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Weighted Gaussian Maps

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Abstract. Motivated by previous work on deformation theory of higher order theta-characteristics, we introduce a weighted Gaussian map $\gamma_{a,b}(X,L)$, where a, b are positive integers, X is a smooth projective variety, L is a line bundle on X and $\gamma_{1,1}(X,L)$ is the ordinary Gaussian map for (X,L). We establish a sharp lower bound on its rank and we investigate the extremal cases for curves.

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1. Introduction

We work over an algebraically closed field $\mathbb{K}$ of characteristic zero.

Let X be a smooth projective variety and let L be a line bundle on X. Fix integers $a > 0$, $b > 0$, and set $M := L^{\otimes a}$, $N := L^{\otimes b}$. We introduce a *weighted Gaussian map* as follows:

$$\begin{aligned} \gamma_{a,b}(X,L) : H^0(X,M) \otimes H^0(X,N) &\longrightarrow H^0(\Omega^1_X \otimes M \otimes N) \\ \sigma \otimes \tau &\longmapsto b\tau d\sigma - a\sigma d\tau \end{aligned} \tag{1}$$

It is easy to check that such a definition is well-posed (see Lemma 1). If $a = b = 1$ we recover the usual Gaussian map (see for instance the survey paper [8]). From now on, we are going to assume $0 < a < b$ and $X = C$ a smooth and connected projective curve. The case $a = 1$, $b = 2m - 1$ is also geometrically meaningful according to the following result (see [4], Theorem 3):

Theorem 1. *Set* $\mathfrak{Th}^r_{g,m} = \{C : C$ *is a smooth curve of genus* g *with a line bundle* $\tilde{L}$ *such that* $h^0(C, \tilde{L}) = r + 1$ *and* $2m\tilde{L} = K\}$. *Define*

$$\tilde{\mu} : H^0(C, \tilde{L}) \otimes H^0(C, K - \tilde{L}) \longrightarrow H^0(C, 2K)$$

$$\sigma \otimes \tau \longmapsto (2m - 1)\sigma d\tau - \tau d\sigma.$$

Then $T_C\left(\mathfrak{Th}^r_{g,m}\right) = (\mathrm{Coker}\tilde{\mu})^*$.

Here instead we address the general case, by drawing our inspiration from the classical works [5] by Giuseppe Gherardelli and [7] by Beniamino Segre (see also [2], Proposition (1.2) and Theorem (1.3), for a useful modern translation, and [1], Proposition 1 and Theorem 1, for a completely different generalization). First of all, we obtain a lower bound on the rank of $\gamma_{a,b}(C, L)$ as follows:

Proposition 1. *Fix integers* $0 < a < b$, *a smooth and connected projective curve* C, *a line bundle* L *on* C *and define* $M := L^{\otimes a}$, $N := L^{\otimes b}$, $s := h^0(C, M) - 1$, $t := h^0(C, N) - 1$. *Then*

$$\mathrm{rank}\gamma_{a,b}(C, L) \geq s + t - 1. \tag{2}$$

Moreover, if $bs - at \neq 0$, *then strict inequality holds in* (2).

It turns out that our estimate is sharp (see Example 1) and that extremal cases are sporadic (see Remark 1). More precisely, the following holds:

Proposition 2. *In the notation of Proposition* 1, *assume that* $\frac{b}{a} \in \mathbb{Z}$. *Then* $\gamma_{a,b}(C, L)$ *has minimal rank* $s + t - 1$ *if and only if the image of* C *under the morphism defined by* M *(resp., by* N*) after removing any base point is a rational normal curve.*

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2. The results

Lemma 1. *The weighted Gaussian map* (1) *is well-defined.*

Proof. Just notice that

$$\begin{aligned} d\left(\frac{\sigma^b}{\tau^a}\right) &= \frac{\tau^a d\sigma^b - \sigma^b d\tau^a}{\tau^{2a}} = \frac{\tau^a b\sigma^{b-1} d\sigma - \sigma^b a\tau^{a-1} d\tau}{\tau^{2a}} = \\ &= \frac{\sigma^{b-1}}{\tau^{a+1}}(b\tau d\sigma - a\sigma d\tau). \end{aligned}$$

Hence we have

$$\gamma_{a,b}(X, L)(\sigma \otimes \tau) = \frac{\tau^{a+1}}{\sigma^{b-1}} d\left(\frac{\sigma^b}{\tau^a}\right)$$

where $\frac{\sigma^b}{\tau^a}$ is a rational function on X. $\square$

Proof of Proposition 1. Let $p \in C$ be a general point, so that the vanishing sequence of $H^0(C, M)$ (resp., of $H^0(C, N)$) at p is the standard one $(0, 1, \dots, s)$ (resp., $(0, 1, \dots, t)$). Choose bases $(v_0, \dots, v_s)$ of $H^0(C, M)$ and $(w_0, \dots, w_t)$ of $H^0(C, N)$ which realize the above vanishing sequence. Therefore if z is a local parameter at p we have local descriptions $v_i = z^i + \text{higher}$, $w_j = z^j + \text{higher}$ and

$$\begin{aligned}\gamma_{a,b}(C, L)(v_i \otimes w_j) &= bz^j dz^i - az^i dz^j = bz^j iz^{i-1}dz - az^i jz^{j-1}dz = \\ &= (bi - aj)z^{i+j-1}dz\end{aligned}$$

up to higher order terms. As a consequence, the rank of $\gamma_{a,b}(C, L)$ is at least the cardinality of the set

$$S_{a,b} := \{i + j : 0 \leq i \leq s, 0 \leq j \leq t, bi - aj \neq 0\}.$$

We claim that

$$\{1, 2, \dots, s + t - 1\} \subseteq S_{a,b},$$

hence (2) follows. Indeed, for $1 \leq n \leq t$ we have $n = 0 + n$ and $b0 - an \neq 0$; if instead $t+1 \leq n \leq s+t-1$, we have $n = r+t = s+(t-s+r)$ with $1 \leq r \leq s-1$. Assume by contradiction that both $br - at = 0$ and $bs - a(t - s + r) = 0$. By subtracting we obtain $b(s - r) + a(s - r) = 0$, hence $b + a = 0$, contradiction. Finally, if $bs - at \neq 0$, then also $s + t \in S_{a,b}$ and $\gamma_{a,b}(C, L) > s + t - 1$. $\square$

We stress that the assumption $bs - at \neq 0$ cannot be removed from the second part of Proposition 1:

Example 1. In the notation of Proposition 1, let $C = \mathbb{P}^1$. If $\deg(L) = d$, then $L = \mathcal{O}_{\mathbb{P}^1}(d)$ and the image of $\gamma_{a,b}(\mathbb{P}^1, d)$ is a non-zero $SL(2)$-invariant linear subspace of $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(ad + bd - 2))$. It follows that $\gamma_{a,b}(\mathbb{P}^1, d)$ is surjective with $\mathrm{rank}\gamma_{a,b}(\mathbb{P}^1, d) = h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(ad + bd - 2)) = ad + bd - 1 = s + t - 1$.

We also point out that almost all extremal cases arise as in Example 1:

Remark 1. In the notation of Proposition 1, if $bs-at = 0$ and both M and N are non-special, then we have $g(C) = 0$ and $C = \mathbb{P}^1$. Indeed, from the Riemann-Roch Theorem we obtain $s = ad - g$ and $t = bd - g$, where $d = \deg(L)$. Hence from $bs - at = 0$ we deduce $ag = bg$ and $g = 0$, as claimed.

Proof of Proposition 2. The "if" part is a direct consequence of Example 1. Conversely, let $b = ma$ with $m \in \mathbb{Z}$. From Proposition 1 it follows that $at = bs$. On the other hand, by the Hopf Theorem and its refinements (see for instance [3] and [6]) we have:

$$\begin{aligned}t &= h^0(C, L^{\otimes b}) - 1 = h^0(C, L^{\otimes ma}) - 1 \geq h^0(C, L^{\otimes (m-1)a}) + \\ &\quad + h^0(C, L^a) - 2 \geq mh^0(C, L^a) - m = m(h^0(C, L^a) - 1) = \frac{b}{a}s\end{aligned}$$

and equality holds if and only if there is a morphism $f : C \to \mathbb{P}^1$ such that $f^*(\mathcal{O}^1_{\mathbb{P}}(k)) = L^{\otimes a}(-B)$ and $f^*H^0(\mathbb{P}^1, \mathcal{O}^1_{\mathbb{P}}(k)) = H^0(C, L^{\otimes a}(-B))$, where B denotes the base locus of $L^{\otimes a}$ and $k := \frac{a\deg(L)-\deg(B)}{\deg(f)}$, and the same holds for $L^{\otimes b}$. $\square$

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