to prove that ord_R cannot be a Rees valuation of I.

To this end note that $\mathcal{M}$ does not divide I, thus $s(I) = ord_R(I)$ where $s(I)$ denotes the degree of the gcd of the elements of $\frac{I+\mathcal{M}^{a+1}}{\mathcal{M}^{a+1}}$ ([8], Proposition 3, p. 368). From this it follows that I has an ideal basis $(x_0, x_1, \ldots, x_n)$ such that $ord_R(\frac{x_i}{x_0}) > 0$ for $i = 1, \ldots, n$.

Consequently $R[\frac{I}{x_0}]$ is contained in the valuation ring W of ord_R and the elements $\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}$ belong to its unique maximal ideal $\mathcal{M}_W$. This implies that the unique local ring of $B\ell_I R$ dominated by W is 2-dimensional. So W does not belong to $B\ell_I R$, which means that ord_R is not a Rees valuation of I.

Finally, since $T(I) = \{v\}$ and I_1 is the transform of I in R_1, Proposition 3.1 implies $d(I, v) = d(I_1, v)$. □

In the next proposition we will prove among other things that a 2-dimensional regular local ring (with algebraically closed residue field) satisfies condition (N), i.e. for every prime divisor v of R, there exists an $\mathcal{M}$-primary ideal I of R with v as its unique Rees valuation.

Proposition 3.5. *Let v be a prime divisor of R. Then:*

(1) *There exists a complete $\mathcal{M}$-primary ideal I of R such that $T(I) = \{v\}$ and $d(I, v) = 1$ (consequently I is simple i.e., not factorable into a product of proper ideals).*

(2) *This ideal I is the only simple complete $\mathcal{M}$-primary ideal of R with v as its unique Rees valuation.*

(3) *The set of all complete $\mathcal{M}$-primary ideals of R with v as unique Rees valuation consists of all powers of I.*

Proof. Let $(V, \mathcal{M}_V)$ be the valuation ring of v. It is readily seen that (1) holds if $v = ord_R$. So let us assume that $v \neq ord_R$. Let $(V, \mathcal{M}_V)$ denote the valuation ring of v.

Since v is a prime divisor of R, Abhyankar ([1], p. 336, Proposition 3) has proved that there exists a unique finite quadratic sequence starting from R and dominated by the valuation ring V of v:

$$(R, \mathcal{M}) = (R_0, \mathcal{M}_0) < (R_1, \mathcal{M}_1) < \cdots < (R_n, \mathcal{M}_n) < (V, \mathcal{M}_V)$$

i.e. $(R_i, \mathcal{M}_i)$ is an immediate quadratic transform of $(R_{i-1}, \mathcal{M}_{i-1})$ for $i = 1, \ldots, n$ and V is the $\mathcal{M}_n$-adic order valuation ring of R_n.

Now $\mathcal{M}_n$ is a complete $\mathcal{M}_n$-primary ideal in the 2-dimensional regular local ring R_n with $T(\mathcal{M}_n) = \{v\}$ and $d(\mathcal{M}_n, v) = 1$. So if I_{n-1} denotes the inverse transform of $\mathcal{M}_n$ in R_{n-1}, then Proposition 3.4 implies that

$$T(I_{n-1}) = \{v\} \quad \text{and} \quad d(I_{n-1}, v) = 1.$$

Descending step by step along the quadratic sequence, we finally obtain a complete $\mathcal{M}$-primary ideal I of R with $T(I) = \{v\}$ and $d(I, v) = 1$. Since $d(I, v) = 1$, I must be simple and this proves (1). It remains to prove (2) and (3).

Suppose J is a complete $\mathcal{M}$-primary ideal of R such that $T(J) = \{v\}$. Denote $n := d(J, v)$. Then $d(J, v) = d(I^n, v)$ and $d(J, v') = d(I^n, v') = 0$ for every prime divisor v' of R distinct from v. Hence, it follows from Section 2 that $\bar{J} = \bar{I^n}$, thus $J = I^n$ because both ideals are complete.

In case J is simple, this implies $J = I$. □

By means of the preceding proposition, we can prove a.o. the well-known result that a simple complete $\mathcal{M}$-primary ideal of the 2-dimensional regular local ring R has only one Rees valuation.

Corollary 3.6. *If I is a simple complete $\mathcal{M}$-primary ideal of a 2-dimensional regular local ring $(R, \mathcal{M})$ with algebraically closed residue field, then I has just one Rees valuation, say v, and $d(I^s, v) = 1$ for each base point S of I.*

Proof. Suppose that $T(I) = \{v_1, \dots, v_n\}$ and let for each i, I_i denote the unique simple complete $\mathcal{M}$-primary ideal which corresponds, according to Proposition 3.5, to v_i. Next, consider the ideal $I_1^{d(I,v_1)} \cdots I_n^{d(I,v_n)}$. Then

$$d(I, v) = d(\prod_{i=1}^{n} I_i^{d(I,v_i)}, v)$$

for each prime divisor v of R, and it follows from Section 2 that

$$\bar{I} = \overline{\prod_{i=1}^{n} I_i^{d(I,v_i)}}$$

and hence

$$I = \prod_{i=1}^{n} I_i^{d(I,v_i)}.$$

Since I is simple, we must have $n = 1$ and $d(I, v_1) = 1$.

Now, the base points of I are the local rings in the unique quadratic sequence starting from R and dominated by the valuating ring $(V_1, \mathcal{M}_{v_1})$ of v_1:

$$(R, \mathcal{M}) = (R_0, \mathcal{M}_0) < (R_1, \mathcal{M}_1) < \cdots < (R_n, \mathcal{M}_n) < (V_1, \mathcal{M}_{v_1}).$$

Therefore it follows from Proposition 3.1 that

$$d(I^s, v_1) = 1$$

for every base point S of I. □

We close this section by showing that in case of a 2-dimensional regular local ring R with algebraically closed residue field, a number of well-known results of Zariski and Lipman follow as quick corollaries from the preceding material of this section.

Corollary 3.7. (Zariski's one-to-one correspondence) *The mapping that associates to each simple complete $\mathcal{M}$-primary ideal of R its unique Rees valuation v, is a one-to-one correspondence between the set of the simple complete $\mathcal{M}$-primary ideals of R and the set of prime divisors of R.*

Proof. This follows immediately from Proposition3.5(1) and (2) and Corollary 3.6. □

Corollary 3.8. (Zariski's unique factorization theorem) *Every complete $\mathcal{M}$-primary ideal of R can be uniquely factored into a product of simple complete ideals (up to order).*

Proof.

- Existence of the factorization. Let $T(I) = \{v_1, \ldots, v_n\}$. Because of Proposition 3.5 we can consider for each v_i a simple complete $\mathcal{M}$-primary ideal I_i of R so that $T(I_i) = \{v_i\}$ and $d(I_i, v_i) = 1$, while $d(I_i, v') = 0$ for all other prime divisors v' of R. If we put $e_i := d(I, v_i)$ for $i = 1, \ldots, n$, then we have

  $$d(I_1^{e_1} \cdots I_n^{e_n}, v) = d(I, v)$$

  for every prime divisor v of R. Applying Section 2, it follows that

  $$\bar{I} = \overline{I_1^{e_1} \cdots I_n^{e_n}}.$$

  The product of complete ideals being complete in a 2-dimensional regular local ring, this implies

  $$I = I_1^{e_1} \cdots I_n^{e_n}.$$

- Uniqueness of the factorization. Suppose that

  $$I = I_1^{e_1} \cdots I_n^{e_n} = J_1^{f_1} \cdots J_m^{f_m}$$

  are two factorizations of I in simple complete $\mathcal{M}$-primary ideals with $I_1, \ldots, I_n$ (resp. $J_1, \ldots, J_m$) distinct simple ideals.

Let $v_1, \ldots, v_n$ (resp. $w_1, \ldots, w_m$) be the corresponding Rees valuations, i.e. $T(I_i) = \{v_i\}$ for $i = 1, \ldots, n$ and $T(J_j) = \{w_j\}$ for $j = 1, \ldots, m$. Then $T(I) = \{v_1, \ldots, v_n\} = \{w_1, \ldots, w_m\}$. It follows that $n = m$ and, after renumbering if necessary, one has $v_1 = w_1, \ldots, v_n = w_n$. Because of Proposition 3.5(2), $T(I_i) = T(J_i) = \{v_i\}$ implies that $I_i = J_i$ for $i = 1, \ldots, n$ and thus

$$I = I_1^{e_1} \cdot \cdots \cdot I_n^{e_n} = I_1^{f_2} \cdot \cdots \cdot I_n^{f_n}.$$

It follows that

$$d(I, v_i) = e_i = f_i$$

for $i = 1, \ldots, n$ and this finishes the proof. □

Corollary 3.9. (Lipman's reciprocity and multiplicity formula)

(1) *If I and J are simple complete $\mathcal{M}$-primary ideals in R with $T(I) = \{v\}$ and $T(J) = \{w\}$, then*

$$v(J) = w(I).$$

(2) *Let $I = I_1^{k_1} \cdot \cdots \cdot I_n^{k_n}$ the unique factorization of the complete $\mathcal{M}$-primary ideal I into a product of simple complete $\mathcal{M}$-primary ideals (with $I_1, \ldots, I_n$ distinct ideals). Suppose that $T(I_i) = \{v_i\}$ for $i = 1, \ldots, n$. Then*

$$e(I) = \sum_{i=1}^{n} k_i v_i(I).$$

Proof. In Section 3 we have seen that

$$d(I,v)\cdot v(J)=d(J,w)\cdot w(I)$$

and using Corollary 3.6 the assertion (1) follows. As for the second statement, from Section 2 we have

$$\begin{aligned} e(I) &= \sum_{v\in P(R)} d(I_1^{k_1},\dots,I_n^{k_n},v)\cdot v(I)\\ &= \sum_{v\in P(R)} (\sum_{i=1}^{n} k_i d(I_i,v))\cdot v(I)\\ &= \sum_{i=1}^{n} k_i d(I_i,v_i)v_i(I). \end{aligned}$$

Since $d(I_i,v_i)=1$, this implies $e(I)=\sum_{i=1}^{n} k_i v_i(I)$. □

4. A characterization of regularity

In Section 3 we have seen that in a 2-dimensional regular local ring $(R,\mathcal{M})$ with algebraically closed residue field the following property $(*)$ holds:

For every prime divisor v of R, there exists a complete $\mathcal{M}$-primary ideal I of R such that

$$T(I)=\{v\} \quad \text{and} \quad d(I,v)=1$$

and $d(I,v')=0$ for all the other prime divisors v' of R.

If the 2-dimensional local ring $(R,\mathcal{M})$ is not regular then this property $(*)$ does not necessarily hold as the following example will show.

Example 4.1. Let $R=\frac{k[X,Y,Z](X,Y,Z)}{(XY-Z^3)(X,Y,Z)}$ with k an algebraically closed field X,Y,Z indeterminates over k and let K be the quotient field of R. Then

$$R=k[x,y,z]_{(x,y,z)},\, xy=z^3$$

with x,y,z the images of X,Y,Z in $\frac{k[X,Y,Z](X,Y,Z)}{(XY-Z^3)(X,Y,Z)}$. It is readily seen that

$$gr_{\mathcal{M}}R=\frac{k[X,Y,Z]}{(XY)}.$$

So putting $S=R[\mathcal{M}t,t^{-1}]$, one sees that $t^{-1}S$ has two minimal primes

$$P_1=(xt,t^{-1})S \quad \text{and} \quad P_2=(yt,t^{-1})S.$$

Since $P_1S_{P_1}=(t^{-1})S_{P_1}$ (resp. $P_2S_{P_2}=(t^{-1}S_{P_2})$, it follows that S_{P_1} (resp. S_{P_2}) is a DVR and hence $V_1:=S_{P_1}\cap K$ and $V_2:=S_{P_2}\cap K$ are the Rees valuation rings of $\mathcal{M}$.

Let v_1, v_2 be the corresponding valuations. One can check that $v_1(x) = 2,\ v_1(y) = v_1(z) = 1$ and $v_2(y) = 2,\ v_2(x) = v_2(z) = 1$.

As R is a 2-dimensional rational singularity, according to Göhner ([3], Corollary 3.11, p. 422) there exist unique complete $\mathcal{M}$-primary ideals A_{v_1} resp. A_{v_2} in R with $T(A_{v_1}) = \{v_1\}$ resp. $T(A_{v_2}) = \{v_2\}$ and such that every other complete $\mathcal{M}$-primary ideal with unique Rees valuation v_1 resp. v_2, is a power of A_{v_1} resp. A_{v_2}. We claim that

$$A_{v_1} = \overline{(x, y^2)} \quad \text{and} \quad A_{v_2} = \overline{(y, x^2)}.$$

To prove this claim, we first remark that

$$\mathcal{M}^3 = (x, (y, z)^2) \cdot (y, (x, z)^2)$$

and thus

$$\mathcal{M}^3 = \overline{(x, y^2)} \cdot \overline{(y, x^2)}.$$

Putting $I = \overline{(x, y^2)}$ and $J = \overline{(y, x^2)}$, one has

$$T(\mathcal{M}^3) = \{v_1, v_2\} = T(I) \cup T(J)$$

and this implies $T(I) = \{v_1\}$ and $T(J) = \{v_2\}$. By Göhner [3] it certainly holds that $I = A_{v_1}^e$ for some positive integer e, consequently

$$d(I, v_1) = e \cdot d(A_{v_1}, v_1)$$

which together with $\mathcal{M}^3 = I \cdot J$ implies that e is a divisor of 3. On the other hand, $d(I, v_1) = e \cdot d(A_{v_1}, v_1)$ in combination with $d_I(A_{v_1}) = d_{A_{v_1}}(I)$ implies that e is also a divisor of 2. Thus $e = 1$, and hence $I = A_{v_1}$. Similarly one proves $J = A_{v_2}$, implying that $\mathcal{M}^3 = A_{v_1} \cdot A_{v_2}$ and $d(A_{v_1}, v_1) = d(A_{v_2}, v_2) = 3$. $\square$

Although a 2-dimensional rational singularity $(R, \mathcal{M})$ essentially of finite type over an algebraically closed field k, is an analytically normal local ring which satisfies a.o. ZPT and condition (N), the previous example shows that nevertheless the property $(*)$ does not necessarily hold in such a local ring R. Since property $(*)$ does not hold even in the "simplest" sort of 2-dimensional singularity, it is natural to ask whether property $(*)$ is characteristic for the regularity of a 2-dimensional normal local ring.

In the following proposition we will show that property (*) completed with a natural condition on the maximal ideal $\mathcal{M}$ of R does imply the regularity of R. More precisely we have the following result.

Proposition 4.2. *Let $(R, \mathcal{M})$ be a 2-dimensional Cohen-Macaulay local domain with algebraically closed residue field. Then R is regular if and only if*

(i) *For every prime divisor v of R, there exists a complete $\mathcal{M}$-primary ideal I of R such that $d(I, v) = 1$ and $d(I, v') = 0$ for all the other prime divisors v' of R. (i.e. property $(*)$ holds).*

(ii) *There is exactly one prime divisor* v *of* R *satisfying* $d(\mathcal{M}, v) \neq 0$ *and there exists a prime divisor* w *of* R *satisfying* $w(\mathcal{M}) = 1$.

Proof. One implication is immediate because of Proposition 3.5. For the converse, suppose v is the unique prime divisor of R satisfying $d(\mathcal{M}, v) \neq 0$. Since $e(m) = d(m, v) \cdot v(\mathcal{M})$, it is sufficient to show that $d(\mathcal{M}, v) = 1$ and $v(\mathcal{M}) = 1$ in order to conclude that R is regular. We first show that $d(\mathcal{M}, v) = 1$. From (i), it follows that there exists a complete $\mathcal{M}$-primary ideal I of R satisfying $d(I, v) = 1$ and $d(I, v') = 0$ for every other prime divisor v' of R. Let $d(\mathcal{M}, v) = e$. Then $d(\mathcal{M}, v) = d(I^e, v)$ and $d(\mathcal{M}, v') = d(I^e, v') = 0$ for all prime divisors $v' \neq v$ of R. This implies that $\mathcal{M} = \overline{I^e}$ and hence $\mathcal{M} = I$. Consequently $d(\mathcal{M}, v) = d(I, v) = 1$. It remains to prove that $v(\mathcal{M}) = 1$. To this end consider a prime divisor w of R satisfying $w(\mathcal{M}) = 1$ (see condition (ii)). According to (i), there exists a complete $\mathcal{M}$-primary ideal J of R such that $d(J, w) = 1$ and $d(J, w') = 0$ for every other prime divisor w' of R. Using the relation $d_J(\mathcal{M}) = d_{\mathcal{M}}(J)$, one has

$$1 = d(J, w) \cdot w(\mathcal{M}) = d(\mathcal{M}, v) \cdot v(J) = v(J) \geq v(\mathcal{M}).$$

It follows that $v(\mathcal{M}) = 1$ and this completes the proof. □

We close this section by giving some examples of non-regular 2-dimensional analytically normal local domains satisfying condition (ii) of the previous proposition. This shows that a.o. that condition (ii) alone is not sufficient to ensure that R is regular.

Example 4.3. Let $R = \frac{k[X,Y,Z]_{(X,Y,Z)}}{(X^2 - ZY^2 - Z^3)(X,Y,Z)}$ with k an algebraically closed field X, Y, Z indeterminates over k and K the quotient field of R. Then

$$R = k[x, y, z]_{(x,y,z)}, \quad x^2 = zy^2 + z^3$$

with x, y, z the images of X, Y, Z in $\frac{k[X,Y,Z]_{(X,Y,Z)}}{(X^2 - ZY^2 - Z^3)(X,Y,Z)}$. First, we look for the Rees valuation of the unique maximal ideal $\mathcal{M}$ of R. It is clear that

$$gr_{\mathcal{M}} R = \frac{k[X, Y, Z]}{(X^2)}.$$

If S denotes the ring $R[\mathcal{M}t, t^{-1}]$, it follows that $(t^{-1})S$ has only one minimal prime ideal

$$P = (xt, t^{-1})S$$

and $PS_P = (xt)S_P$. Hence S_P is a DVR and

$$V := S_p \cap K$$

is the unique Rees valuation ring of $\mathcal{M}$. Let v be the corresponding valuation. We have $v(x) = 3, v(y) = 2, v(z) = 2$, hence $v(\mathcal{M}) = 2$. This implies that $d(\mathcal{M}, v) = 1$ (i.e. v is the unique prime divisor of R such that $d(\mathcal{M}, v) \neq 0$).

Next we show that there exists a prime divisor w of R satisfying $w(\mathcal{M}) = 1$. Consider the ideal $(z, y^2)R$. This is an $\mathcal{M}$-primary ideal whose integral closure I is given by

$$I = (x, y^2, z)R.$$

Then one can check that $(z)R[\frac{I}{z}]$ has only one minimal prime which determines the unique Rees valuation w of I and $w(y) = 1, w(x) = w(z) = 2$. Hence $w(\mathcal{M}) = 1$. This shows that R satisfies condition (ii) of Proposition 4.2 and since R is not regular, Proposition 4.2 implies that condition (i) is not satisfied in R.

The local ring $(R, \mathcal{M})$ of the previous example has quite a number of properties in common with a regular 2-dimensional local ring. However, there is an important exception: its associated graded ring $gr_{\mathcal{M}}R$ is not a domain (equivalently ord_R is not a valuation). Therefore in the next example we consider a 2-dimensional local ring $(R, \mathcal{M})$ whose associated graded ring is a domain.

Example 4.4. Let $R = \frac{k[X,Y,Z](X,Y,Z)}{(XY-Z^2)(X,Y,Z)}$ with k an algebraically closed field and X, Y, Z indeterminates over k. Then

$$R = k[x, y, z]_{(x,y,z)}, z^2 = xy$$

with x, y, z the images of X, Y, Z in $\frac{k[X,Y,Z](X,Y,Z)}{(xy-z^2)(X,Y,Z)}$. One has

$$gr_{\mathcal{M}}R = \frac{k[X, Y, Z]}{(XY - Z^2)}$$

and thus $gr_{\mathcal{M}}R$ is a domain.

Consequently, ord_R is a valuation and it is the only Rees valuation of $\mathcal{M}$. It is clear that $ord_R(\mathcal{M}) = 1$ and $e(\mathcal{M}) = d(\mathcal{M}, ord_R)$ implies that $d(\mathcal{M}, ord_R) = 2$.

References

[1] Abhyankar, S. S.: *On the valuations centered in a local domain.* Am. J. Math. **78**(1) (1956), 321–348. Zbl 0074.26301

[2] Debremaeker, R.; Van Lierde, V.: *A short proof of the length formula of Hoskin and Deligne.* Arch. Math. **78** (2002), 369–371. Zbl 0997.13007

[3] Göhner, H.: *Semifactoriality and Mukly's condition* (N) *in two dimensional local rings.* J. Algebra **34** (1975), 403–429. Zbl 0308.13023

[4] Lipman, J.: *Proximity inequalities for complete ideals in two-dimensional regular local rings.* In: Commutative algebra: syzygies, multiplicities and birational algebra. Contemp. Math. **159**, Am. Math. Soc. 1994. Zbl 0814.13016

[5] Rees, D.: *Degree functions in local rings.* Proc. Camb. Phil. Soc. **57** (1961), 1–7. Zbl 0111.24901

[6] Rees, D.; Sharp, R.: *On a theorem of B. Teissier on multiplicities of ideals in local rings.* J. Lond. Math. Soc. **18**(2) (1978), 449–463. Zbl 0408.13009

[7] Verma, J. K.: *Rees algebras and mixed multiplicities*. Proc. Amer. Math. Soc. **104**(4) (1988), 1036–1044. Zbl 0693.13015

[8] Zariski, O.; Samuel, P.: *Commutative Algebra*. **2**, Van Nostrand, 1960. Zbl 0121.27801

Received April 5, 2004