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Multiplication Modules and Homogeneous Idealization

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Abstract. All rings are commutative with identity and all modules are unital. Let R be a ring, M an R-module and $R(M)$, the idealization of M. Homogeneous ideals of $R(M)$ have the form $I_{(+)}N$ where I is an ideal of R, N a submodule of M and $IM \subseteq N$. The purpose of this paper is to investigate how properties of a homogeneous ideal $I_{(+)}N$ of $R(M)$ are related to those of I and N. We show that if M is a multiplication R-module and $I_{(+)}N$ is a meet principal (join principal) homogeneous ideal of $R(M)$ then these properties can be transferred to I and N. We give some conditions under which the converse is true. We also show that $I_{(+)}N$ is large (small) if and only if N is large in M (I is a small ideal of R).

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0. Introduction

Let R be a commutative ring with identity and M an R-module. M is a *multiplication* module if every submodule N of M has the form IM for some ideal I of R. Equivalently, $N = [N : M]M$, [11]. A submodule K of M is multiplication if $N \cap K = [N : K]K$ for all submodules N of M, [19, Lemma 1.3]. Let N be a submodule of R and I an ideal of R. The *residual* submodule of N by I is $[N :_M I] = \{m \in M : Im \subseteq N\}$, [16] and [17]. If M is multiplication then

$[N :_M I] = [N : IM]\, M$. In particular, if M is faithful and multiplication then $[0 :_M I] = (\mathrm{ann} I)\, M$, [2]. Several properties of residual submodules of multiplication modules are given in [2].

Anderson [9] defined $\theta\,(M) = \sum_{m \in M} [Rm : M]$ and showed the usefulness of this ideal in studying multiplication modules. He proved for example that if M is multiplication then $M = \theta(M)M$ and a finitely generated module M is multiplication (equivalently, locally cyclic) if and only if $\theta\,(M) = R$, [9, Proposition 1 and Theorem 1]. The *trace* ideal of an R-module M is $\mathrm{Tr}(M) = \sum_{f \in \mathrm{Hom}(M,R)} f\,(M)$. If M is faithful multiplication then $\theta\,(M) = \mathrm{Tr}\,(M)$ is a pure ideal of R, (equivalently, multiplication and idempotent, [3, Theorem 1.1]).

Let M be an R-module and P a maximal ideal of R. El-Bast and Smith [12, p. 756] defined $T_P\,(M) = \{m \in M : (1-p)\, m = 0 \text{ for some } p \in P\}$. $T_P(M)$ is a submodule of M. M is P-*torsion* if and only if $T_P\,(M) = M$. They also defined M to be P-*cyclic* if there exist $p \in P$ and $m \in M$ such that $(1-p)\, M \subseteq Rm$. They proved that M is multiplication if and only if for each maximal ideal P of R, either M is P-torsion or M is P-cyclic, [12, Theorem 2.1].

Let R be a commutative ring with identity and M an R-module. The R-module $R\,(M) = R_{(+)}M$ (called the *idealization* of M) becomes a commutative ring with identity if multiplication is defined by $(r_1, m_1)\,(r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1)$. $0_{(+)}M$ is an ideal of $R\,(M)$ satisfying $(0_{(+)}M)^2 = 0$, and the structure of $0_{(+)}M$ as an ideal of $R(M)$ is essentially the same as the R-module structure of M. Every ideal contained in $0_{(+)}M$ has the form $0_{(+)}N$ for some submodule N of M and every ideal contains $0_{(+)}M$ has the form $I_{(+)}M$ for some ideal I of R. Since $R \cong R\,(M)\,/0_{(+)}M, I \to I_{(+)}M$ gives a one-to-one correspondence between the ideals of R and the ideals of $R\,(M)$ containing $0_{(+)}M$. Thus prime (maximal) ideals of $R\,(M)$ have the form $P_{(+)}M$ where P is a prime (maximal) ideal of R.

Let R be a ring and M an R-module. Let I be an ideal of R and N a submodule of M. Then $I_{(+)}N$ is an ideal of $R\,(M)$ if and only if $IM \subseteq N$, [13, Theorem 25(1)] and [6, Theorem 3.1]. The *homogeneous* ideals of $R\,(M)$ have the form $I_{(+)}N$ where I is an ideal of R, N a submodule of M and $IM \subseteq N$. If H is a homogeneous ideal then $H = I_{(+)}N$ where $I = \{r \in R : (r, b) \in H \text{ for some } b \in M\}$ and $N = \{m \in M : (s, m) \in H \text{ for some } s \in R\}$. Ideals of $R\,(M)$ need not have the form $I_{(+)}N$, that is, need not be homogeneous. For example, it is easily checked that the principal ideal of $\mathbb{Z}_{(+)}\mathbb{Z}$ which is generated by (2,1) is not homogeneous. Some facts about homogeneous ideals of $R\,(M)$ are given in [6] and [13, Section 25]. In this paper we say that $R\,(M)$ is a *homogeneous* ring if every ideal of $R\,(M)$ is homogeneous. It is shown, [6, Theorem 3.3], that if R is an integral domain then $R\,(M)$ is homogeneous if and only if M is a divisible R-module. Thus $\mathbb{Z}_{(+)}Q$, where Q is the field of rational numbers, is a homogeneous ring.

Idealization is useful for reducing results concerning submodules to the ideal case and generalizing results from rings to modules. D. D. Anderson, [7] and [8], investigated the idealization of modules. He proved that a submodule N of an R-module M is multiplication (weak cancellation) if and only if $0_{(+)}N$ is a multiplication (weak cancellation) ideal of $R\,(M)$. Thus the study of multiplication (weak cancellation) modules can often be reduced to the study of multiplication (weak cancellation) ideals I with $I^2 = 0$, [7, Theorem 3.1] and [8, Theorem 3.1].

In this paper we investigate homogeneous ideals of $R(M)$. Let $I_{(+)}N$ be a homogeneous ideal of $R(M)$. In Section 1 we show that if $I_{(+)}N$ is meet principal then so too is I and if we assume further that M is meet principal then N is meet principal. We also prove that if I and N are meet principal such that $\operatorname{ann}I + [IM : N] = R$ then so too is $I_{(+)}N$, Theorem 3 and Proposition 7.

In section 2 we study the idealization of join principal submodules. Theorem 9 proves that if $I_{(+)}N$ is join principal then so too is I and if we assume further that M is finitely generated multiplication then N is join principal. In Theorem 13 we show that if $R(M)$ is homogeneous, I join principal, N weak cancellation and $\operatorname{ann}I + [IK : N] = R$ for each submodule K of M then $I_{(+)}N$ is weak cancellation.

Section 3 is concerned with the idealization of large and small submodules. Among several results we show that $I_{(+)}N$ is large if and only if N is large in M and $I_{(+)}N$ is small if and only if I is a small ideal of R, Proposition 17.

All rings are assumed to be commutative with identity and all modules are unital. For the basic concepts used, we refer the reader to [13]–[17].

1. Meet principal submodules

Let M be an R-module and N a submodule of M. Then N is *meet principal* if $K \cap IN = ([K : N] \cap I)N$ for all ideals I of R and all submodules K of M. Setting $I = R$ we define N to be *weak meet principal* if $K \cap N = [K : N]N$ for all submodules K of M, [7]. Hence multiplication modules are in fact weak meet principal modules. The following conditions are equivalent for a submodule N of M: (1) N is meet principal, (2) N is multiplication, (3) if $P \supseteq \theta(N)$ is a maximal ideal of R then $N_P = 0_P$, [9, Theorem 2]. In this section we investigate the idealization of meet principal submodules. We start by the following lemma which plays a main role in our paper.

Lemma 1. *Let R be a ring and M an R-module. If $I_{(+)}N$ and $J_{(+)}K$ are homogeneous ideals of $R(M)$ then*

$$\left[I_{(+)}N :_{R(M)} J_{(+)}K\right] = ([I : J] \cap [N : K])_{(+)} [N :_M J].$$

Furthermore, it is a homogeneous ideal of $R(M)$.

Proof. The proof of the first assertion is straightforward. To show that the ideal is homogeneous, we have that

$$([I : J] \cap [N : K])M \subseteq [I : J]M \subseteq [IM : JM]M \subseteq [N : JM]M \subseteq [N :_M J]. \quad \square$$

As a consequence of the above lemma, we note that if $I_{(+)}N$ is a homogeneous ideal of $R(M)$ then

$$\operatorname{ann}(I_{(+)}N) = (\operatorname{ann}I \cap \operatorname{ann}N)_{(+)} [0 :_M I].$$

Let M be faithful. Since $IM \subseteq N$, we infer that $\mathrm{ann}N \subseteq \mathrm{ann}I$, and hence $\mathrm{ann}\,(I_{(+)}N) = \mathrm{ann}N_{(+)}\,[0 :_M I]$. Assuming further that M is multiplication, we obtain that $\mathrm{ann}(I_{(+)}N) = \mathrm{ann}N_{(+)}\,(\mathrm{ann}I)\,M$. Compare the next result with [8, Theorem 3.1].

Proposition 2. *Let R be a ring and N a submodule of an R-module M. Then N is meet principal if and only if $0_{(+)}N$ is a meet principal ideal of $R\,(M)$.*

Proof. Let $0_{(+)}N$ be meet principal. Suppose K is a submodule of M and I an ideal of R. Then

$$(0_{(+)}K) \cap (I_{(+)}M)\,(0_{(+)}N) = (0_{(+)}K) \cap (0_{(+)}IN) = 0_{(+)}\,(K \cap IN)\,.$$

On the other hand, we get from Lemma 1 that

$$\begin{aligned}\big((I_{(+)}M) \cap \big[0_{(+)}K :_{R(M)} 0_{(+)}N\big]\big)\,(0_{(+)}N) &= ((I_{(+)}M) \cap ([K : N]_{\,(+)}M))\,(0_{(+)}N)\\ &= ((I \cap [K : N])_{\,(+)}M)\,(0_{(+)}N) = 0_{(+)}\,(I \cap [K : N])\,N.\end{aligned}$$

Hence $K \cap IN = (I \cap [K : N])\,N$, and N is meet principal. Conversely, suppose N is meet principal. Let H_1 and H_2 be ideals of $R\,(M)$. We prove that

$$H_1 \cap H_2\,(0_{(+)}N) = \big(\big[H_1 :_{R(M)} 0_{(+)}N\big] \cap H_2\big)\,(0_{(+)}N)\,.$$

Now

$$H_1 \cap H_2\,(0_{(+)}N) = (H_1 \cap 0_{(+)}N) \cap (H_2 + 0_{(+)}M)\,(0_{(+)}N)\,.$$

Assume $H_1 \cap 0_{(+)}N = 0_{(+)}K$ for some submodule K of N and $H_2 + 0_{(+)}M = I_{(+)}M$ for some ideal I of R. It follows that

$$\begin{aligned}H_1 \cap H_2\,(0_{(+)}N) &= 0_{(+)}\,(K \cap IN) = 0_{(+)}\,([K : N] \cap I)\,N\\ &= (([K : N] \cap I)_{\,(+)}M)\,(0_{(+)}N) = ([K : N]_{\,(+)}M \cap I_{(+)}M)\,(0_{(+)}N)\\ &= \big(\big[0_{(+)}K :_{R(M)} 0_{(+)}N\big] \cap I_{(+)}M\big)\,(0_{(+)}N)\\ &= \big(\big[(H_1 \cap 0_{(+)}N) :_{R(M)} 0_{(+)}N\big] \cap (H_2 + 0_{(+)}M)\big)\,(0_{(+)}N)\\ &= \big(\big[H_1 :_{R(M)} 0_{(+)}N\big] \cap (H_2 + 0_{(+)}M)\big)\,(0_{(+)}N)\,.\end{aligned}$$

We verify that

$$\big(\big[H_1 :_{R(M)} 0_{(+)}N\big] \cap (H_2 + 0_{(+)}M)\big)\,(0_{(+)}N) = \big(\big[H_1 :_{R(M)} 0_{(+)}N\big] \cap H_2\big)\,(0_{(+)}N)\,.$$

Let $x \in \big(\big[H_1 :_{R(M)} 0_{(+)}N\big] \cap (H_2 + 0_{(+)}M)\big)\,(0_{(+)}N)$. Then

$$x = \sum_{i=1}^{k} ((r_i, m_i) + (0, m_i'))\,(0, n_i) = \sum_{i=1}^{k} (0, r_i n_i) = \sum_{i=1}^{k} (r_i, m_i)\,(0, n_i)\,,$$

where $(r_i, m_i) \in H_2, m_i' \in M$ and $n_i \in N$. Since $(r_i, m_i + m_i') \in \big[H_1 :_{R(M)} 0_{(+)}N\big]$, it follows that $(r_i, m_i)\,(0, n_i') = (0, r_i n_i') = (r_i, m_i + m_i')\,(0, n_i') \in H_1$ for all $n_i' \in N$. Hence $(r_i, m_i) \in \big[H_1 :_{R(M)} 0_{(+)}N\big]$. This implies that $x \in \big(\big[H_1 :_{R(M)} 0_{(+)}N\big] \cap H_2\big)$ $(0_{(+)}N)$. The reverse inclusion is obvious, and this finishes the proof of the proposition. $\square$

The next result shows that the meet principal property of a homogeneous ideal of $R\,(M)$ can be transferred to its components.

Theorem 3. *Let R be a ring and M an R-module. If $I_{(+)}N$ is a meet principal homogeneous ideal of $R(M)$ then I is a meet principal ideal of R. Assuming further that M is meet principal then N is a meet principal submodule of M.*

Proof. Let A and B be ideals of R. Since $I_{(+)}N$ is meet principal,

$$(A_{(+)}M) \cap (B_{(+)}M)(I_{(+)}N) = \left(\left[A_{(+)}M :_{R(M)} I_{(+)}N\right] \cap (B_{(+)}M)\right)(I_{(+)}N).$$

But

$$\begin{aligned}(A_{(+)}M) \cap (B_{(+)}M)(I_{(+)}N) &= (A_{(+)}M) \cap (IB_{(+)}(BN + IM)) \\ &= (A \cap IB)_{(+)}(BN + IM),\end{aligned}$$

and

$$\begin{aligned}&\left(\left[A_{(+)}M :_{R(M)} I_{(+)}N\right] \cap (B_{(+)}M)\right)(I_{(+)}N) = (([A : I]_{(+)}M) \cap (B_{(+)}M))(I_{(+)}N) \\ &\quad = (([A : I] \cap B)_{(+)}M)(I_{(+)}N) = ([A : I] \cap B) I_{(+)}([A : I] \cap B) N + IM.\end{aligned}$$

Thus $A \cap IB = ([A : I] \cap B) I$, and I is meet principal.

Now, suppose M is meet principal. Let K be a submodule of M and A an ideal of R. Then

$$\begin{aligned}&(A_{(+)}AM)(I_{(+)}N) \cap ([K : M]_{(+)}K) \\ &= (A_{(+)}AM) \cap \left[[K : M]_{(+)}K :_{R(M)} I_{(+)}N\right](I_{(+)}N).\end{aligned}$$

But

$$\begin{aligned}(A_{(+)}AM)(I_{(+)}N) \cap ([K : M]_{(+)}K) &= (AI_{(+)}AN) \cap ([K : M]_{(+)}K) \\ &= (AI \cap [K : M])_{(+)}(AN \cap K),\end{aligned}$$

and

$$\begin{aligned}&\left((A_{(+)}AM) \cap \left[[K : M]_{(+)}K :_{R(M)} I_{(+)}N\right]\right)(I_{(+)}N) \\ &= ((A_{(+)}AM) \cap (([[K : M] : I] \cap [K : N])_{(+)}[K :_M I]))(I_{(+)}N) \\ &= ((A_{(+)}AM) \cap (([K : IM] \cap [K : N])_{(+)}[K :_M I]))(I_{(+)}N) \\ &= ((A_{(+)}AM) \cap ([K : N]_{(+)}[K :_M I]))(I_{(+)}N) \\ &= ((A \cap [K : N])_{(+)}(AM \cap [K :_M I]))(I_{(+)}N) \\ &= (A \cap [K : N]) I_{(+)}(A \cap [K : N]) N + I(AM \cap [K :_M I]).\end{aligned}$$

Hence

$$AI \cap [K : M] = (A \cap [K : N]) I,$$

and

$$AN \cap K = (A \cap [K : N]) N + I(AM \cap [K :_M I]).$$

Since I is meet principal, we infer that

$$(A \cap [K : N]) I = AI \cap [K : M] = (A \cap [[K : M] : I]) I = (A \cap [K : IM]) I.$$

As I and M are meet principal, we obtain from, [9, Corollary to Theorem 2], that IM is meet principal, and hence

$$\begin{aligned} I\,(AM \cap [K :_M I]) &= I\,(AM \cap [K : IM]\,M) \\ &\subseteq IAM \cap [K : IM]\,IM \subseteq A\,(IM) \cap K \\ &= (A \cap [K : IM])\,IM = (A \cap [K : N])\,IM \\ &\subseteq (A \cap [K : N])\,N. \end{aligned}$$

This finally gives that $AN \cap K = (A \cap [K : N])\,N$, and N is meet principal. $\square$

We make two observations on our theorem. First, since the homomorphic image of a meet principal submodule has the same property, the first part of the theorem follows since I is a homomorphic image of $I_{(+)}N$. Second, the condition that M is meet principal (multiplication) in the above theorem is required. Let $R = \mathbb{Z}$ and $M = Q$, where $\mathbb{Z}$ is the ring of integers and Q is the field of rational numbers. Then $\mathbb{Z}\,(Q)\,(2,0) = 2\mathbb{Z}_{(+)}Q$ is a principal (hence meet principal) ideal of $\mathbb{Z}\,(Q)$, but Q is not meet principal.

Compare the next result with [8, Theorem 3.2 (2)].

Proposition 4. *Let R be a ring and M an R-module. Let I be an ideal of R and P a maximal ideal of R.*

(1) $T_P\,(I)_{(+)}T_P\,(I)\,M \subseteq T_{P(+)M}\,(I_{(+)}IM)$.

(2) $T_{P(+)M}\,(I_{(+)}IM) \subseteq T_P\,(I)_{(+)}T_P\,(IM)$.

(3) *I is P-torsion if and only if $I_{(+)}IM$ is $P_{(+)}M$-torsion.*

(4) *I is P-principal if and only if $I_{(+)}IM$ is $P_{(+)}M$-principal.*

Proof. (1) Let $(a,n) \in T_P(I)_{(+)}T_P\,(I)\,M$. Then $a \in T_P(I)$ and $n \in T_P\,(I)\,M$. Hence there exists $p \in P$ such that $(1-p)\,a = 0$. Now, let $n = \sum_{i=1}^{r} a_i m_i$, where $a_i \in T_P\,(I)$ and $m_i \in M$. It follows that there exist $p_i \in P$ such that $(1-p_i)\,a_i = 0$. Let $q = 1-(1-p)\prod_{i=1}^{r}(1-p_i)$. Then $q \in P$ and $(1-q)\,a = 0 = (1-q)\,a_i$. This implies that $(1-q)\,n = 0$, and hence $((1,0)-(q,0))\,(a,n) = (1-q,0)\,(a,n) = ((1-q)\,a, (1-q)\,n) = (0,0)$, and $(a,n) \in T_{P(+)M}\,(I_{(+)}IM)$.

(2) Let $(a,n) \in T_{P(+)M}\,(I_{(+)}IM)$. There exist $p \in P$ and $m \in M$ such that $((1,0)-(p,m))\,(a,n) = (0,0)$. Hence $(0,0) = (1-p,-m)\,(a,n) = ((1-p)\,a, (1-p)\,n - am)$. It follows that $(1-p)\,a = 0$, (and hence $a \in T_P\,(I)$) and $(1-p)\,n = am$. Let $q = 2p - p^2$. Then $q \in P$, and $(1-q)\,n = 0$. Hence $n \in T_P\,(IM)$, and $(a,n) \in T_P\,(I)_{(+)}T_P\,(IM)$.

(3) If I is P-torsion then $I = T_P\,(I)$ and by (1) we get that

$$I_{(+)}IM = T_P\,(I)_{\,(+)}T_P\,(I)\,M \subseteq T_{P(+)M}\,(I_{(+)}IM) \subseteq I_{(+)}IM,$$

so that $I_{(+)}IM = T_{P(+)M}\,(I_{(+)}IM)$, and $I_{(+)}IM$ is $P_{(+)}M$-torsion. Conversely, suppose $I_{(+)}IM$ is $P_{(+)}M$-torsion. It follows by (2) that

$$I_{(+)}IM = T_{P(+)M}\,(I_{(+)}IM) \subseteq T_P\,(I)_{\,(+)}T_P\,(IM) \subseteq I_{(+)}IM,$$

so that $I_{(+)}IM = T_P(I)_{(+)}T_P(IM)$. Hence $I = T_P(I)$, and I is P-torsion.
(4) For an ideal $I = \sum_{\alpha} Ra_\alpha$ of a ring R, $I_{(+)}IM = \sum_{\alpha} Ra_{\alpha(+)}a_\alpha M = \sum_{\alpha} R(M)(a_\alpha, 0)$. Hence $\{a_\alpha\}$ generates I as an ideal of R if and only if $\{(a_\alpha, 0)\}$ generates $I_{(+)}IM$ as an ideal of $R(M)$. Suppose I is P-principal. Then there exist $a \in I$, $p \in P$ such that $(1-p)I \subseteq Ra$, and hence

$$\begin{aligned} &((1,0)-(p,0))(I_{(+)}IM) = (1-p,0)(I_{(+)}IM) \\ &= (1-p)I_{(+)}(1-p)IM \subseteq Ra_{(+)}aM = R(M)(a,0). \end{aligned}$$

This shows that $I_{(+)}IM$ is $P_{(+)}M$-principal. Conversely, let $I_{(+)}IM$ be $P_{(+)}M$-principal . There exist $a \in I$, $p \in P$ and $m \in M$ such that $((1,0)-(p,m))(I_{(+)}IM) \subseteq R(M)(a,0)$. Then $((1-p)x, (1-p)y - mx) = ((1,0)-(p,m))(x,y) \in Ra_{(+)}aM$ for all $x \in I$ and all $y \in IM$. Hence $(1-p)x \in Ra$ for all $x \in I$. This implies that $(1-p)I \subseteq Ra$, and I is P-principal. □

The next result gives necessary and sufficient conditions for the homogeneous ideal $I_{(+)}IM$ of $R(M)$ to be meet principal.

Proposition 5. *Let R be a ring, M an R-module and I an ideal of R. Then $I_{(+)}IM$ is a meet principal (equivalently multiplication) ideal of $R(M)$ if and only if I is meet principal (equivalently multiplication).*

Proof. The maximal ideals of $R(M)$ have the form $P_{(+)}M$ where P is a maximal ideal P of R, [13, Theorem 25(1)]. The result follows by the above fact, Proposition 4 and [12, Theorem 1.2]. Note that, if $I_{(+)}IM$ is meet principal then the fact that I is meet principal also follows by Theorem 3. □

The next result gives a condition under which the converse of Theorem 3 is true. First, we give a lemma.

Lemma 6. *Let R be a ring and K, N meet principal submodules of an R-module M. If $[K:N] + [N:K] = R$ then $K + N$ is meet principal.*

Proof. Let $[K:N] + [N:K] = R$. Then

$$\begin{aligned} K &= [K:N]K + [N:K]K = [K:N]K + (K \cap N) \\ &= [K:N]K + [K:N]N = [K:N](K+N) = [K:(K+N)](K+N). \end{aligned}$$

Similarly, $N = [N:(K+N)](K+N)$. Let A be an ideal of R and L a submodule of M. Since $[K:N] + [N:K] = R$, it is easily verified that $[AK:AN] + [AN:AK] = R$. It follows by [20, Proposition 4], that

$$\begin{aligned} &L \cap A(K+N) = L \cap (AK + AN) \\ &= (L \cap AK) + (L \cap AN) = ([L:K] \cap A)K + ([L:N] \cap A)N \\ &= ([L:K] \cap A)[K:(K+N)](K+N) + ([L:N] \cap A)[N:(K+N)](K+N) \\ &\subseteq ([L:K][K:(K+N)] \cap A)(K+N) + ([L:N][N:(K+N)] \cap A)(K+N) \\ &\subseteq ([L:(K+N)] \cap A)(K+N) + ([L:(K+N)] \cap A)(K+N) \\ &= ([L:(K+N)] \cap A)(K+N), \end{aligned}$$

obviously,

$$([L:(K+N)]\cap A)(K+N)\subseteq L\cap A(K+N).$$

Hence

$$([L:(K+N)]\cap A)(K+N)=L\cap A(K+N),$$

and $K+N$ is a meet principal submodule of M. $\square$

Proposition 7. *Let R be a ring and M an R-module. Let $I_{(+)}N$ be a homogeneous ideal of $R(M)$. If I is a meet principal ideal of R and N a meet principal submodule of M such that* $\mathrm{ann}I+[IM:N]=R$ *then $I_{(+)}N$ is meet principal.*

Proof. By Propositions 2 and 5, $0_{(+)}N$ and $I_{(+)}IM$ are meet principal ideals of $R(M)$. Next,

$$\begin{aligned}&[0_{(+)}N:_{R(M)} I_{(+)}IM]+[I_{(+)}IM:_{R(M)} 0_{(+)}N]=(\mathrm{ann}I_{(+)}M)\\ &+([IM:N]_{(+)}M)=(\mathrm{ann}I+[IM:N])_{(+)}M=R(M).\end{aligned}$$

The result follows by Lemma 6. $\square$

2. Join principal submodules

Let M be an R-module and N a submodule of M. Then N is *join principal* if $[(IN+K):N]=I+[K:N]$ for all ideals I of R and all submodules N of M. Setting $K=0$, we define N to be *weak join principal* if $[IN:N]=I+\mathrm{ann}N$ for all ideals I of R, [7].

A submodule N of an R-module M is called *cancellation* (resp. *weak cancellation*) if $[IN:N]=N$ (resp. $[IN:N]=I+\mathrm{ann}N$) for all ideals I of R, [18]. Hence weak join principal submodules are weak cancellation submodules. While meet principal and weak meet principal submodules coincide, join principal submodules are obviously weak cancellation but not conversely. For example, let R be an almost Dedekind domain that is not Dedekind. Hence R has a maximal ideal P that is not finitely generated. So P is a cancellation ideal and hence a weak cancellation ideal of R but not join principal, [7]. D. D. Anderson, [7], defined restricted cancellation modules: A submodule N of an R-module M is a *restricted cancellation submodule* if $0\neq IN=JN$ for all ideals I and J of R implies $I=J$. He proved that a submodule N of M is restricted cancellation if and only if it is weak cancellation and $\mathrm{ann}N$ is comparable to every ideal of R, [7, Theorem 2.5]. In [1], we investigated join principal submodules and gave several properties of such modules. We gave necessary and sufficient conditions for the sum, intersection, product and tensor product of join principal submodules (ideals) to be join principal. In this section we investigate the idealization of join principal submodules. We start by a result proved by D. D. Anderson, [7, Theorem 3.1]. We give it here for completeness.

Proposition 8. *Let R be a ring and N a submodule of an R-module M.*

(1) *N is weak cancellation if and only if $0_{(+)}N$ is a weak cancellation ideal of $R(M)$.*

(2) *N is cancellation if and only if $0_{(+)}N$ is a weak cancellation ideal of $R(M)$ and $\operatorname{ann}(0_{(+)}N) = 0_{(+)}M$.*

(3) *$0_{(+)}N$ is a restricted cancellation ideal of $R(M)$ if and only if N is a restricted cancellation submodule of M and for all ideals J of R, $JN \neq 0$ implies $JM = M$.*

(4) *N is join principal if and only if $0_{(+)}N$ is a join principal ideal of $R(M)$.*

The next result gives some conditions under which cancellation properties of a homogeneous ideal of $R\,(M)$ transfer to its components.

Theorem 9. *Let R be a ring and M an R-module. Let $I_{(+)}N$ be a homogeneous ideal of $R(M)$.*

(1) *If M is cancellation and $I_{(+)}N$ is cancellation then I is a cancellation ideal of R and N is a cancellation submodule of M.*

(2) *If M is finitely generated, faithful and multiplication and $I_{(+)}N$ is weak cancellation then I is a weak cancellation ideal of R and N is a weak cancellation submodule of M.*

(3) *If M is finitely generated faithful multiplication and $I_{(+)}N$ is restricted cancellation then I is a restricted cancellation ideal of R and N is a restricted cancellation submodule of M.*

(4) *If $I_{(+)}N$ is join principal then I is a join principal ideal of R. Assuming further that M is finitely generated multiplication then N is a join principal submodule of M.*

Proof. (1) Let A be an ideal of R. Then

$$\begin{aligned} 0_{(+)}AM &= [(0_{(+)}AM)(I_{(+)}N) :_{R(M)} I_{(+)}N] = [0_{(+)}AIM :_{R(M)} I_{(+)}N] \\ &= (\operatorname{ann} I \cap [AIM : N])_{(+)}[AIM :_M I]. \end{aligned}$$

It follows that $[AIM : IM]M \subseteq [AIM :_M I] = AM$. Since M is cancellation, we infer that $[AI : I] \subseteq A$. The reverse inclusion is always true and I is cancellation. Next,

$$\begin{aligned} A_{(+)}AM &= [(A_{(+)}AM)(I_{(+)}N) :_{R(M)} I_{(+)}N] = [AI_{(+)}AN :_{R(M)} I_{(+)}N] \\ &= ([AI : I] \cap [AN : N])_{(+)}[AN :_M I]. \end{aligned}$$

But I is cancellation. Thus $[AI : I] \cap [AN : N] = A \cap [AN : N] = A$. Hence $A_{(+)}AM = A_{(+)}[AN :_M I]$, and hence

$$AM = [AN :_M I] \supseteq [AN : IM]\, M \supseteq [AN : N]\, M.$$

As M is cancellation, we obtain that $[AN : N] \supseteq A \supseteq [AN : N]$, so that $A = [AN : N]$ and N is cancellation. Alternatively, M is a cancellation module and hence $0_{(+)}M$ is a weak cancellation ideal of $R(M)$. It follows that $0_{(+)}IM = (I_{(+)}N)(0_{(+)}M)$ is a weak cancellation ideal of $R(M)$. By Proposition 8, IM is a weak cancellation submodule of M. But M is cancellation. Thus I is a weak cancellation ideal of R. Since $I_{(+)}N$ is faithful, we infer from Lemma 1 that $\mathrm{ann}(I)M = \mathrm{ann}(IM)M \subseteq [0:_M I] = 0$. Hence $\mathrm{ann}I \subseteq \mathrm{ann}M = 0$, and hence I is faithful. This implies that I is cancellation. Next, $0_{(+)}IN = (I_{(+)}N)^2(0_{(+)}M)$ is a weak cancellation ideal of $R(M)$, and hence IN is a weak cancellation submodule of M. Since $IM \subseteq N$ and IM is faithful, N is faithful, and hence IN is faithful. This implies that IN is cancellation and hence N is cancellation.

(2) Suppose M is finitely generated, faithful and multiplication and suppose A is an ideal of R. Then

$$[(0_{(+)}AM)(I_{(+)}N) :_{R(M)} I_{(+)}N] = 0_{(+)}AM + \mathrm{ann}(I_{(+)}N).$$

But

$$\begin{aligned}[(0_{(+)}AM)(I_{(+)}N) :_{R(M)} I_{(+)}N] &= [0_{(+)}AIM :_{R(M)} I_{(+)}N] \\ &= (\mathrm{ann}I \cap [AIM : N])_{(+)}[AIM :_M I],\end{aligned}$$

and

$$\begin{aligned}0_{(+)}AM + \mathrm{ann}(I_{(+)}N) &= 0_{(+)}AM + \mathrm{ann}N_{(+)}(\mathrm{ann}I)M \\ &= \mathrm{ann}N_{(+)}(A + \mathrm{ann}I)M.\end{aligned}$$

Thus $[AIM :_M I] = (A + \mathrm{ann}I)M$. Since M is finitely generated, faithful and multiplication (hence cancellation), it follows that$[AI : I]M = [AIM : IM]M = [AIM :_M I] = (A + \mathrm{ann}I)M$,and this finally gives that $[AI : I] = A + \mathrm{ann}I$, and I is weak cancellation. Next, we have that

$$[(A_{(+)}AM)(I_{(+)}N) :_{R(M)} I_{(+)}N] = A_{(+)}AM + \mathrm{ann}(I_{(+)}N).$$

It follows that

$$[(A_{(+)}AM)(I_{(+)}N) :_{R(M)} I_{(+)}N] = ([AI : I] \cap [AN : N])_{(+)}[AN :_M I],$$

and

$$A_{(+)}AM + \mathrm{ann}(I_{(+)}N) = (A + \mathrm{ann}N)_{(+)}(A + \mathrm{ann}I)M.$$

Hence $[AI : I] \cap [AN : N] = A + \mathrm{ann}N$ and $[AN :_M I] = [AN : IM]M = (A{+}\mathrm{ann}I)M$ from which it follows that $[AN : IM] = A + \mathrm{ann}I$. Since I is weak cancellation, we get that

$$\begin{aligned}A + \mathrm{ann}N = [AI : I] \cap [AN : N] &= (A + \mathrm{ann}I) \cap [AN : N] \\ &= [AN : IM] \cap [AN : N] = [AN : N],\end{aligned}$$

and N is weak cancellation.

(3) Let M be finitely generated faithful multiplication and $I_{(+)}N$ restricted cancellation. By [7, Theorem 2.5], $I_{(+)}N$ is weak cancellation and by (2) I and N are weak cancellation. Suppose A is an ideal of R. Then either

$$0_{(+)}AM \subseteq \mathrm{ann}(I_{(+)}N) = \mathrm{ann}N_{(+)}(\mathrm{ann}I)M$$

from which it follows that $AM \subseteq (\mathrm{ann}I)\,M$, and hence $A \subseteq \mathrm{ann}I$ or

$$\mathrm{ann}N_{(+)}\,(\mathrm{ann}I)\,M = \mathrm{ann}\,(I_{(+)}N) \subseteq 0_{(+)}AM.$$

From the latter case we get that $(\mathrm{ann}I)\,M \subseteq AM$, and hence $\mathrm{ann}I \subseteq A$. Hence $\mathrm{ann}I$ is comparable to A, and by [7, Theorem 2.5], I is a restricted cancellation ideal of R. To show N is restricted cancellation, we have either

$$A_{(+)}AM \subseteq \mathrm{ann}\,(I_{(+)}N) = \mathrm{ann}N_{(+)}\,(\mathrm{ann}I)\,M,$$

and hence $A \subseteq \mathrm{ann}N$ or

$$\mathrm{ann}N_{(+)}\,(\mathrm{ann}I)\,M = \mathrm{ann}\,(I_{(+)}N) \subseteq A_{(+)}AM,$$

and hence $\mathrm{ann}N \subseteq A$. This shows that $\mathrm{ann}N$ is comparable to A and the result follows by [7, Theorem 2.5].

(4) Let A and B be ideals of R. Then

$$\begin{aligned}&\left[\left((A_{(+)}M)\,(I_{(+)}N) + B_{(+)}M\right) :_{R(M)} I_{(+)}N\right]\\ &= A_{(+)}M + \left[B_{(+)}M :_{R(M)} I_{(+)}N\right].\end{aligned}$$

But

$$\begin{aligned}&\left[\left((A_{(+)}M)\,(I_{(+)}N) + B_{(+)}M\right) :_{R(M)} I_{(+)}N\right]\\ &= \left[\left((AI+B)_{(+)}M\right) :_{R(M)} I_{(+)}N\right] = [(AI+B) : I]_{(+)}M,\end{aligned}$$

and

$$A_{(+)}M + \left[B_{(+)}M :_{R(M)} I_{(+)}N\right] = A_{(+)}M + [B : I]_{(+)}M = (A + [B : I])_{(+)}M.$$

Thus

$$[(AI+B) : I] = A + [B : I]\,,$$

and this shows that I is a join principal ideal of R. Suppose now M is a finitely generated multiplication module. Let A be an ideal of R and K a submodule of M. Then

$$\begin{aligned}&\left[\left((A_{(+)}AM)\,(I_{(+)}N) + [K : M]_{(+)}K\right) :_{R(M)} I_{(+)}N\right]\\ &= A_{(+)}AM + \left[[K : M]_{(+)}K :_{R(M)} I_{(+)}N\right].\end{aligned}$$

Now,

$$\begin{aligned}&\left[\left((A_{(+)}AM)\,(I_{(+)}N) + [K : M]_{(+)}K\right) :_{R(M)} I_{(+)}N\right]\\ &= \left[(AI + [K : M])_{(+)}\,(AN + K) :_{R(M)} I_{(+)}N\right]\\ &= \left([(AI + [K : M]) : I] \cap [(AN + K) : N]\right)_{(+)}\,[(AN+K) :_M I]\,.\end{aligned}$$

On the other hand,

$$\begin{aligned}&A_{(+)}AM + \left[[K : M]_{(+)}K :_{R(M)} I_{(+)}N\right]\\ &= A_{(+)}AM + \left([[K : M] : I] \cap [K : N]\right)_{(+)}\,[K :_M I]\\ &= A_{(+)}AM + \left([K : IM] \cap [K : N]\right)_{(+)}\,[K :_M I]\\ &= A_{(+)}AM + [K : N]_{(+)}\,[K :_M I] = (A + [K : N])_{(+)}\,(AM + [K :_M I])\,.\end{aligned}$$

Hence

$$[(AI + [K : M]) : I] \cap [(AN + K) : N] = A + [K : N],$$

and

$$[(AN + K) :_M I] = AM + [K :_M I].$$

Since I is join principal, we have that

$$\begin{aligned} A + [K : N] &= [(AI + [K : M]) : I] \cap [(AN + K) : N] \\ &= (A + [K : IM]) \cap [(AN + K) : N]. \end{aligned}$$

Since M is finitely generated multiplication, we infer that

$$[(AN + K) : IM]\, M = [(AN + K) :_M I] = AM + [K :_M I] = (A + [K : IM])\, M,$$

and hence

$$[(AN + K) : IM] + \mathrm{ann}M = A + [K : IM] + \mathrm{ann}M.$$

But $\mathrm{ann}M \subseteq [(AN + K) : IM]$ and $\mathrm{ann}M \subseteq [K : IM]$. Thus $[(AN + K) : IM] = A + [K : IM]$. It follows that

$$[(AN + K) : N] \subseteq [(AN + K) : IM] = A + [K : IM],$$

and this finally gives that $[(AN + K) : N] = A + [K : N]$, and N is a join principal submodule of M. This completes the proof of the theorem. □

The next two results show how cancellation properties of $I_{(+)}IM$ are related to those of I.

Proposition 10. *Let R be a ring, M an R-module and I an ideal of R.*

(1) *If $I_{(+)}IM$ is a weak cancellation ideal of $R(M)$ then I is a weak cancellation ideal of R.*

(2) *If $I_{(+)}IM$ is a cancellation ideal of $R(M)$ then I is a cancellation ideal of R.*

(3) *If $I_{(+)}IM$ is a restricted cancellation ideal of $R(M)$ then I is a restricted cancellation ideal of R. Assuming further that $IM \neq 0$ then I is faithful (and hence cancellation).*

(4) *If $I_{(+)}IM$ is a join principal ideal of $R(M)$ then I is a join principal ideal of R.*

Proof. (1) Suppose A is an ideal of R. Then

$$\left[(A_{(+)}M)\,(I_{(+)}IM) :_{R(M)} I_{(+)}IM\right] = A_{(+)}M + \mathrm{ann}\,(I_{(+)}IM).$$

But

$$\left[(A_{(+)}M)\,(I_{(+)}IM) :_{R(M)} I_{(+)}IM\right] = \left[AI_{(+)}IM :_{R(M)} I_{(+)}IM\right] = [AI : I]_{(+)}M,$$

and

$$A_{(+)}M + \operatorname{ann}(I_{(+)}IM) = A_{(+)}M + (\operatorname{ann}I_{(+)}[0:_M I]) = (A + \operatorname{ann}I)_{(+)}M.$$

Thus $[AI : I] = A + \operatorname{ann}I$, and I is weak cancellation.

(2) If $I_{(+)}IM$ is cancellation, it is faithful weak cancellation. By Lemma 1 and part (1), I is faithful weak cancellation and hence it is cancellation.

(3) Suppose $I_{(+)}IM$ is restricted cancellation. By, [7, Theorem 2.5], $I_{(+)}IM$ is weak cancellation and by (1), I is weak cancellation. Let A be an ideal of R. It follows by, [7, Theorem 2.5], that either

$$\operatorname{ann}I_{(+)}[0:_M I] = \operatorname{ann}(I_{(+)}IM) \subseteq A_{(+)}M,$$

from which it follows that $\operatorname{ann}I \subseteq A$ or

$$A_{(+)}M \subseteq \operatorname{ann}(I_{(+)}IM) = \operatorname{ann}I_{(+)}[0:_M I].$$

From the latter case we infer that $A \subseteq \operatorname{ann}I$. Hence $\operatorname{ann}I$ is comparable to A, and by, [7, Theorem 2.5], I is restricted cancellation. Assume now that $IM \neq 0$. Then $0 \neq (0_{(+)}M)(I_{(+)}IM) = (\operatorname{ann}I_{(+)}M)(I_{(+)}IM)$, and hence $0_{(+)}M = \operatorname{ann}I_{(+)}M$. It follows that $\operatorname{ann}I = 0$, and hence I is cancellation.

(4) Let $I_{(+)}IM$ be join principal. Let A and B be ideals of R. Then

$$\left[((A_{(+)}M)(I_{(+)}IM) + B_{(+)}M) :_{R(M)} I_{(+)}IM\right] = A_{(+)}M + \left[B_{(+)}M :_{R(M)} I_{(+)}IM\right].$$

Now,

$$\begin{aligned}&\left[((A_{(+)}M)(I_{(+)}IM) + B_{(+)}M) :_{R(M)} I_{(+)}IM\right]\\ &= \left[(AI+B)_{(+)}M :_{R(M)} I_{(+)}IM\right] = [(AI+B) : I]_{(+)}M,\end{aligned}$$

and

$$A_{(+)}M + \left[B_{(+)}M :_{R(M)} I_{(+)}IM\right] = A_{(+)}M + [B : I]_{(+)}M = (A + [B : I])_{(+)}M.$$

Hence $[(AI + B) : I] = A + [B : I]$, and I is join principal. □

Theorem 11. *Let R be a ring, M a multiplication R-module and $R(M)$ a homogeneous ring. Let I be an ideal of R.*

1. *If M is faithful and I is weak cancellation then $I_{(+)}IM$ is a weak cancellation ideal of $R(M)$.*
2. *If M is faithful and I is cancellation then $I_{(+)}IM$ is a cancellation ideal of $R(M)$.*
3. *If M is finitely generated and I is join principal then $I_{(+)}IM$ is a join principal ideal of $R(M)$.*

Proof. (1) Let $A_{(+)}K$ be an ideal of $R(M)$. Then

$$\begin{aligned}\left[(A_{(+)}K)(I_{(+)}IM):_{R(M)} I_{(+)}IM\right] &= \left[AI_{(+)}IK:_{R(M)} I_{(+)}IM\right]\\ &= ([AI:I]\cap[IK:IM])_{(+)}[IK:_M I].\end{aligned}$$

Since $AM \subseteq K$, $[AI:I] \subseteq [AIM:IM] \subseteq [IK:IM]$, and hence $[AI:I]\cap[IK:IM] = [AI:I] = A+\operatorname{ann}I$. Next, we show that $[IK:_M I] = K+(\operatorname{ann}I)M$. Obviously, $K+(\operatorname{ann}I)M \subseteq [IK:_M I]$. Let $m \in [IK:_M I]$. Then $Im \subseteq IK$, and hence $I[Rm:M]M \subseteq I[K:M]M$. Hence $I[Rm:M]\mathrm{Tr}(M) \subseteq I[K:M]\mathrm{Tr}(M)$. As I is weak cancellation, $[Rm:M]\mathrm{Tr}(M) \subseteq [K:M]\mathrm{Tr}(M)+\operatorname{ann}I$. Since M is faithful multiplication, it follows by [10, Theorem 2.6], that

$$\begin{aligned}Rm = [Rm:M]M = [Rm:M]\,\mathrm{Tr}(M)M &\subseteq [K:M]\,\mathrm{Tr}(M)M+(\operatorname{ann}I)M\\ &= K+(\operatorname{ann}I)M,\end{aligned}$$

so that $m \in K+(\operatorname{ann}I)M$ and hence $[IK:_M I] = K+(\operatorname{ann}I)M$. This finally gives that

$$\begin{aligned}\left[(A_{(+)}K)(I_{(+)}IM):_{R(M)} I_{(+)}IM\right] &= (A+\operatorname{ann}I)_{(+)}(K+(\operatorname{ann}I)M)\\ &= A_{(+)}K+\operatorname{ann}I_{(+)}(\operatorname{ann}I)M = A_{(+)}K+\operatorname{ann}(I_{(+)}IM),\end{aligned}$$

and hence $I_{(+)}IM$ is weak cancellation.

(2) Since M is faithful multiplication, I is faithful if and only if $I_{(+)}IM$ is faithful. The result follows by (1) and the fact that every ideal is cancellation if and only if it is faithful weak cancellation.

(3) Suppose $A_{(+)}K$ and $B_{(+)}L$ are ideals of $R(M)$. Then

$$\begin{aligned}&\left[((A_{(+)}K)(I_{(+)}IM)+B_{(+)}L):_{R(M)} I_{(+)}IM\right]\\ &= ([(AI+B):I]\cap[(IK+L):IM])_{(+)}[(IK+L):_M I].\end{aligned}$$

Since I is join principal, $AM \subseteq K$ and $BM \subseteq L$, we infer that

$$A+[B:I] = [(AI+B):I] \subseteq [(AIM+BM):IM] \subseteq [(IK+L):IM].$$

Since M is finitely generated multiplication, M is weak cancellation, [20, Corollary 2 to Theorem 9]. It follows by [18, Proposition 1.4], that

$$\begin{aligned}[(IK+L)_{:M} I] = [(IK+L):IM]M &= [(I[K:M]+[L:M])M:IM]M\\ &= [(I[K:M]+[L:M]+\operatorname{ann}M):I]M\\ &= [(I[K:M]+[L:M]):I]M.\end{aligned}$$

As I is join principal, we get that

$$[(IK+L)_{:M} I] = ([K:M]+[[L:M]:I])M = K+[L:IM]M = K+[L:_M I].$$

Hence

$$\begin{aligned}&\left[((A_{(+)}K)(I+IM)+B_{(+)}L)_{:R(M)} I_{(+)}IM\right] = (A+[B:I])_{(+)}(K+[L:_M I])\\ &= A_{(+)}K+[B:I]_{(+)}[L:_M I] = A_{(+)}K+([B:I]\cap[L:IM])_{(+)}[L:_M I]\\ &= A_{(+)}K+\left[B_{(+)}L:_{R(M)} I_{(+)}IM\right],\end{aligned}$$

as required. This finishes the proof of the theorem. □

A submodule N of an R-module M is called *locally join principal* if N_P is a join principal submodule of the R_P-module M_P for each maximal ideal P of R. Suppose N is a finitely generated locally join principal submodule of an R-module M. Then N is join principal. For let A be an ideal of R, P a maximal ideal of R and K a submodule of M. Then

$$[(AN+K):N]_P = [(A_PN_P+K_P):N_P] = A_P + [K_P:N_P] = (A+[K:N])_P\,.$$

Since P is arbitrary, $[(AN+K):N] = A+[K:N]$. We use this fact to give some conditions under which finitely generated ideals of a homogeneous ring $R\,(M)$ are join principal.

Proposition 12. *Let R be a ring and M a finitely generated multiplication R-module. Let $R\,(M)$ be homogeneous and $I_{(+)}N$ a finitely generated ideal of $R\,(M)$. If I is a join principal ideal of R and N a join principal submodule of M such that* $\mathrm{ann}I + [IM:N] = R$ *then $I_{(+)}N$ is join principal.*

Proof. Since $I_{(+)}N$ is finitely generated, it is enough to prove the result locally. Thus we may assume $R\,(M)$ is a local ring. Since $R = \mathrm{ann}I + [IM:N]$, we infer that

$$R\,(M) = \left[0_{(+)}N :_{R(M)} I_{(+)}N\right] + \left[I_{(+)}IM :_{R(M)} I_{(+)}N\right].$$

Hence, either $I_{(+)}N = 0_{(+)}N$ or $I_{(+)}N = I_{(+)}IM$. The result follows by Proposition 8 and Theorem 11. □

The next result gives some conditions under which the ideal $I_{(+)}N$ (not necessarily finitely generated) is weak cancellation.

Theorem 13. *Let R be a ring, M a finitely generated multiplication R-module and $R\,(M)$ a homogeneous ring. Let $I_{(+)}N$ be an ideal of $R\,(M)$. If I is a join principal ideal of R and N a weak cancellation submodule of M such that* $\mathrm{ann}I+[IK:N] = R$ *for each submodule K of M then $I_{(+)}N$ is weak cancellation.*

Proof. Let $A_{(+)}K$ be an ideal of $R\,(M)$. We need to show that

$$\begin{aligned}\left[(A_{(+)}K)\,(I_{(+)}N) :_{R(M)} I_{(+)}N\right] &= A_{(+)}K + \mathrm{ann}\,(I_{(+)}N)\\ &= A_{(+)}K + (\mathrm{ann}I\cap\mathrm{ann}N)_{(+)}\,[0:_M I]\\ &= (A+(\mathrm{ann}I\cap\mathrm{ann}N))_{(+)}\,(K+[0:_M I])\,.\end{aligned}$$

Since I is join principal (and hence weak cancellation) and M is multiplication, we infer that

$$\begin{aligned}\left[(A_{(+)}K)\,(I_{(+)}N) :_{R(M)} I_{(+)}N\right] &= \left[AI_{(+)}\,(AN+IK) :_{R(M)} I_{(+)}N\right]\\ &= ([AI:I]\cap[(AN+IK):N])_{(+)}\,[(AN+IK):_M I]\\ &= ((A+\mathrm{ann}I)\cap[(AN+IK):N])_{(+)}\,[(AN+IK):IM]\,M.\end{aligned}$$

Next, since

$$R = \mathrm{ann}I + [IK : N] \subseteq [AN : IK] + [IK : AN] \subseteq R,$$

it follows by [4, Corollary 1.2], that

$$[(AN + IK) : N] = [AN : N] + [IK : N] \, .$$

But N is weak cancellation. Thus

$$[(AN + IK) : N] = A + [0 : N] + [IK : N] = A + [IK : N] \, .$$

Now

$$R = \mathrm{ann}I + [IK : N] \subseteq [\mathrm{ann}I : [IK : N]] + [[IK : N] : \mathrm{ann}I] \, ,$$

so that

$$R = [\mathrm{ann}I : [IK : N]] + [[IK : N] : \mathrm{ann}I] \, .$$

We obtain from [4, Corollary 1.2], that

$$(A + \mathrm{ann}I) \cap (A + [IK : N]) = A + (\mathrm{ann}I \cap [IK : N]) \, .$$

Again, since $R = \mathrm{ann}I + [IK : N]$, we get that

$$\mathrm{ann}I \cap [IK : N] = (\mathrm{ann}I) \, [IK : N] \subseteq [0 : IK] \, [IK : N] \subseteq [0 : N] = \mathrm{ann}N.$$

But $\mathrm{ann}I \cap [IK : N] \subseteq \mathrm{ann}I$. Thus

$$\mathrm{ann}I \cap [IK : N] \subseteq \mathrm{ann}I \cap \mathrm{ann}N \subseteq \mathrm{ann}I \cap [IK : N] \, ,$$

so that $\mathrm{ann}I \cap [IK : N] = \mathrm{ann}I \cap \mathrm{ann}N$, and hence

$$(A + \mathrm{ann}I) \cap [(AN + IK) : N] = A + (\mathrm{ann}I \cap \mathrm{ann}N) \, .$$

On the other hand,

$$[(AN + IK) : IM] \, M = [AN : IM] \, M + [IK : IM] \, M.$$

Since

$$R = \mathrm{ann}I + [IK : N] \subseteq \mathrm{ann}I + [IM : N] \subseteq R,$$

we get that

$$\begin{gathered} [AN : IM] = [AN : IM] \, \mathrm{ann}I + [AN : IM] \, [IM : N] \\ \subseteq \mathrm{ann}I + [AN : N] \subseteq [0 : IM] + A + [0 : N] = A + [0 : IM] \, . \end{gathered}$$

It follows that

$$[AN : IM] \, M = AM + [0 : IM] \, M \subseteq K + [0 :_M I] \, .$$

Finally, since M is finitely generated multiplication (hence weak cancellation) and I is join principal, we infer from [18, Proposition 1.4] that

$$[IK : IM]\, M = [I\,[K : M]\, M : IM]\, M = [(I\,[K : M] + \operatorname{ann} M) : I]\, M$$
$$= ([K : M] + [\operatorname{ann} M : I])\, M \subseteq [K : M]\, M + [0 : IM]\, M = K + [0 :_M I]\,.$$

Hence $[(AN + IK) :_M I] \subseteq K + [0 :_M I]$. This shows that

$$\left[(A_{(+)}K)\,(I_{(+)}N) :_{R(M)} I_{(+)}N\right] \subseteq (A_{(+)}K) + \operatorname{ann}(I_{(+)}N)\,.$$

The reverse inclusion is always true, and hence $I_{(+)}N$ is weak cancellation. This completes the proof of the theorem. □

3. Large and small submodules

A submodule N of an R-module M is said to be *large in M* if for all submodules K of M, $K \cap N = 0$ implies that $K = 0$. Dually, N is *small in M* if for all submodules K of $M, K + N = M$ implies that $K = M$. If I is a faithful ideal of a ring R then I is large. In particular, every non-zero ideal of an integral domain R is large. If N is a faithful submodule of a multiplication R-module M then N is large in M, [1]. For all submodules K and N of M with $K \subseteq N$, if K is large in M then so too is N and if N is small in M then so too is K. For properties of large and small modules, see [15]. The next theorem gives some properties of idealization of large and small modules.

Theorem 14. *Let R be a ring and M an R-module. Let I be an ideal of R and N a submodule of M.*

1. *$0_{(+)}M$ is a small ideal of $R(M)$ and, in particular, $0_{(+)}N$ is a small ideal of $R(M)$. If M is faithful then $0_{(+)}M$ is a large ideal of $R(M)$ and, in particular, $I_{(+)}M$ is a large ideal of $R(M)$.*
2. *If $0_{(+)}N$ is a large ideal of $R(M)$ then N is large in M, and the converse is true if M is faithful.*
3. *If I is a small ideal of R then $I_{(+)}M$ is a small ideal of $R(M)$.*
4. *I is a small ideal of R if and only if $I_{(+)}IM$ is a small ideal of $R(M)$.*
5. *If $I_{(+)}M$ is a small ideal of $R(M)$ then I is a small ideal of R.*
6. *Let M be faithful multiplication. Then I is a large ideal of R if and only if $I_{(+)}IM$ is a large ideal of $R(M)$.*

Proof. (1) Let H be an ideal of $R(M)$ such that $H + 0_{(+)}M = R(M)$. Then $(0_{(+)}M)\, H = 0_{(+)}M$. Hence $0_{(+)}M \subseteq H$ and hence $H + 0_{(+)}M = H$. It follows that $H = R(M)$ and $0_{(+)}M$ is a small ideal of $R(M)$. As $0_{(+)}N \subseteq 0_{(+)}M$, $0_{(+)}N$ is a small ideal of $R(M)$. Suppose now M is faithful and suppose H an ideal of $R(M)$ such that $H \cap 0_{(+)}M = 0$. Then $H\,(0_{(+)}M) = 0$, and hence $H \subseteq \operatorname{ann}(0_{(+)}M) = 0_{(+)}M$. This implies that $0 = H \cap 0_{(+)}M = H$, and hence $0_{(+)}M$ is a large ideal of $R(M)$. Since $0_{(+)}M \subseteq I_{(+)}M$, $I_{(+)}M$ is a large ideal of $R(M)$.

(2) Let $0_{(+)}N$ be a large ideal of $R(M)$. Let K be a submodule of M such that $K \cap N = 0$. Then $0_{(+)}K \cap 0_{(+)}N = 0_{(+)}(K \cap N) = 0$. Hence $0_{(+)}K = 0$, and hence $K = 0$. This implies that N is large in M. Conversely, suppose H is an ideal of $R(M)$ such that $H \cap 0_{(+)}N = 0$. Then $(H \cap 0_{(+)}M) \cap 0_{(+)}N = 0$. Assume $H \cap 0_{(+)}M = 0_{(+)}K$ for some submodule K of M. Then $0 = 0_{(+)}(K \cap N) = 0_{(+)}K \cap 0_{(+)}N$, and hence $K \cap N = 0$. Since N is large in M, $K = 0$, and hence $0 = 0_{(+)}K = H \cap 0_{(+)}M$. As M is faithful, we obtain by (1) that $0_{(+)}M$ is large and hence $H = 0$. This shows that $0_{(+)}N$ is a large ideal of $R(M)$.

(3) Suppose I is a small ideal of R and H an ideal of $R(M)$ such that $H + I_{(+)}M = R(M)$. Then $H + 0_{(+)}M + I_{(+)}M = R(M)$. Let $H + 0_{(+)}M = J_{(+)}M$ for some ideal J of R. Then $(J + I)_{(+)}M = R(M)$, and hence $J + I = R$. It follows that $J = R$, and hence $H + 0_{(+)}M = R(M)$. As $0_{(+)}M$ is a small ideal of $R(M)$, $H = R(M)$ and $I_{(+)}M$ is a small ideal of $R(M)$.

(4) Let I be a small ideal of R. By (3), $I_{(+)}M$ is a small ideal of $R(M)$ and hence $I_{(+)}IM$ is a small ideal of $R(M)$. Conversely, let $I_{(+)}IM$ be a small ideal of $R(M)$. Let J be an ideal of R such that $J + I = R$. Then $J_{(+)}M + I_{(+)}IM = R(M)$. Hence $J_{(+)}M = R(M)$ and hence $J = R$. This shows that I is a small ideal of R.

(5) If $I_{(+)}M$ is a small ideal of $R(M)$ then so too is $I_{(+)}IM$. The result follows by (4).

(6) Suppose M is faithful and multiplication. Let $I_{(+)}IM$ be a large ideal of $R(M)$. Let J be an ideal of R such that $J \cap I = 0$. It follows by [12, Corollary 1.7], that $0 = (J \cap I)M = JM \cap IM$, and hence

$$0 = (J \cap I)_{(+)}(JM \cap IM) = (J_{(+)}JM) \cap (I_{(+)}IM).$$

It follows that $J_{(+)}JM = 0$, and hence $J = 0$. This implies that I is a large ideal of R. Conversely, suppose I is large and suppose H an ideal of $R(M)$ such that $H \cap I_{(+)}IM = 0$. Hence $H \cap 0_{(+)}M \cap I_{(+)}IM = 0$. Suppose $H \cap 0_{(+)}M = 0_{(+)}K$ for some submodule K of M. It follows that $0_{(+)}(K \cap IM) = 0_{(+)}K \cap I_{(+)}IM = 0$. Hence $K \cap IM = 0$. Since M is faithful multiplication, we infer, from [12, Corollary 1.7], that $([K : M] \cap I)M = 0$, and hence $[K : M] \cap I = 0$. As I is large, we obtain that $[K : M] = 0$, and hence $K = [K : M]M = 0$. This gives that $H \cap 0_{(+)}M = 0_{(+)}K = 0$. Finally, since $0_{(+)}M$ is large, $H = 0$, and hence $I_{(+)}IM$ is a large ideal of $R(M)$. This finishes the proof of the theorem. $\square$

The *socle* of an R-module M, soc$M$, is the intersection of all large submodules of M, while the *Jacobson radical* of M, $J(M)$, is the sum of all small submodules of M, [15]. The next corollary shows that the socle but not Jacobson radical behaves well with respect to idealization.

Corollary 15. *Let R be a ring and N a submodule of an R-module M.*

(1) $J(0_{(+)}M) = 0_{(+)}M$.

(2) *If M is faithful then* $\mathrm{soc}(0_{(+)}M) = 0_{(+)}\mathrm{soc}M$.

(3) *If M is faithful and multiplication then* $\mathrm{soc}(0_{(+)}M) = \theta(0_{(+)}M)\mathrm{soc}(0_{(+)}M)$.

Proof. (1) Follows by Theorem 14(1).

(2) By Theorem 14(1), the large submodules of $0_{(+)}M$ are exactly those of the form $0_{(+)}N$ where N is large in M. Hence the result follows.

(3) By [5, Corollary 1.4] and [8, Theorem 3.2], $\mathrm{soc}M = \theta(M)\,\mathrm{soc}M$. Hence, $\mathrm{soc}\,(0_{(+)}M) = 0_{(+)}\mathrm{soc}M = 0_{(+)}\theta(M)\,\mathrm{soc}M = (\theta(M)\,_{(+)}M)\,(0_{(+)}\mathrm{soc}M) = \theta(0_{(+)}M)$ $\mathrm{soc}\,(0_{(+)}M)$. □

Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_4$. Then $J(0_{(+)}M) = 0_{(+)}\mathbb{Z}_4$ while $0_{(+)}J(M) = 0_{(+)}2\mathbb{Z}_4$. Hence $J(0_{(+)}M) \neq 0_{(+)}J(M)$.

An R-module M is called *finitely cogenerated* if for every non-empty collection of submodules $N_\lambda\,(\lambda \in \Lambda)$ of M with $\bigcap_{\lambda\in\Lambda} N_\lambda = 0$, there exists a finite subset Λ' of Λ such that $\bigcap_{\lambda\in\Lambda'} N_\lambda = 0$. M is called *uniform* if the intersection of any two non-zero submodules of M is non-zero, and M has *finite uniform dimension* if it does not contain an infinite direct sum of non-zero submodules, [15]. The next result gives some properties of idealization of finitely cogenerated and uniform modules.

Proposition 16. *Let R be a ring and M an R-module. Let I be an ideal of R and N a submodule of M.*

(1) *N is finitely cogenerated if and only if $0_{(+)}N$ is a finitely cogenerated ideal of $R(M)$.*

(2) *N is uniform if and only if $0_{(+)}N$ is a uniform ideal of $R(M)$.*

(3) *N has finite uniform dimension if and only if $0_{(+)}N$ has finite uniform dimension.*

(4) *If M is faithful multiplication and $I_{(+)}IM$ is finitely cogenerated then so too is I.*

(5) *If M is faithful multiplication and $I_{(+)}IM$ is uniform then so too is I.*

(6) *If M is faithful multiplication and $I_{(+)}IM$ has finite uniform dimension then so too has I.*

Proof. (1) Suppose $0_{(+)}N$ is finitely cogenerated. Let $N_\lambda\,(\lambda \in \Lambda)$ be a non-empty collection of submodules of N such that $\bigcap_{\lambda\in\Lambda} N_\lambda = 0$. Then $0 = 0_{(+)}\bigcap_{\lambda\in\Lambda} N_\lambda = \bigcap_{\lambda\in\Lambda} 0_{(+)}N_\lambda$, and hence there exists a finite subset Λ' of Λ such that $\bigcap_{\lambda\in\Lambda} 0_{(+)}N_\lambda = 0$. Hence $0_{(+)}\bigcap_{\lambda\in\Lambda'} N_\lambda = 0$, and hence $\bigcap_{\lambda\in\Lambda'} N_\lambda = 0$. This implies that N is finitely cogenerated. The converse is now clear since every submodule of $0_{(+)}N$ has the form $0_{(+)}K$ for some submodule K of N.

(2) Follows by (1).

(3) Suppose $0_{(+)}N$ has finite uniform dimension. If N contains a direct sum of submodules $N_\lambda\,(\lambda \in \Lambda)$ then $\sum_{\lambda\in\Lambda} 0_{(+)}N_\lambda$ is direct, and hence all but a finite number of ideals $0_{(+)}N_\lambda$ is zero. If $0_{(+)}N_\lambda = 0$, then $N_\lambda = 0$ and N has finite uniform dimension. The converse is routine.

(4) Let $I_{(+)}IM$ be finitely cogenerated. Let $I_\lambda\,(\lambda \in \Lambda)$ be a non-empty collection of ideals of R contained in I such that $\bigcap_{\lambda\in\Lambda} I_\lambda = 0$. Since M is faithful multiplication, we infer from [12, Corollary 1.7], that $\bigcap_{\lambda\in\Lambda} I_\lambda M = \left(\bigcap_{\lambda\in\Lambda} I_\lambda\right) M = 0$, and hence $\bigcap_{\lambda\in\Lambda} I_{\lambda(+)} I_\lambda M = 0$. It follows that there exists a finite subset Λ' of Λ such that $\bigcap_{\lambda\in\Lambda'} I_{\lambda(+)} I_\lambda M = 0$. Hence $\bigcap_{\lambda\in\Lambda'} I_\lambda = 0$, and I is finitely cogenerated.

(5) Follows by (4).

(6) Suppose $I_{(+)}IM$ has finite uniform dimension. If I contains a direct sum of subideals $I_\lambda\,(\lambda \in \Lambda)$ then $\sum_{\lambda\in\Lambda} I_{\lambda(+)} I_\lambda M$ is direct and hence all but a finite number of the ideals $I_{\lambda(+)} I_\lambda M$ is zero. If $I_{\lambda(+)} I_\lambda M = 0$, then $I_\lambda = 0$ and I has finite uniform dimension. □

For all submodules K and N of an R-module M with $K \subseteq N$, if N is finitely cogenerated (resp. uniform, has finite uniform dimension) then so too is (has) K. The following result shows how large, small, finitely cogenerated and uniform properties of a homogeneous ideal $I_{(+)}N$ are related to those of I and N.

Proposition 17. *Let R be a ring and M an R-module. Let $I_{(+)}N$ be a homogeneous ideal of $R(M)$.*

(1) *If $I_{(+)}N$ is large then N is large in M. The converse is true if M is faithful.*
(2) *If M is faithful multiplication and I is a large ideal of R then $I_{(+)}N$ is large.*
(3) *$I_{(+)}N$ is small if and only if I is a small ideal of R.*
(4) *If M is finitely generated faithful and N is small in M then $I_{(+)}N$ is small.*
(5) *If $I_{(+)}N$ is finitely cogenerated (resp. uniform, has finite uniform dimension) then so too is (has) N.*
(6) *Assuming further to the assumption of* (5) *that M is faithful multiplication then I is finitely cogenerated (resp. uniform, has finite uniform dimension).*

Proof. (1) Suppose $I_{(+)}N$ is large. Let K be a submodule of M such that $K \cap N = 0$. Then $(0_{(+)}K) \cap (I_{(+)}N) = 0$. Hence $0_{(+)}K = 0$ and hence $K = 0$. This shows that N is large in M. Assume M is faithful and N is large in M. By Theorem 14 (2), $0_{(+)}N$ is a large ideal of $R(M)$ and hence $I_{(+)}N$ is large.

(2) Suppose M is faithful multiplication and I is a large ideal of R. By Theorem 14 (4), $I_{(+)}IM$ is a large ideal of $R(M)$ and hence $I_{(+)}N$ is large. Alternatively, since M is faithful and multiplication, IM is large in M. Since $IM \subseteq N$, N is large in M. Hence $0_{(+)}N$ is large and therefore $I_{(+)}N$ is large.

(3) Let $I_{(+)}N$ be small. Let J be an ideal of R such that $J + I = R$. Then $J_{(+)}M + I_{(+)}N = R(M)$. Hence $J_{(+)}M = R(M)$, and hence $J = R$. This implies that I is a small ideal of R. Conversely, if I is a small ideal of R, it follows by Theorem 14 (3) that $I_{(+)}M$ is a small ideal of $R(M)$. Hence $I_{(+)}N$ is small.

(4) Let M be finitely generated faithful and N small in M. Let H be an ideal of $R\,(M)$ such that $H + I_{(+)}N = R\,(M)$. Then

$$H\,(0_{(+)}M) + 0_{(+)}IM = (H + I_{(+)}N)\,(0_{(+)}M) = 0_{(+)}M.$$

Let $H\,(0_{(+)}M) = 0_{(+)}K$ for some submodule K of M. Then $0_{(+)}(K + IM) = 0_{(+)}M$ and hence $K + IM = M$. Since $IM \subseteq N$, $K + N = M$, and hence $K = M$. This implies that $H\,(0_{(+)}M) = 0_{(+)}M$. As M is finitely generated and faithful, it follows that $0_{(+)}M$ is finitely generated and $\mathrm{ann}(0_{(+)}M) = 0_{(+)}M$, see [7, Theorem 3.1] and Lemma 1. By [14, Theorem 76], we obtain that $R\,(M) = H + 0_{(+)}M$. But $0_{(+)}M$ is a small ideal of $R\,(M)$. Thus $H = R\,(M)$, and $I_{(+)}N$ is a small ideal of $R\,(M)$.

(5) and (6) follow by Proposition 16, Theorem 14 and the remarks made before the proposition. □

We close by the following result which gives conditions under which the converse of parts (5) and (6) of Proposition 17 is true.

Proposition 18. *Let R be a ring, M a faithful multiplication R-module and $R\,(M)$ a homogeneous ring. Let $I_{(+)}N$ be an ideal of $R\,(M)$.*

1. *If N is a finitely cogenerated submodule of M then $I_{(+)}N$ is finitely cogenerated.*
2. *If N is a uniform submodule of M then $I_{(+)}N$ is uniform.*
3. *If N has finite uniform dimension then so too has $I_{(+)}N$.*

Proof. $IM \subseteq N$. If N is finitely cogenerated (resp. uniform, has finite uniform dimension) then so too is (has) IM. Since M is faithful multiplication, it is easily verified that I is finitely cogenerated (resp. uniform, has finite uniform dimension).

(1) Let $J_{\lambda(+)}K_\lambda\,(\lambda \in \Lambda)$ be a non-empty collection of ideals of $R\,(M)$ contained in $I_{(+)}N$ such that $\bigcap_{\lambda\in\Lambda} J_{\lambda(+)}K_\lambda = 0$. Then $\bigcap_{\lambda\in\Lambda} J_\lambda = 0$ and $\bigcap_{\lambda\in\Lambda} K_\lambda = 0$. It follows that there exist finite subsets Λ' and Λ'' of Λ such that $\bigcap_{\lambda\in\Lambda'} J_\lambda = 0$ and $\bigcap_{\lambda\in\Lambda''} K_\lambda = 0$. Without loss of generality we may assume that $\Lambda' \subseteq \Lambda''$. Then $\bigcap_{\lambda\in\Lambda'} J_{\lambda(+)}K_\lambda = 0$, and hence $I_{(+)}N$ is finitely cogenerated.

(2) Follows by (1).

(3) If $I_{(+)}N$ contains a direct sum of subideals $J_{\lambda(+)}K_\lambda\,(\lambda \in \Lambda)$ then $\sum_{\lambda\in\Lambda} J_\lambda$ and $\sum_{\lambda\in\Lambda} K_\lambda$ are direct. Hence all but a finite number of each of J_λ and K_λ is zero. If $J_\lambda = 0$ and $K_\lambda = 0$ then $J_{\lambda(+)}K_\lambda = 0$ and hence $I_{(+)}N$ has finite uniform dimension. □

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