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General Projections in Spaces of Pencils

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Abstract. The notion of the central projection in spaces of pencils is generalized and new concepts of projections are introduced. The category of projectivities with segment subspaces as objects arises. These general projectivities are collineations given by linear maps.
Properties of pencils of segment subspaces and projections between segments are investigated. Classical projections of lines onto pencils of hyperplanes are considered in terms of spaces of pencils as projections of lines onto pencils of segment subspaces.

Introduction

The paper deals with projections defined (in possibly the most general way) in spaces of pencils. Let us recall briefly that the *space of k-pencils* $\mathfrak{R} = \mathbf{P}_k(\mathfrak{P})$ is an incidence structure whose points are all k-dimensional subspaces of a projective space $\mathfrak{P}$, and whose lines are pencils of such subspaces (in the paper we use this definition expressed in the language of vector subspaces of a vector space V, so $\mathbf{P}_k(\mathfrak{P}) \cong \mathbf{P}_{k+1}(V)$, while $\mathfrak{P} \cong \mathbf{P}_1(V)$). Spaces of pencils are partial linear spaces (a synthetic characterization of their geometry can be found, e.g. in [2]). Let us stress that we do not assume that $\mathfrak{P}$ (or equivalently V) is finite-dimensional, though, of course, k is finite ($1 \le k,\ k+1 < \dim(\mathfrak{P})$). Moreover, we do not assume that $\mathfrak{P}$is pappian.

The following notions: projection (between two subspaces), projective correspondence, perspectivity, and projectivity play crucial role in the classical projective geometry when it comes to determine linear collineations, linear correlations or quadrics (cf. e.g. [1, Ch. II.10], [7], [9, Ch. 4]), as well as in foundations of plane projective geometry (cf. e.g. [15]). Spaces of pencils, which are theatrum of our investigations, generalize projective spaces and projections in spaces of pencils appear as important as they are in projective geometry.

The question is how to define projections.
An arbitrary space of pencils $\mathfrak{R}$ satisfies both Veblen and Shult (none, one or all) axioms and therefore there is no difficulty in defining central projections between the lines of $\mathfrak{R}$ (cf. [14]). However, the standard definition of a central projection applied to (linear) subspaces $\mathcal{X}_1, \mathcal{X}_2$ of higher dimension of $\mathfrak{R}$ yields that $\mathcal{X}_1, \mathcal{X}_2$ are contained in some strong subspace (i.e. subspace where every two points are collinear) of $\mathfrak{R}$. Since the intersection of any two distinct strong subspaces of $\mathfrak{R}$ is contained in a line, classical central projections are insufficient to characterize a projective map which moves a strong subspace onto another one. This is the reason to generalize the standard construction.

Projections considered in the paper are (partial) maps between segment subspaces of $\mathfrak{R}$. In terms of the underlying projective space $\mathfrak{P}$ a segment subspace $[Z, Y]_k$ is defined to be the family of all k-subspaces of $\mathfrak{P}$ which are contained in a fixed subspace Y and contain a fixed subspace Z. Evidently, this definition generalizes the definition of a pencil. From view of the geometry of $\mathfrak{R}$ segment subspaces are exactly these subspaces of $\mathfrak{R}$ that carry the geometry of a space of pencils. So, the choice of domains of projections seems natural.

The following general idea characterizes projections: projected subspace $\mathcal{X}_1$, its image $\mathcal{X}_2$, and a subspace $\mathcal{X}_3$ which contains the "center" $\mathcal{Y}$ of the projection are in one pencil, and through any point on $\mathcal{X}_1 \setminus \mathcal{X}_2$ there is exactly one line which crosses both $\mathcal{X}_2$ and $\mathcal{Y}$. To make this idea a strict definition first we have to precise the notion of a pencil of segment subspaces. This is done in (2). After that, nearly all the remaining notions related to "projection" can be defined pretty "automatically" and analogously as it is done in the classical projective geometry.
We do not know if our notion of a pencil of subspaces and, consequently, the notion of a projection are the most general in the theory of spaces of pencils. It is however general enough to produce well known types of projections in the case when we start from a pappian finite-dimensional projective space $\mathfrak{P}$. It is also general enough to characterize collineations and correlations acting on segment subspaces and determined by linear maps.

In the context of non-pappian and dimension-free projective geometry some techniques and results commonly used to investigate projections become more or less useless. In particular, the analytical methods involving coordinates (and matrices) cannot be easily used here. Similarly, the method involving Plücker coordinates, which in the pappian geometry enables us to embed the space of pencils $\mathbf{P}_k(V)$ into the projective space $\mathbf{P}_1(\bigwedge^k V)$ (cf.+[5]) cannot be applied. What is more, the Fundamental Theorem of Projective Geometry, which states that a projectivity defined on a line (on a pencil) is uniquely determined by the images of three points of this line (resp.: of three elements of this pencil), fails in non-pappian geometry (cf. [3, Ch. I], [5, Ch. I.15], [6]). Therefore, we have chosen as a basic language the language of the lattice of subspaces with the dimension function defined on it. This adds some complexity, but within this more general framework we are able to prove analogues of most of the classical results. In particular, we can provide a classification of pencils, characterize linear ("analytically" linear) maps as projectivities (compositions of projections) of various types, and study perspectivities.

From our perspective projections in the paper are (partial, "locally linear") point transformations of some particular partial linear space – a space of pencils. On the other hand our projections can be viewed as transformations of some families of subspaces of a projective space $\mathfrak{P}$, it is just a matter of taste. From this view one can immediately find connections with various "generalized projections" considered in the projective geometry (here, in the case of pappian projective geometry, one can refer to many older and newer works including [3, 4, 5, 6, 7, 12]).

Perhaps some portion of the theory of our projections could be presented in a more synthetic way, based on an axiomatic characterization of spaces of pencils, but such an approach is not so common.

1. Notations and generalities

Let V be a vector space over a not necessarily commutative field F. We write Θ for the zero subspace of V and $\mathrm{Sub}_k(V)$ for the set of all k-subspaces of V. If Z, Y are subspaces of V and $Z \subseteq Y$, then $[Z, Y]$, that is, the set of subspaces U such that $Z \subseteq U \subseteq Y$, is a *segment* of the lattice $\mathfrak{L}(V)$ of subspaces of V (comp. [8]), and $[Z, Y]_k = [Z, Y] \cap \mathrm{Sub}_k(V)$. If $0 < k < \dim V$ we write $\mathbf{p}(H, B)$ for $[H, B]_k$ such that B is a $(k+1)$-subspace of V and H is a $(k-1)$-subspace of B. We call $\mathbf{p}(H, B)$ a k-pencil. The family of all k-pencils is $\mathcal{P}_k(V)$, and the space of pencils $\mathbf{P}_k(V)$ is the structure:

$$\mathbf{P}_k(V) = \big\langle \mathrm{Sub}_k(V), \mathcal{P}_k(V) \big\rangle.$$

The space of pencils as defined above, is sometimes called *Grassmann space*, or more precisely: a *Grassmann space representing* $(k-1)$*-dimensional subspaces of the projective space* $\mathbf{P}_1(V)$ (cf. [16]).

The fundamental notion used in the paper is a segment subspace of a space of pencils. Segment subspaces are exactly those subspaces which have the structure of a space of pencils, that is, they are isomorphic images of a space of pencils (cf. [17]). *Strong subspaces*, i.e. those where every two points are collinear, are segments $[Z, Y]_k$ with $\dim Z = k-1$, or $\dim Y = k+1$, respectively *stars* and *tops*. Every line p extends to the maximal star and maximal top uniquely (cf. [16]), which we denote by $\boldsymbol{S}(p)$, $\boldsymbol{T}(p)$ respectively.

Let for a moment $\mathfrak{A} = \langle S, \mathcal{L} \rangle$ be an arbitrary partial linear space. We write $a \sim b$ if points $a, b \in S$ are collinear, and $a \not\sim b$ if not. A subset $\mathcal{X}_1$ of S *adheres weakly* a subset $\mathcal{X}_2$, in symbols $\mathcal{X}_1 \lhd\!| \ \mathcal{X}_2$ or $\mathcal{X}_2 \ |\!\rhd \mathcal{X}_1$, iff for any point x_1 of $\mathcal{X}_1$ there are some points in $\mathcal{X}_2$ collinear with x_1. To exclude trivial cases, where $\mathcal{X}_1 \subseteq \mathcal{X}_2$, we say that $\mathcal{X}_1$ *adheres strongly* $\mathcal{X}_2$, and write $\mathcal{X}_1 \lhd\!|\!\rhd \mathcal{X}_2$, iff $\mathcal{X}_1 \setminus \mathcal{X}_2$ mutually adheres $\mathcal{X}_2 \setminus \mathcal{X}_1$.

2. Pencils of segment subspaces

In a projective space a pencil of subspaces is a family of all m-subspaces which share a $(m-1)$-subspace, the *vertex*, and lie in some $(m+1)$-subspace, the *base* of that pencil. In the geometry of spaces of pencils the definition gets complex as there are various classes of subspaces. In the paper we deal with pencils of segment subspaces. An analytical definition of such pencils is given in (2) and their geometrical characterization in 2.13.

We adopt the following conventions that $\infty - n = \infty$, $\infty + n = \infty$, and the like for a finite n. For Z, Y such that $Z \subseteq Y$ we also identify $\dim Y - \dim Z$ with $\dim Y/Z$.

The *index* of a segment subspace $\mathcal{X} = [Z, Y]_k$ of $\mathbf{P}_k(V)$ is $\mathrm{idx}(\mathcal{X}) = k - \dim Z$, and *co-index* is $\mathrm{coidx}(\mathcal{X}) = \dim Y - k$, in other words, index and co-index of a corresponding space of pencils $\mathbf{P}_{k-\dim Z}(Y/Z)$ (cf. [16]). Geometrically, the index (co-index) is the projective dimension of a maximal top (star) that lie in $\mathcal{X}$. The pair $\mathrm{pdim}(\mathcal{X}) = (\mathrm{idx}\,\mathcal{X}, \mathrm{coidx}\,\mathcal{X})$ is the *pencil (geometrical) dimension* of the segment subspace $\mathcal{X}$. We call two segments *similar* if they are of the same dimension. Note that segments of $\mathbf{P}_k(V)$ are similar if their vertices and bases are of equal linear dimensions, and conversely.

A linear subspace Z is said to be a *predecessor* of a linear subspace Y, in symbols $Z \prec Y$, iff $\mathrm{codim}_Y Z = \dim Y/Z = 1$. We also say that Y is a *successor* of Z. Sometimes we also write $Z \preccurlyeq Y$ when $Z \prec Y$ or $Z = Y$. Let us recall one basic lattice theoretical fact valid for all modular lattices including $\mathfrak{L}(V)$.

Fact 2.1. (Grätzer [8, Th. 4, Ch. IV.1]) *Let H, U, W, B be linear subspaces of V. If $H \preccurlyeq U$ and $H \subseteq W$, then $W \preccurlyeq U + W$. Dually, if $U \preccurlyeq B$ and $W \subseteq B$, then $U \cap W \preccurlyeq W$.*

Subspaces U, W are said to be *adjacent* if they have a common predecessor or a successor. For distinct and adjacent U, W the line through U, W is the set

$$\overline{U, W} = \big\{X \in \mathrm{Sub}(V) \colon U \cap W \prec X \prec U + W\big\}, \tag{1}$$

if $U = W$, then $\overline{U, W} = \{U\}$.

Segment subspaces $\mathcal{X}_i = [Z_i, Y_i]_k$, $i = 1, 2$, of $\mathbf{P}_k(V)$ are adjacent if their vertices Z_1, Z_2 are adjacent and bases Y_1, Y_2 are adjacent. Trivially, adjacent segments are similar. A *quasi-pencil* determined by adjacent $\mathcal{X}_1, \mathcal{X}_2$ is the set

$$\overline{\mathcal{X}_1, \mathcal{X}_2} = \big\{[Z, Y]_k \colon Z \in \overline{Z_1, Z_2}, Y \in \overline{Y_1, Y_2}\big\}. \tag{2}$$

We call a subspace of a space of pencils *non-trivial* if it contains a line. In the remainder of the paper we consider non-trivial segment subspaces $\mathcal{X}_i = [Z_i, Y_i]_k$ $i = 1, 2, 3$ in $\mathbf{P}_k(V)$. For convenience we use the following notation:

$$Z' = Z_1 \cap Z_2, \quad Z'' = Z_1 + Z_2, \quad Y' = Y_1 \cap Y_2, \quad Y'' = Y_1 + Y_2, \quad \text{and}$$
$$\mathcal{X}' = \mathcal{X}_1 \cap \mathcal{X}_2 = [Z'', Y']_k, \quad \mathcal{X}'' = \langle \mathcal{X}_1, \mathcal{X}_2 \rangle = [Z', Y'']_k.$$

Note that our notation permits to write a set $\{U\}$ or $\emptyset$ as a segment $[U, Y]_k$ (or $[Z, U]_k$) when $\dim U = k$ or $U \nsubseteq Y$ ($Z \nsubseteq U$), respectively.

Further investigations are focused on classification of quasi-pencils of segment subspaces. We begin with two technical facts.

Lemma 2.2. *Let U, W_1, W_2 be points such that $W_1, W_2 \in \mathcal{X} = [Z, Y]_k$, $W_1 \neq W_2$, and $U \sim W_1, W_2$. If $W_1 \nsim W_2$, then $U \in \mathcal{X}$.*

Proof. We have $U \cap W_1 \neq U \cap W_2$, since otherwise U, W_1, W_2 would lie on some strong subspace. Note that $U \cap W_i \subseteq Y$, and hence $U = (U \cap W_1) + (U \cap W_2) \subseteq Y$. Similarly, $U + W_1 \neq U + W_2$, and thus $Z \subseteq (U + W_1) \cap (U + W_2) = U$. □

Lemma 2.3. *Let U, W_1, W_2 be points such that $W_1, W_2 \in \mathcal{X} = [Z, Y]_k$, $W_1 \neq W_2$, and $U \sim W_1, W_2$. If $W_1 \sim W_2$, then $Z \subseteq U$ or $U \subseteq Y$.*

Proof. Points U, W_1, W_2 are coplanar. Hence, either $U = (U \cap W_1) + (U \cap W_2) \subseteq Y$ or $Z \subseteq (U + W_1) \cap (U + W_2) = U$, as the plane is a top or a star, respectively. □

There are three possible ways that two distinct and adjacent segment subspaces $\mathcal{X}_i$ may lie with respect to each other:

(W1) $\mathcal{X}_1 \cap \mathcal{X}_2 \neq \emptyset$, which is equivalent to $Z_1, Z_2 \subseteq Y_1, Y_2$.

(W2) $\mathcal{X}_1 \cap \mathcal{X}_2 = \emptyset$ and either $Z_1 \subseteq Y_2$, or $Z_2 \subseteq Y_1$.

(W3) None of the above inclusions hold in this case.

Accordingly, we obtain a classification of quasi-pencils determined by suitable pairs of segment subspaces:

Lemma 2.4. *Let $\mathcal{X}_1, \mathcal{X}_2$ be distinct adjacent segment subspaces, and let $\mathsf{G} = \overline{\mathcal{X}_1, \mathcal{X}_2}$.*

(i) *$\mathcal{X}_1, \mathcal{X}_2$ are of the type* (W1) *iff $Z'' \subseteq Y'$ iff $Z'' \cap Y' = Z''$ iff $Y' + Z'' = Y'$.*

(ii) *$\mathcal{X}_1, \mathcal{X}_2$ are of the type* (W2) *iff $Z' \prec Z'' \cap Y' \prec Z''$ iff $Y' \prec Y' + Z'' \prec Y''$.*

(iii) *$\mathcal{X}_1, \mathcal{X}_2$ are of the type* (W3) *iff $Z'' \cap Y' = Z'$ iff $Y' + Z'' = Y''$.*

Proof. Set $Z_0 = Z'' \cap Y'$ and $Y_0 = Y' + Z''$.

(i) is evident.

(ii) Assume $Z_1 \subseteq Y_2$ and $Z_2 \not\subseteq Y_1$. Then $Z_1 \subseteq Y'$, so $Z_1 \subseteq Z_0 \subseteq Z''$ and, by (i), $Z_0 = Z_1$. Analogously we prove that $Y_2 = Y_0$.

Assume $Z' \prec Z_0 \prec Z''$, so $Z_0 \in \overline{Z_1, Z_2}$, consequently, $[Z_0, Y_i]_k \in \mathcal{G}$. Note that $Z_0 = Z'' \cap Y' \subseteq Z'' \cap Y_i \subseteq Z''$ for $i = 1, 2$. If there were $Z'' \cap Y_i = Z''$ for $i = 1$ and $i = 2$ we would have $Z'' \subseteq Y'$; thus $Z_0 = Z'' \cap Y_i$ for some i, say: $i = 1$. Then $Z_1 \subseteq Y_1$ and $Z_1 \subseteq Z''$ yields $Z_1 \subseteq Z_0$, so $Z_1 = Z_0$ and we are through. Similarly, we prove that $[Z, Y_i]_k \in \mathcal{G}$ for some i and all $Z \in \overline{Z_1, Z_2}$.

(iii) Since $Z' \subseteq Z_0 \subseteq Z''$ and $Y' \subseteq Y_0 \subseteq Y''$, the claim follows by (i) and (ii). □

Generally, quasi-pencils of segment subspaces are not transitive, that is, there may be pairs of distinct elements of a quasi-pencil that span different quasi-pencils. Indeed, if $Z_1 \subseteq Y_2$ then for $\mathcal{X}_0 = [Z_1, Y_2]_k$ we have $\mathcal{X}_0 \in \overline{\mathcal{X}_1, \mathcal{X}_2}$. But if $Z_1 \neq Z_2$ then $\mathcal{X}_2 \notin \overline{\mathcal{X}_1, \mathcal{X}_0}$, if $Y_1 \neq Y_2$ then $\mathcal{X}_1 \notin \overline{\mathcal{X}_2, \mathcal{X}_0}$. We avoid such cases and distinguish the following subclasses of quasi-pencils:

proper pencil is a quasi-pencil $\overline{\mathcal{X}_1, \mathcal{X}_2}$ with $Z_1 = Z_2$ or $Y_1 = Y_2$, i.e. iff $Z' = Z''$ or $Y' = Y''$. This condition may be expressed in pure geometrical terms as: $\mathcal{X}'$ and $\mathcal{X}''$ have equal indexes or co-indexes;

wafer is a quasi-pencil $\overline{\mathcal{X}_1, \mathcal{X}_2}$ determined by a pair of type (W3);

pencil means proper pencil or wafer;

projective pencil is a pencil such that $\mathcal{X}''$ is (up to an isomorphism) a projective space (actually it is a star or a top).

Arrangement of elements of a pencil in the lattice as well as geometrical arrangement is visualized in the following diagrams.

$$\begin{tikzcd}
Z'' \arrow[r] & B \arrow[r] & Y'' \\
Z_3 \arrow[u] \arrow[r, dotted] & U_3 \arrow[u, dotted] \arrow[r, dotted] & Y_3 \arrow[u] \\
Z' \arrow[u] \arrow[r] & H \arrow[u, dotted] \arrow[r] & Y' \arrow[u]
\end{tikzcd}$$

Diagram 1

$$\begin{tikzcd}
Z_1 \arrow[r, equal] \arrow[d, dash, dashed] & U_1 \arrow[r, equal] \arrow[d, dash, dotted] & Y_1 \arrow[d, dash, dashed] \\
Z_2 \arrow[r, equal] \arrow[d, dash, dashed] & U_2 \arrow[r, equal] \arrow[d, dash, dotted] & Y_2 \arrow[d, dash, dashed] \\
\overline{Z_1, Z_2} & \overline{U_1, U_2} & \overline{Y_1, Y_2}
\end{tikzcd}$$

Diagram 2

In some latter propositions we claim that Diagrams 1 and 2 can be suitably completed.

Now, pure geometrical characterization of pencils of segment subspaces can be given.

Proposition 2.5. *Let $\mathcal{X}_1, \mathcal{X}_2$ be adjacent segment subspaces of $\mathbf{P}_k(V)$. Either, $\mathcal{X}' \neq \emptyset$ and $\overline{\mathcal{X}_1, \mathcal{X}_2}$ is not a proper pencil, or, for every point $U_1 \in \mathcal{X}_1 \setminus \mathcal{X}_2$ there is a point $U_2 \in \mathcal{X}_2 \setminus \mathcal{X}_1$ collinear with U_1 and, consequently, $\mathcal{X}_1 \lhd\!|\!\rhd \mathcal{X}_2$.*

Proof. Let $U_1 \in \mathcal{X}_1 \setminus \mathcal{X}_2$. Since $\mathcal{X}' = [Z'', Y']_k$, either $Z'' \nsubseteq U_1$, or $U_1 \nsubseteq Y'$. Assume that $Z'' \nsubseteq U_1$. Then $Z_1 \neq Z_2$ and by 2.1 $U_1 \prec U_1 + Z''$. Since $Y_2 \preccurlyeq Y''$ and $U_1 + Z'' \subseteq Y''$ we have $Y_2 \cap (U_1 + Z'') \preccurlyeq U_1 + Z''$ again by 2.1. Therefore $\mathcal{X} = [Z_2, Y_2 \cap (U_1 + Z'')]_k \neq \emptyset$. Every element of $\mathcal{X}$ belongs to $\mathcal{X}_2$ and is collinear with U_1. If $\mathcal{X}' = \emptyset$, we are through.

Otherwise $Z'' \subseteq Y'$. Suppose that $\mathcal{X} \subseteq \mathcal{X}'$. Since however $Z'' \nsubseteq Z_2$ and $\dim Z_2 \neq k$, it has to be $\dim Y_2 \cap (U_1 + Z'') = k$. In case the pencil $\overline{\mathcal{X}_1, \mathcal{X}_2}$ is proper we have $Y_1 = Y_2$, so $Y_2 \cap (U_1 + Z'') = Y_1 \cap (U_1 + Z'') = U_1 + Z''$ which contradicts that $U_1 \prec U_1 + Z''$. Hence either $\mathcal{X} \setminus \mathcal{X}' \neq \emptyset$, or $\overline{\mathcal{X}_1, \mathcal{X}_2}$ is not a proper pencil. $\square$

In case where $\mathcal{X}' \neq \emptyset$ and the quasi-pencil $\overline{\mathcal{X}_1, \mathcal{X}_2}$ is not proper all points of $\mathcal{X}_2$ collinear with points on $\mathcal{X}_1 \setminus \mathcal{X}_2$ lie on $\mathcal{X}'$. Still $\mathcal{X}_1$ mutually adheres $\mathcal{X}_2$ though.

Proposition 2.6. *Let $\mathcal{X}_1, \mathcal{X}_2$ be similar segments of $\mathbf{P}_k(V)$. If $\mathcal{X}_1 \lhd\!|\!\rhd \mathcal{X}_2$, then $\mathcal{X}_1, \mathcal{X}_2$ lie in a strong subspace of $\mathbf{P}_k(V)$ or they are adjacent.*

Proof. Assume that there is a linear subspace D such that $2 \leq \dim D$, $D \subseteq Y_1$ and $D \cap Y_2 = \Theta$. Consider two cases. First, suppose that there is no $U_1 \in \mathcal{X}_1$ with $D \subseteq U_1$. In such a case we would have $\dim Z_i = k - 1$, and because for $U_1 \in \mathcal{X}_1$ with $U_1 \subseteq Z_1 + D$ there is $U_2 \in \mathcal{X}_2$ collinear with U_1, we would find that $Z_1 = Z_2$. Then $\mathcal{X}_1, \mathcal{X}_2 \subseteq [Z_1, V]_k$, which is a star in $\mathbf{P}_k(V)$.

Now, take $U_1 \in \mathcal{X}_1$ with $D \subseteq U_1$. In consequence, $\dim U_1 \cap Y_2 \leq k - 2$. On the other hand, there is $U_2 \in \mathcal{X}_2$ collinear with U_1. Since $U_1 \cap U_2 \subseteq U_1 \cap Y_2$, we have $k - 1 \leq \dim U_1 \cap Y_2$ and contradiction arises. For vertices Z_1, Z_2 the reasoning is dual. $\square$

In segments that belong to a pencil a vertex determines uniquely the base and conversely.

Lemma 2.7. *Let $\mathcal{X}_1, \mathcal{X}_2$ be segments of $\mathbf{P}_k(V)$ such that $\mathsf{G} = \overline{\mathcal{X}_1, \mathcal{X}_2}$ is a pencil.*

(i) *If $Z_1 \neq Z_2$, then for every $Z_3 \in \overline{Z_1, Z_2}$ there is a unique $Y_3 \in \overline{Y_1, Y_2}$ such that $[Z_3, Y_3]_k \in \mathsf{G}$.*

(ii) *If $Y_1 \neq Y_2$, then for every $Y_3 \in \overline{Y_1, Y_2}$ there is a unique $Z_3 \in \overline{Z_1, Z_2}$ such that $[Z_3, Y_3]_k \in \mathsf{G}$.*

In consequence G is transitive and every two elements of G are of the same type.

Proof. (i) For G a proper pencil the claim is trivial, for a wafer take $Y_3 = Y' + Z_3$. Suppose it is not unique. Then $Z_3 \subseteq Y'$, and hence $Z_3 \subseteq Z'' \cap Y' \subseteq Z''$. Since $Z_3 \prec Z''$ contradiction with 2.4(iii) arises.

(ii) The reasoning is the same but we take $Z_3 = Y_3 \cap Z''$. □

Lemma 2.8. *Let segment subspaces $\mathcal{X}_1, \mathcal{X}_2$ determine a quasi-pencil $\mathsf{G} = \overline{\mathcal{X}_1, \mathcal{X}_2}$, and let $p = \mathbf{p}(H, B)$ be a line in $\mathbf{P}_k(V)$.*

(i) *If p crosses $\mathcal{X}_1$, $\mathcal{X}_2$ in distinct points, then $Z' \subseteq H \subseteq Y'$ and $Z'' \subseteq B \subseteq Y''$.*

(ii) *If $Z' \subseteq H \subseteq Y'$ and $Z'' \subseteq B \subseteq Y''$, then p crosses every $\mathcal{X} \in \mathsf{G}$.*

Proof. (i) Straightforward.

(ii) Let $\mathcal{X} = [Z, Y]_k \in \mathsf{G}$. Then $Z + H \subseteq B \cap Y$. Since $Z' \preccurlyeq Z$ and $Y \preccurlyeq Y''$, we have $\dim(Z + H) \leq k \leq \dim(B \cap Y)$ by 2.1. Thus $p \cap \mathcal{X} = [Z + H, B \cap Y]_k \neq \emptyset$. □

Lemma 2.9. *Let G be a wafer in $\mathbf{P}_k(V)$ and $\mathcal{X}_1, \mathcal{X}_2 \in \mathsf{G}$ distinct segments. Then*

(i) *for every point $U_1 \in \mathcal{X}_1$ there is a unique point $U_2 = (U_1 + Z_2) \cap Y_2$ in $\mathcal{X}_2$, collinear with U_1,*

(ii) *$Z' \prec U \cap Z'' \prec Z''$ and $Y' \prec U + Y' \prec Y''$ for all $U \in \bigcup \mathsf{G}$.*

Proof. (i) By 2.5 for every point $U_1 \in \mathcal{X}_1$ there is a point $U_2 \in \mathcal{X}_2$ collinear with U_1. It is unique for if not, we would have either $Z_2 \subseteq U_1 \subseteq Y_1$, or $Z_1 \subseteq U_1 \subseteq Y_2$, in view of 2.2 and 2.3, which contradicts 2.7.

(ii) Wafers are transitive by 2.7, hence we can assume that $U \in \mathcal{X}_1$ without loss of generality. Evidently $Z' \subseteq U \cap Z'' \subseteq Z''$, and it suffices to show that $U \cap Z'' \prec Z''$. By 2.5 there is a line $p = \mathbf{p}(H, B)$ through U that crosses $\mathcal{X}_2$. Consequently by 2.8(i) we have $Z'' \subseteq B$. This together with $U \prec B$ gives $U \cap Z'' \preccurlyeq Z''$ by 2.1. Note that if $Z'' \subseteq U$ then $Z_2 \subseteq U \cap Y_2 \subseteq Y_1$ which contradicts 2.7.

One proves $Y' \prec U + Y' \prec Y''$ dually. □

This is a crucial feature of wafers, which resemble to some extent nets or reguli (cf. [3, Ch. 10], in case of classical projective geometry comp. also e.g. [10, 11]). We will give characterization (ii) from 2.9 in terms of projections later.

Lemma 2.10. *Let G be a quasi-pencil in $\mathbf{P}_k(V)$ and $\mathcal{X}_1, \mathcal{X}_2 \in \mathsf{G}$ distinct segments.*

(i) *If $\mathcal{X}_3 \in \mathsf{G}$ and a line p crosses $\mathcal{X}_1, \mathcal{X}_2$, then p crosses $\mathcal{X}_3$.*

(ii) *If $\mathcal{X}_3 \in \mathsf{G}$, then for every $U_3 \in \mathcal{X}_3$ there is a line p through U_3 crossing $\mathcal{X}_1, \mathcal{X}_2$.*

(iii) *If a line p crosses $\mathcal{X}_1, \mathcal{X}_2$ in distinct points and $U_3 \in p$, then there is $\mathcal{X}_3 \in \mathsf{G}$ such that $U_3 \in \mathcal{X}_3$.*

Proof. (i) Let $U_i \in p \cap \mathcal{X}_i$ for $i = 1, 2$. If $U_1 \neq U_2$, we are through by 2.8. Otherwise, $U_1 = U_2 \in \mathcal{X}_3$.

(ii) Let $U_3 \in \mathcal{X}_3 \in \mathsf{G}$. Since $Z_3 \preccurlyeq Z''$ and $Y' \preccurlyeq Y_3$, we have $U_3 \preccurlyeq U_3 + Z''$ and $U_3 \cap Y' \preccurlyeq U_3$ by 2.1. For this reason we can take $H \in [Z', U_3 \cap Y']_{k-1}$ and $B \in [U_3 + Z'', Y'']_{k+1}$. The line $p = \mathbf{p}(H, B)$ crosses $\mathcal{X}_1$ and $\mathcal{X}_2$ by 2.8(ii).

(iii) By 2.8 $p = \mathbf{p}(H, B)$, where $Z' \subseteq H \subseteq Y'$ and $Z'' \subseteq B \subseteq Y''$. If $Z'' \nsubseteq U_3$, then in view of 2.1 $Z_3 := U_3 \cap Z'' \prec Z''$ since $U_3 \prec B$. Otherwise we take $Z_3 := Z_1$. Analogously, either $Y_3 := U_3 + Y'$ or $Y_3 := Y_1$. In any of these cases $\mathcal{X}_3 := [Z_3, Y_3]_k$ satisfies required conditions. □

The above lemma is an announcement of geometrical characterization of pencils of segment subspaces in $\mathbf{P}_k(V)$. Two conditions are critical to that description:

($*_1$) if a line p crosses $\mathcal{X}_1$ and $\mathcal{X}_2$, then p crosses $\mathcal{X}_3$,

($*_2$) for every $U_3 \in \mathcal{X}_3$ there is a line p through U_3 crossing $\mathcal{X}_1, \mathcal{X}_2$ in distinct points.

Lemma 2.10 says that quasi-pencils satisfy ($*_1$), and ($*_2$) in a weaker form.

Lemma 2.11. *Let $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3$ be segment subspaces of $\mathbf{P}_k(V)$ such that $\mathsf{G} = \overline{\mathcal{X}_1, \mathcal{X}_2}$ is a pencil. Then:*

(i) *If $\mathcal{X}_3 \in \mathsf{G}$, then ($*_1$) and ($*_2$) hold.*

(ii) *If G is not projective, the condition ($*_1$) implies $Z_3 \subseteq Z''$ and $Y' \subseteq Y_3$.*

(iii) *The condition ($*_2$) implies $Z' \subseteq Z_3$ and $Y_3 \subseteq Y''$.*

Proof. (i) By 2.10, ($*_1$) holds. It is clear that ($*_2$) holds for wafers by definition and 2.8. Therefore we can assume that $Z_1 = Z_2$ or $Y_1 = Y_2$.

Let $U_3 \in \mathcal{X}_3$. Evidently $\mathcal{X}_1 \cap \mathcal{X}_2 \subseteq \mathcal{X}_3$. Hence, if $U_3 \in \mathcal{X}_1 \cap \mathcal{X}_2$, then we take any line $p \subseteq \mathcal{X}_1$ through U_3, and $U_1 \in p$ such that $U_1 \neq U_3$. As $U_3 \in p \cap \mathcal{X}_2$ we are through.

In case $U_3 \notin \mathcal{X}_1 \cap \mathcal{X}_2$, by 2.5 we have $U_1 \in \mathcal{X}_1 \setminus \mathcal{X}_3$ collinear with U_3. Since $\mathcal{X}_2 \in \mathsf{G} = \overline{\mathcal{X}_1, \mathcal{X}_3}$ by transitivity, the line $\overline{U_1, U_3}$ crosses $\mathcal{X}_2$ in some point U_2 by ($*_1$).

(ii) Let $B \in [Z'', Y'']_{k+1}$. Observe that $B \neq Y''$ as G is not projective. Estimation of dimensions gives that $k - 1 \leq \dim(B \cap Y')$. Evidently $Z' \subseteq B$, hence there is $H \in [Z', B \cap Y']_{k-1}$. According to 2.8(ii) the line $p = \mathbf{p}(H, B)$ crosses $\mathcal{X}_1, \mathcal{X}_2$. By ($*_1$) there is a point $U_3 \in p \cap \mathcal{X}_3$. We have shown that $Z_3 \subseteq \bigcap\{B\colon B \in [Z'', Y'']_{k+1}\} = Z''$. The proof of $Y' \subseteq Y_3$ runs dually.

(iii) Let $U_3 \in \mathcal{X}_3$. By ($*_2$) there is a line $p = \mathbf{p}(H, B)$ crossing $\mathcal{X}_1, \mathcal{X}_2$ in distinct points. Hence by 2.8 $Z' \subseteq H \subseteq U_3$. Since U_3 is arbitrary we have $Z' \subseteq \bigcap\{U_3\colon U_3 \in [Z_3, Y_3]_k\} = Z_3$. The proof of $Y_3 \subseteq Y''$ runs dually. □

Lemma 2.12. *If $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3$ are segment subspaces of $\mathbf{P}_k(V)$ satisfying ($*_1$), ($*_2$) such that $\mathsf{G} = \overline{\mathcal{X}_1, \mathcal{X}_2}$ is a pencil, then either $\mathcal{X}_3 \in \mathsf{G}$, or G is a projective pencil.*

Proof. Assume that G is a pencil but not projective. In view of 2.11 $Z' \subseteq Z_3 \subseteq Z''$ and $Y' \subseteq Y_3 \subseteq Y''$.

We start with showing that $\mathcal{X}_3 \neq \mathcal{X}'$. In view of 2.8 every line $p = \mathbf{p}(H, B)$ such that $Z' \subseteq H \subseteq Y'$ and $Z'' \subseteq B \subseteq Y''$ crosses both $\mathcal{X}_1$ and $\mathcal{X}_2$. In particular we can take p so that $Z'' \nsubseteq H$, that is, $p \cap \mathcal{X}_i \nsubseteq \mathcal{X}_{3-i}$, $i = 1, 2$. On the other hand, there is $U_3 \in p \cap \mathcal{X}_3$ by $(*_1)$. If $\mathcal{X}_3 = \mathcal{X}_1 \cap \mathcal{X}_2$, we would have $p \subseteq \mathcal{X}_1 \cap \mathcal{X}_2$ as $U_3 \neq U_i$ and $U_i, U_3 \in p \cap \mathcal{X}_i$ for $i = 1, 2$.

Now, assume that $\mathcal{X}_3 = \mathcal{X}''$ and $Y' \neq Y''$. Then consider a point $U_3 \in \mathcal{X}_3$ such that $U_3 + Y' = Y''$. Such a point exists since $\dim Z' < k - 1$. By $(*_2)$ and 2.8 we have a line $p = \mathbf{p}(H, B)$ with $Z' \subseteq H \subseteq Y'$ and $Z'' \subseteq B \subseteq Y''$. Since $H \prec U_3$ and $H \subseteq Y'$, we have $Y' \preccurlyeq U_3 + Y' = Y''$ by 2.1, which contradicts with previous assumption. In case $Z' \neq Z''$ we will lead to contradiction dually.

If G is a proper pencil, then $\mathcal{X}_3 = \mathcal{X}'$, $\mathcal{X}_3 = \mathcal{X}''$ or $\mathcal{X}_3 \in \mathsf{G}$ and we are through by the above reasoning.

Now, consider the case where G is a wafer. Note that $\dim Z' < k - 1$ and $\dim Y'' > k + 1$ give $[Z', Y']_k, [Z'', Y'']_k \neq \emptyset$. Let $U_3 \in \mathcal{X}_3 = [Z', Y']_k$. We will show that there is no line through U_3 which crosses $\mathcal{X}_1$ and $\mathcal{X}_2$ in distinct points. Indeed, if such a line exists, by 2.8 $U_3 \subseteq U_3 + Z'' \subseteq B$ for some B such that $Z'' \subseteq B \subseteq Y''$. Hence either $U_3 + Z'' = U_3$, or $U_3 + Z'' = B$. In the first case $Z'' \subseteq U_3 \subseteq Y'$, which means that $\mathcal{X}'$ is non-empty and thus G is not a wafer. In the second case, since $U_3 \prec B$ and $U_3 \subseteq Y'$, we have $Y' \preccurlyeq Y' + B = Y' + U_3 + Z'' = Y''$ by 2.1. This contradicts that G is a wafer. Dually, we prove the same for $\mathcal{X}_3 = [Z'', Y'']_k$.
Accordingly, if $\mathcal{X}_3 \cap [Z', Y']_k \neq \emptyset$ or $\mathcal{X}_3 \cap [Z'', Y'']_k \neq \emptyset$, then $\mathcal{X}_3$ does not satisfy $(*_2)$. In particular, this yields that $Z_3 \neq Z', Z''$ and $Y_3 \neq Y', Y''$. □

Theorem 2.13. *Let $\mathcal{X}_1, \mathcal{X}_2$ be distinct, similar segment subspaces in $\mathbf{P}_k(V)$.*

(i) *$\mathcal{X}_1, \mathcal{X}_2$ determine a proper pencil iff $\mathcal{X}' \neq \emptyset$ and* $\mathrm{idx}(\mathcal{X}') = \mathrm{idx}(\mathcal{X}'')$ *or* $\mathrm{coidx}(\mathcal{X}') = \mathrm{coidx}(\mathcal{X}'')$. *In this case $\mathcal{X}_3 \in \overline{\mathcal{X}_1, \mathcal{X}_2}$ iff $\mathcal{X}_3$ is similar to $\mathcal{X}_1$ and $\mathcal{X}' \subseteq \mathcal{X}_3 \subseteq \mathcal{X}''$.*

(ii) *$\mathcal{X}_1, \mathcal{X}_2$ determine a wafer iff for every $U_1 \in \mathcal{X}_1$ there is a unique $U_2 \in \mathcal{X}_2$ with $U_1 \sim U_2$, and conversely.*

(iii) *If $\overline{\mathcal{X}_1, \mathcal{X}_2}$ is a non-projective pencil, then $\mathcal{X}_3 \in \overline{\mathcal{X}_1, \mathcal{X}_2}$ iff $\mathcal{X}_3$ satisfies $(*_1)$ and $(*_2)$.*

Proof. (i) Straightforward by the definition of a proper pencil.

(ii) $\Rightarrow$: by 2.9(i).

$\Leftarrow$: Contrary to the definition of a wafer assume that $Z_1 \subseteq Y_2$. Then $Z_1 \subseteq Y'$. Consider points U_1, U_2 such that $Z_1 \subseteq U_1 \subseteq Y'$, $U_2 \in \mathcal{X}_2$, and $U_1, U_2 \in p = \mathbf{p}(H, B)$. Observe that $B \subseteq Y_2$ and all points in $[Z_2, B]_k \subseteq \mathcal{X}_2$ are collinear with U_1. In case $Z_2 \subseteq Y_1$ we proceed the same way.

(iii) Immediate by 2.11 and 2.12. □

3. Shift projections

In [13] general projections in projective spaces have been introduced. Let X_1, X_2, C be subspaces of V such that $\dim X_1 = \dim X_2$, $X_1 \subseteq C + X_2$. Then, according to the definition, a projection with center C, from subspace X_1 onto subspace X_2, in symbols $\overset{\top X_1}{\underset{\downarrow X_2}{\circ\, C}}$, is a map $U \rightsquigarrow (U + C) \cap X_2$ for $U \subseteq X_1$. This projection may be considered a composition of two maps $U \rightsquigarrow U + C$ and $U \rightsquigarrow U \cap X_2$. They are commonly known in lattice theory as *shifts*, though they have also their pure geometric nature as meet and join of points, lines, planes and so on. In this context, shift projections defined by (3) and (4) can be thought of as analogues of a perspective in a projective geometry. They are technical but very useful tools in this paper. Another approach to perspectivity determined by the intrinsic geometry of a space of pencils is discussed in Section 6.

Let us begin with a known fact for modular lattices, slightly reformulated.

Fact 3.1. (Grätzer [8, Th. 2, Ch. IV.1]) *If* $Z \cap F = Y \cap F$ *and* $Z + G = Y + G$, *then maps* $f, g\colon [Z, Y] \longrightarrow \mathrm{Sub}(V)$ *such that* $f = U \rightsquigarrow U + F$, *and* $g = U \rightsquigarrow U \cap G$ *are bijections.*

The maximal segment on which f from 3.1 is a bijection is $[\Theta, Y]$, where Y is a linear complement of F in V. Dually, the maximal segment for g is $[Z, V]$, where Z is a linear complement of G in V. Note that f is always a bijection on $[Y \cap F, Y]$ and g on $[Z, Z + G]$.

We define two projections, J-*projection*

$$\mathrm{J}_{Y,Z}\colon [Z \cap Y, Y] \longrightarrow [Z, Z + Y], \quad U \rightsquigarrow U + Z, \tag{3}$$

and M-*projection*

$$\mathrm{M}_{Y,Z}\colon [Z, Z + Y] \longrightarrow [Z \cap Y, Y], \quad U \rightsquigarrow U \cap Y. \tag{4}$$

By 3.1 they are bijections. We use notation $\mathrm{J}_{Y,Z} = \big\downarrow^{[Z\cap Y,Y]}_{[Z,Z+Y]}$ and $\mathrm{M}_{Y,Z} = \big\downarrow^{[Z,Z+Y]}_{[Z\cap Y,Y]}$. Observe that if $\mathcal{X}_i = [Z_i, Y_i]$, $i = 1, 2$ and there is a J-projection (M-projection) $f = \big\downarrow^{\mathcal{X}_1}_{\mathcal{X}_2}\colon \mathcal{X}_1 \longrightarrow \mathcal{X}_2$ then, $f = \mathrm{J}_{Y_1,Z_2}$ ($f = \mathrm{M}_{Y_2,Z_1}$). Note also that f is a lattice isomorphism between $\mathcal{X}_1, \mathcal{X}_2$, thus, preserves the height (rank) of elements in the lattice $\mathcal{X}_1$. Evidently, $\left(\big\downarrow^{\mathcal{X}_1}_{\mathcal{X}_2}\right)^{-1} = \big\downarrow^{\mathcal{X}_2}_{\mathcal{X}_1}$. We use notion J or M-projection with respect to a map $f = \big\downarrow^{[Z_1,Y_1]}_{[Z_2,Y_2]}$ between two sublattices of $\mathfrak{L}(V)$, as well as to the map $f' = \big\downarrow^{[Z_1,Y_1]_k}_{[Z_2,Y_2]_m} = f \big| \mathrm{Sub}_k(V)$ between corresponding spaces of pencils (cf. 3.2).

Lemma 3.2. *Let* Y_1, Z_2 *be subspaces of* V *and* $Z_1 = Z_2 \cap Y_1$, $Y_2 = Z_2 + Y_1$. J-*projection* J_{Y_1,Z_2} *maps* $[Z_1, Y_1]_k$ *onto* $[Z_2, Y_2]_m$ *and* M-*projection* M_{Y_1,Z_2} *maps* $[Z_2, Y_2]_m$ *onto* $[Z_1, Y_1]_k$, *where* $k - \dim Z_1 = m - \dim Z_2$. *Moreover,* $\dim Y_1 - k = \dim Y_2 - m$ *and thus* $\mathrm{pdim}([Z_1, Y_1]_k) = \mathrm{pdim}([Z_2, Y_2]_m)$.

Proof. Since $U \cap Z_2 = Z_1$ for every $U \in \mathcal{X}_1$, we have $m = \dim(U + Z_2) = k + \dim Z_2 - \dim Z_1$. Analogously we find that $U + Y_1 = Y_2$ for every $U \in \mathcal{X}_2$, and $k = \dim(U \cap Y_1) = m + \dim Y_1 - \dim Y_2$. □

Note that if f is a J or M-projection and $\mathrm{dm}(f)$ is a line, then $\mathrm{rg}(f)$ is also a line. Such J-projections map lines of $\mathbf{P}_k(V)$ onto lines of $\mathbf{P}_m(V)$, and M-projections act conversely.

Lemma 3.3. *Let $\mathcal{X}_1$, $\mathcal{X}_2$ be segment subspaces of $\mathbf{P}_k(V)$, $\mathbf{P}_m(V)$ respectively, and let $f\colon \mathcal{X}_1 \longrightarrow \mathcal{X}_2$ be a* J*-projection or* M*-projection. If $\mathcal{E}_1$ is a segment subspace such that $\mathcal{E}_1 \subseteq \mathcal{X}_1$, then $f\big|\mathcal{E}_1$ is a* J*-projection or an* M*-projection, respectively, of $\mathcal{E}_1$ onto some segment subspace $\mathcal{E}_2 \subseteq \mathcal{X}_2$.*

Proof. Let $\mathcal{X}_i = [Z_i, Y_i]_{k_i}$, $i = 1, 2$. Assume that $f = \mathrm{J}_{Y_1,Z_2}$ and $\mathcal{E}_1 = [Z_1', Y_1']_k$. We have $Z_1 = Z_2 \cap Y_1$, $Y_2 = Z_2 + Y_1$ and $f(U) = U + Z_2$ for all $Y \in \mathcal{X}_1$. Accordingly, f maps $[Z_1', Y_1']_k$ onto $[Z_1'+Z_2, Y_1'+Z_2]_m$ and $f(U) = U+(Z_1'+Z_2)$ for $U \in \mathcal{E}_1$. Clearly, $Y_1'+(Z_1'+Z_2) = Y_1'+Z_2$, and $Y_1' \cap (Z_1'+Z_2) = Z_1'+(Y_1' \cap Z_2) = Z_1'$ since $Y_1' \cap Z_2 \subseteq Z_1$. In result, $f\big|\mathcal{E}_1$ is a J-projection. For M-projections the reasoning runs dually. □

The straightforward consequence is that every J- and M-projection is a collineation (cf. 3.6). Next two lemmas give criteria for how to compose J- and M-projections.

Lemma 3.4. *If f, g are* J*-projections, or* M*-projections, such that $\mathrm{dm}(g) = \mathrm{rg}(f)$, then gf is a* J*-projection or an* M*-projection respectively.*

Proof. Assume that $f = \big\downarrow^{[Z_1,Y_1]}_{[Z_2,Y_2]}$ and $g = \big\downarrow^{[Z_2,Y_2]}_{[Z_3,Y_3]}$. Then, $Z_1 = Z_2 \cap Y_1$, $Y_2 = Z_2 + Y_1$, $Z_2 = Z_3 \cap Y_2$ and $Y_3 = Z_3 + Y_2$. It is seen that $Z_1 = Z_2 \cap Y_3$ and $Y_3 = Z_3 + Y_1$, which means that $gf = \mathrm{J}_{Y_1,Z_3}$.

Note that when f, g are M-projections, then f^{-1}, g^{-1} are J-projections and $\mathrm{dm}(f^{-1}) = \mathrm{rg}(g^{-1})$. Hence, $(gf)^{-1}$ is a J and fg an M-projection. □

Lemma 3.5. *Let $\mathcal{X}$ be a segment in $\mathbf{P}_k(V)$, and let l, m be such that $l \le k \le m$ and there are subspaces similar to $\mathcal{X}$ in $\mathbf{P}_l(V)$ and $\mathbf{P}_m(V)$. Then, there is an* M*-projection g and a* J*-projection f on $\mathcal{X}$ such that $\mathrm{rg}(g), \mathrm{rg}(f)$ are segments in $\mathbf{P}_l(V)$, $\mathbf{P}_m(V)$ respectively.*

Proof. We will find g only, as procedure for f is dual. Let $\mathcal{X} = [Z, Y]_k$. Observe that $(*)$: $k - \dim Z \le l \le k$. First, take $Q \subseteq Y$ such that $Q + Z = Y$ and $\dim Q - l = \dim Y - k$ (cf. 3.2). Such Q exists in view of $(*)$. Then g is an M-projection $U \rightsquigarrow U \cap Q$ on $\mathcal{X}$, in other words $g = \mathrm{M}_{Y \cap Q, Z}$. Clearly, $Z + (Y \cap Q) = Y$ and $Z \cap (Y \cap Q) = Z \cap Q$ which means that $g(\mathcal{X}) = [Z \cap Q, Y \cap Q]_l$. □

For convenience we assign to every linear map $\varphi\colon Y_1/Z_1 \longrightarrow Y_2/Z_2$ a map $\tilde{\varphi}\colon [Z_1, Y_1] \longrightarrow [Z_2, Y_2]$ such that $\tilde{\varphi}(U) = W$ iff $\varphi(U/Z_1) = W/Z_2$ for $U \in [Z_1, Y_1]$, $W \in [Z_2, Y_2]$. In the context of spaces of pencils $\tilde{\varphi}_k = \tilde{\varphi}\big|\mathrm{Sub}_k(V)$.

Proposition 3.6. *Let Y_1, Z_2 be subspaces of V with $Z_1 = Z_2 \cap Y_1$, $Y_2 = Z_2 + Y_1$.*

(i) *Shift projections $f = \big\downarrow^{[Z_1,Y_1]_k}_{[Z_2,Y_2]_m}$, $g = \big\downarrow^{[Z_2,Y_2]_m}_{[Z_1,Y_1]_k}$ determine collineation between $\mathbf{P}_r(Y_1/Z_1)$ and $\mathbf{P}_r(Y_2/Z_2)$, where $r = k - \dim Z_1 = m - \dim Z_2$.*

(ii) *There is a linear bijection $\varphi\colon Y_1/Z_1 \longrightarrow Y_2/Z_2$ with $\tilde{\varphi}_k = f$ and $\tilde{\varphi}_k^{-1} = g$.*

Proof. (i) Immediate consequence of 3.3.

(ii) Assume that $Y_1 = Z_1 \oplus M$ for some subspace M. Then, it is seen that $Z_2 \cap M = \Theta$ and $Y_2 = Z_2 + Y_1 = Z_2 \oplus M$, since $Z_1 = Y_1 \cap Z_2$. By 3.2 we have $k - \dim Z_1 = m - \dim Z_2 =: r$.

In case of $f = \mathrm{J}_{Y_1,Z_2}$, consider $U \in [Z_1, Y_1]_k$. Then $U = Z_1 \oplus N$, where $N \in \mathrm{Sub}_r(M)$, and $f(U) = Z_2 + N$ as $Z_1 \subseteq Z_2$. For $g = \mathrm{M}_{Y_1,Z_2}$ and $W \in [Z_2, Y_2]_m$ we find that $W = Z_2 + N$ and $g(W) = Z_1 + N$, where $N \in \mathrm{Sub}_r(M)$. Hence, the map $\varphi = v + Z_1 \rightsquigarrow v + Z_2$, where $v \in M$, is a requested linear bijection with $f(U) = W$ iff $\varphi(U/Z_1) = W/Z_2$ iff $g(W) = U$. □

4. Projections between segment subspaces

Usually, in projective geometry, projections are considered maps between lines. In this paper we deal with more general projections, namely, projections between segment subspaces. In this section we give geometrical definitions of our two projections: a *generalized central projection* (5) and a *slide* (6). Also their analytical interpretations are given (4.2, 4.6).

According to general theory, a subspace of codimension 1 is a maximal proper subspace. In case $\mathcal{X}, \mathcal{Y}$ are segment subspaces of $\mathbf{P}_k(V)$, and $\mathcal{X}$ is not strong, $\mathcal{Y}$ is of codimension 1 in $\mathcal{X}$ iff $\mathcal{Y} \subseteq \mathcal{X}$ and (i) $\operatorname{pdim} \mathcal{Y} = \operatorname{pdim} \mathcal{X} - (0,1)$, or (ii) $\operatorname{pdim} \mathcal{Y} = \operatorname{pdim} \mathcal{X} - (1,0)$. If $\mathcal{X}$ is a star $\mathcal{Y}$ needs to satisfy (i), if $\mathcal{X}$ is a top (ii). Note that if G is a proper pencil and $\mathcal{X} \in \mathsf{G}$, then $\mathcal{X}'$ is of codimension 1 in $\mathcal{X}$ and $\mathcal{X}$ is of codimension 1 in $\mathcal{X}''$.

In a partial linear space, we say that subspaces $\mathcal{Y}_1, \mathcal{Y}_2$ are *complementary* in a subspace $\mathcal{X}$ if $\mathcal{Y}_1 \cap \mathcal{Y}_2 = \emptyset$ and $\langle \mathcal{Y}_1, \mathcal{Y}_2 \rangle = \mathcal{X}$.

Lemma 4.1. *Let $\mathcal{Y}_1, \mathcal{Y}_2$ be complementary segment subspaces of codimension 1 in a segment $\mathcal{X}$ in $\mathbf{P}_k(V)$.*

(i) *For every point $U_1 \in \mathcal{Y}_1$ there is a point $U_2 \in \mathcal{Y}_2$ collinear with U_1.*

(ii) *Every line through $U_1 \in \mathcal{Y}_1$ misses $\mathcal{Y}_2$ or meets $\mathcal{Y}_2$ in a single point.*

Proof. (i) Let $\mathcal{X} = [Z, Y]_k$, $\mathcal{Y}_i = [Z_i, Y_i]_k$ and $U_1 \in \mathcal{Y}_1$. Since $\mathcal{Y}_1, \mathcal{Y}_2$ are complementary of codimension 1 in $\mathcal{X}$ we can assume without loss of generality that $Z_1 = Z \prec Z_2$, $Y_1 \prec Y = Y_2$ and $Z_2 \nsubseteq Y_1$. Clearly $Z_2 \nsubseteq U_1$ and hence $U_1 \prec Z_2 + U_1$ by 2.1. We find that $\mathcal{Y} = [Z_2, Z_2 + U_1]_k$ is non empty, and $\mathcal{Y} \subseteq \mathcal{Y}_2$ as $U_1 \subseteq Y = Y_2$. Every point of $\mathcal{Y}$ is adjacent, i.e. collinear, with U_1.

(ii) Assume contrary to our claim that U_1, U_2, U_3 lie on some line q, $U_1 \in \mathcal{Y}_1$ and $U_2, U_3 \in \mathcal{Y}_2$ are distinct. Hence $q \subseteq \mathcal{Y}_2$ and consequently $U_1 \in \mathcal{Y}_1 \cap \mathcal{Y}_2 = \emptyset$. □

Lemma 4.2. *Let $\mathsf{G} = \overline{\mathcal{X}_1, \mathcal{X}_2}$ be a proper pencil, $\mathcal{X}_3 \in \mathsf{G}$, $\mathcal{X}_3 \neq \mathcal{X}_1, \mathcal{X}_2$, and $\mathcal{Y}$ a subspace of codimension 1 in $\mathcal{X}_3$ such that $\mathcal{X}', \mathcal{Y}$ are complementary in $\mathcal{X}_3$.*

(i) *If $Z_1 = Z_2 = Z$ then $\mathcal{Y} = [W, Y_3]_k$, where $Z \prec W \subseteq Y_3$, and $Z = Y' \cap W$, $Y_3 = Y' + W$.*

(ii) *If $Y_1 = Y_2 = Y$ then $\mathcal{Y} = [Z_3, W]_k$, where $Z_3 \subseteq W \prec Y$, and $Y = Z'' + W$, $Z_3 = Z'' \cap W$.*

(iii) *For $U_1 \in \mathcal{X}_1 \setminus \mathcal{X}'$ there is a unique $U_3 \in \mathcal{Y} \setminus \mathcal{X}'$ collinear with U_1.*

(iv) *For $U_1 \in \mathcal{X}_1$ there is a unique $U_2 \in \mathcal{X}_2$ collinear with U_1, such that a line through U_1, U_2 crosses $\mathcal{Y}$. If $U_1 \in \mathcal{X}_1 \setminus \mathcal{X}'$, then $U_2 \in \mathcal{X}_2 \setminus \mathcal{X}'$, if $U_1 \in \mathcal{X}'$, then $U_2 = U_1$. Moreover, if $Z_1 = Z_2$, then $U_2 = (U_1 + W) \cap Y_2$, if $Y_1 = Y_2$, then $U_2 = (U_1 \cap W) + Z_2$.*

Proof. (i) We have $\mathcal{X}' = [Z, Y']_k$, $\mathcal{X}_3 = [Z, Y_3]_k$. Segments $\mathcal{X}', \mathcal{Y}$ are complementary, hence $\mathcal{Y} = [W, Y_3]_k$ and $Z \prec W$. Since $Y' \prec Y_3$, $W \subseteq Y_3$ and $W \nsubseteq Y'$ we find that $Y' \cap W \prec W$ by

2.1. It is easily seen that $Z \subseteq Y' \cap W$ which yields $Z = Y' \cap W$. Equality $Y_3 = Y' + W$ can be shown by similar argument.

(ii) is a dual case to (i).

(iii) Assume that $Z_1 = Z_2 = Z$. Then $\mathcal{Y} = [W, Y_3]_k$ by (i). Let $U_1 \in \mathcal{X}_1 \setminus \mathcal{X}'$. By 2.1 we have $H = U_1 \cap Y' \prec U_1$ since $Y' \prec Y_1$, $U_1 \subseteq Y_1$ and $U_1 \nsubseteq Y'$. Take $U_3 = H + W$ and note that $H \prec U_3$ again by 2.1, as $Z \prec W$, $Z \subseteq H$ and $W \nsubseteq H$ by (i). Hence U_1, U_3 are collinear. Evidently, $U_3 \in \mathcal{Y} = \mathcal{Y} \setminus \mathcal{X}'$. Suppose that U_3 is not unique. Then $W \subseteq U_1$ or $U_1 \subseteq Y_3$ by 2.2 and 2.3. In the first case $W \subseteq Y'$, in the latter $U_1 \in \mathcal{X}_3$, but both are invalid. For $Y_1 = Y_2$ we proceed dually.

(iv) Let $U_1 \in \mathcal{X}_1 \setminus \mathcal{X}'$. By (iii) there is a unique $U_3 \in \mathcal{Y} \subseteq \mathcal{X}_3$ collinear with U_1, and by 2.11(i) there is $U_2 \in \mathcal{X}_2$ such that $U_2 \in \overline{U_1, U_3}$. Suppose that $U_2 \in \mathcal{X}'$. Then $U_1 \in \overline{U_2, U_3} \subseteq \mathcal{X}_3$ and hence $U_1 \in \mathcal{X}_1 \cap \mathcal{X}_3 = \mathcal{X}'$ which is false. The point U_2 is unique, for if not, there would be another $U_2' \in \mathcal{X}_2$ and U_1, U_2, U_2', U_3 would be collinear as U_3 is unique. In that case $U_1 \in \overline{U_2, U_2'} \subseteq \mathcal{X}_2$ which is false.

If $U_1 \in \mathcal{X}'$, and $U_2 \neq U_1$, then $U_3 \in \overline{U_1, U_2} \subseteq \mathcal{X}_2$ which is impossible as $\mathcal{Y} \cap \mathcal{X}_2 = \emptyset$.

Now, we shall find the formula for U_2 when $Z_1 = Z_2 = Z$. First note that $U_1 \prec U_1 + W$ by 2.1 as $Z \prec W$ by (i), $Z \subseteq U_1$ and $W \nsubseteq U_1$. Indeed, if $W \subseteq U_1$, then $W \subseteq Y'$ and contradiction with (i) arises. Since $Y_2 \prec Y''$, $U_1 + W \subseteq Y''$ and $U_1 + W \nsubseteq Y_2$ we have $U_2 := (U_1 + W) \cap Y_2 \prec U_1 + W$. Hence U_1, U_2 are collinear and $U_2 \in \mathcal{X}_2$. Note that $\overline{U_1, U_2} = \mathbf{p}(U_1 \cap Y_2, U_1 + W)$. Therefore, $\overline{U_1, U_2} \cap \mathcal{Y} = [(U_1 \cap Y_2) + W, (U_1 + W) \cap Y_3]_k$ by (i). It is easily seen that $(U_1 \cap Y_2) + W \subseteq (U_1 + W) \cap Y_3$. Moreover, $\dim\big((U_1 + W) \cap Y_3\big) = k$ similarly as above for U_2. Finally, the line $\overline{U_1, U_2}$ crosses $\mathcal{Y}$ and we are through. For $Y_1 = Y_2$ we proceed analogously. □

Under assumptions of 4.2 we can define a *generalized central projection* $\overset{\mathcal{X}_1}{\underset{\mathcal{X}_2}{\downarrow\mathcal{Y}}} \colon \mathcal{X}_1 \longrightarrow \mathcal{X}_2$ by condition

$$\overset{\mathcal{X}_1}{\underset{\mathcal{X}_2}{\downarrow\mathcal{Y}}}(U_1) = U_2 \quad \text{iff} \quad U_1, U_2, U_3 \text{ are collinear for some } U_3 \in \mathcal{Y}. \tag{5}$$

Lemma 4.2 actually says that for all $\mathcal{X}_1, \mathcal{X}_2$ that determine a proper pencil there is a generalized central projection of $\mathcal{X}_1$ onto $\mathcal{X}_2$. If G consists of lines then $\mathcal{Y}$ is a single point $\mathcal{Y} = \{C\}$, and $\overset{\mathcal{X}_1}{\underset{\mathcal{X}_2}{\downarrow\mathcal{Y}}} = \overset{\mathcal{X}_1}{\underset{\mathcal{X}_2}{\downarrow C}}$ is an ordinary central projection of the line $\mathcal{X}_1$ onto $\mathcal{X}_2$ with the center C. Let us define generally that $\overset{\mathcal{X}_1}{\underset{\mathcal{X}_2}{\downarrow C}} := \overset{\mathcal{X}_1}{\underset{\mathcal{X}_2}{\downarrow \{C\}}}$, whenever it is a function. One can see, if $\overset{\mathcal{X}_1}{\underset{\mathcal{X}_2}{\downarrow C}}$ is defined, then C adheres $\mathcal{X}_i$ for $i = 1, 2$. Note that if $\mathcal{X}$ is a strong subspace of $\mathbf{P}_k(V)$, $\mathcal{Y}_1$ is of codimension 1 in $\mathcal{X}$ and $\mathcal{Y}_2$ is complementary to $\mathcal{Y}_1$ in $\mathcal{X}$, then $\mathcal{Y}_2$ is a point. In particular, if a pencil G is projective with $Z' = Z'' = Z$, then $\mathcal{X}' = [Z, Y']_k$ and $\mathcal{Y} = \{C\}$ complementary to $\mathcal{X}'$ satisfies $C \nsubseteq Y'$. Under these circumstances the claim of 4.2 remains valid for $\mathcal{Y} = [C, Y_3]_k$, and $\overset{\mathcal{X}_1}{\underset{\mathcal{X}_2}{\downarrow\mathcal{Y}}} = \overset{\mathcal{X}_1}{\underset{\mathcal{X}_2}{\downarrow C}}$ is a plain projection. Recall that in view of [13] the formula 4.2(iv), which defines the projection, remains valid as well. Dually for $Y' = Y''$.

Lemma 4.3. *Let $\mathcal{X} = [Z, Y]_k$ be a non-trivial segment subspace and C be a point not in $\mathcal{X}$ such that $C \mid\!\rhd \mathcal{X}$. Then there is a maximal strong subspace containing both C and $\mathcal{X}$, and either*

(i) *$\mathcal{X}$ is a star and $Z = C \cap Y$, or*

(ii) *$\mathcal{X}$ is a top and $Y = Z + C$.*

Proof. Immediately by 2.2 $\mathcal{X}$ is strong. In view of 2.3 either $\mathcal{X}$ is a star and $Z \subseteq C \cap Y \subsetneq C$, or $\mathcal{X}$ is a top and $C \subsetneq C + Z \subseteq Y$. □

As an immediate consequence of 4.3 we obtain

Proposition 4.4. *Let $\mathcal{X}_i$ be segment subspaces and $f = \updownarrow\begin{smallmatrix}\mathcal{X}_1\\ C\\ \mathcal{X}_2\end{smallmatrix}$ be a central projection. Then $\mathcal{X}_1$, $\mathcal{X}_2$ and C lie in a strong subspace Π of $\mathbf{P}_k(V)$ and f is an ordinary central projection in the projective space Π.*

Proposition 4.5. *For every generalized central projection $h = \updownarrow\begin{smallmatrix}\mathcal{X}_1\\ \mathcal{Y}\\ \mathcal{X}_2\end{smallmatrix}$, if $Z_1 = Z_2$ and $m = k+1$, or $Y_1 = Y_2$ and $m = k - 1$, then there are shifts f, g such that $h = fg$ and f, g have their ranges and domains in $\mathbf{P}_k(V)$ or $\mathbf{P}_m(V)$.*

Proof. Consider the case where $Z_1 = Z_2$ and $m = k + 1$. In view of 4.2(iv) we take $g = \downarrow^{\mathcal{X}_1}_{[Z_1+W, Y_1+W]_k}$ and $f = \downarrow^{[Z_1+W, Y_1+W]_k}_{\mathcal{X}_2}$. If $Y_1 = Y_2$ and $m = k - 1$ we take $g = \downarrow^{\mathcal{X}_1}_{[Z_1 \cap W, Y_1 \cap W]_k}$ and $g = \downarrow^{[Z_1 \cap W, Y_1 \cap W]_k}_{\mathcal{X}_2}$. □

For a wafer G, according to 2.9(i), one can define a *slide* $\zeta^{\mathcal{X}_1}_{\mathcal{X}_2} = \updownarrow^{\mathcal{X}_1}_{\mathcal{X}_2} \colon \mathcal{X}_1 \longrightarrow \mathcal{X}_2$, where $\mathcal{X}_1, \mathcal{X}_2 \in \mathsf{G}$ are distinct segments, by

$$\updownarrow^{\mathcal{X}_1}_{\mathcal{X}_2}(U_1) = U_2 \quad \text{iff} \quad U_1, U_2 \text{ are collinear.} \tag{6}$$

It is seen that for all $\mathcal{X}_1, \mathcal{X}_2$ that determine a wafer there is a slide of $\mathcal{X}_1$ onto $\mathcal{X}_2$. As long as central projections confirm projective nature of spaces of pencils, slides indicate features of ruled quadrics. In the context of classical projective geometry slides are referred to as net projections (cf. [3, Ch. 10]). Further, we use the short term projection for slides, central projections and generalized central projections, that is those projections which act within $\mathbf{P}_k(V)$.

Proposition 4.6. *For every slide $h = \zeta^{\mathcal{X}_1}_{\mathcal{X}_2}$, and $m = k \pm 1$ there are shifts f, g such that $h = fg$ and f, g have their ranges and domains in $\mathbf{P}_k(V)$ or $\mathbf{P}_m(V)$.*

Proof. In view of 2.9(i), $h(U_1) = (U_1 + Z_2) \cap Y_2 = (U_1 \cap Y_2) + Z_2$ by modularity, for all $U_1 \in \mathcal{X}_1$. Clearly, $U_1 + Z_2 = U_1 + Z''$ and $U_1 \cap Y_2 = U_1 \cap Y'$. Therefore, if $m = k + 1$ we take $g = \downarrow^{\mathcal{X}_1}_{[Z'', Y'']_{k+1}}$ and $f = \downarrow^{[Z'', Y'']_{k+1}}_{\mathcal{X}_2}$. If $m = k - 1$, then $g = \downarrow^{\mathcal{X}_1}_{[Z', Y']_{k-1}}$ and $f = \downarrow^{[Z', Y']_{k-1}}_{\mathcal{X}_2}$. □

Combining 4.5, 4.6, and 3.2 we find that *a projection of $\mathcal{X}_1$ onto $\mathcal{X}_2$ is a collineation between corresponding subspaces of our space of pencils.*

For any segment $\mathcal{X} = [Z, Y]_k$ with $Z \neq \Theta$ or $Y \neq V$ we can find a proper pencil G such that $\mathcal{X} \in \mathsf{G}$. To find a wafer we need $Z \neq \Theta$ and $Y \neq V$. These conditions are equivalent to existence of a (generalized) central projection or a slide h, respectively, with $\mathrm{dm}(h) = \mathcal{X}$ or $\mathrm{rg}(h) = \mathcal{X}$. Note also that $Z \neq \Theta$ means $\mathrm{idx}(X) \neq k$, and $Y \neq V$ for finite-dimensional V means that $\mathrm{coidx}(\mathcal{X}) \neq \dim V - k$.

Proposition 4.7. *Let $\mathcal{X}_i$ be segments such that $Z_i \neq \Theta$ and $Y_i \neq V$, $i = 1, 2$. If* (i)*: $h = \begin{smallmatrix}\mathcal{X}_1\\ C\\ \mathcal{X}_2\end{smallmatrix}$ is a central projection, or* (ii)*: $h = \begin{smallmatrix}\mathcal{X}_1\\ \mathcal{Y}\\ \mathcal{X}_2\end{smallmatrix}$ is a generalized central projection, then there are slides f, g such that $h = gf$.*

Proof. Since central projections are valid for proper pencils only, there are two cases to examine.

(1) Set $Z_1 = Z_2 = Z' = Z'' =: Z$, then $Z \neq \Theta$. We shall construct a segment $\mathcal{X}_0$ which forms wafers with $\mathcal{X}_1, \mathcal{X}_2$. So, consider Y_0 such that $Z \nsubseteq Y_0 \prec Y''$.

In case (i) $C \in [Z, Y'']_k$ and we take $Z_0 := C \cap Y_0$. Since $C \subseteq Y''$ and $C \nsubseteq Y_0$ we find that $Z_0 \prec C$ by 2.1. Due to 4.4 which says that $Z \prec C$ we have $Z + Z_0 = C$. Note that $Z_0 \nsubseteq Y_1, Y_2$ since otherwise, if $Z_0 \subseteq Y_i$, then $C \subseteq Y_i$ and thus $C \in \mathcal{X}_i$, which is false as C is the center of projection between $\mathcal{X}_1, \mathcal{X}_2$. We set $\mathcal{Y} = \{C\}$.

In case (ii) $\mathcal{Y} = [W, Y_3]_k$ such that $Y_3 \in \overline{Y_1, Y_2}$, $Z \prec W$ and $Y' \cap W = Z$ by 4.2(i). We take $Z_0 := W \cap Y_0$. As $W \subseteq Y''$ and $W \nsubseteq Y_0$ we have $Z_0 \prec W$ by 2.1. Note that $Z_0 \nsubseteq Y_1, Y_2$. Indeed, if $Z_0 \subseteq Y_i$, where i is 1 or 2, then $Z_0 \subseteq Y'$ as $Z_0 \subseteq Y_3$. Moreover, $Z_0 \subseteq Y' \cap W = Z$, thus $Z_0 = Z$, and the contradiction arises since $Z \nsubseteq Y_0$.

Eventually, we take $\mathcal{X}_0 = [Z_0, Y_0]_k$. It is seen that $\mathcal{X}_0, \mathcal{X}_i$ are of type (W3), hence $\mathcal{X}_0$, $\mathcal{X}_i$ determine a wafer. Therefore, one can take slides $f := \zeta^{\mathcal{X}_1}_{\mathcal{X}_0}$, $g := \zeta^{\mathcal{X}_0}_{\mathcal{X}_2}$. For every $U_i \in \mathcal{X}_i$, such that $U_2 = h(U_1)$ the line $\overline{U_1, U_2} = \mathbf{p}(H, B)$ contains some $U_3 \in \mathcal{Y}$, so $C \subseteq B$ in (i), and $W \subseteq B$ in (ii). Moreover, $Y_0 \prec Y''$, $B \subseteq Y''$ and $B \nsubseteq Y_0$, therefore $B \cap Y_0 \prec B$ by 2.1. Consequently $U_0 := B \cap Y_0$ is a point in $\mathcal{X}_0$. It is easily seen that $U_0, U_1, U_2 \prec B$, hence U_0 is collinear with U_1 and U_2. Then $f(U_1) = U_0$, $g(U_0) = U_2$ and finally $h = gf$.

(2) In case $Y_1 = Y_2 = Y' = Y''$ the proof runs dually. □

For convenience we apply the following convention:

$$\mathcal{E}_i = [T_i, S_i]_k, \quad \mathcal{E}' = \mathcal{E}_1 \cap \mathcal{E}_2 = [T'', S']_k \quad \text{and} \quad \mathcal{E}'' = \langle \mathcal{E}_1 \cup \mathcal{E}_2 \rangle = [T', S'']_k.$$

Lemma 4.8. *Let f be a projection of $\mathcal{X}_1$ on $\mathcal{X}_2$. For a segment $\mathcal{E}_1 \subseteq \mathcal{X}_1$, and $\mathcal{E}_2 = f(\mathcal{E}_1)$, either $\mathcal{E}_1 = \mathcal{E}_2$ or $\mathcal{E}_1, \mathcal{E}_2$ determine a pencil. If $\overline{\mathcal{E}_1, \mathcal{E}_2}$ is a proper pencil, then $\overline{\mathcal{X}_1, \mathcal{X}_2}$ is a proper pencil, and additionally, if* $\mathrm{idx}(\mathcal{E}') = \mathrm{idx}(\mathcal{E}'')$*, then* $\mathrm{idx}(\mathcal{X}') = \mathrm{idx}(\mathcal{X}'')$*, if* $\mathrm{coidx}(\mathcal{E}') = \mathrm{coidx}(\mathcal{E}'')$*, then* $\mathrm{coidx}(\mathcal{X}') = \mathrm{coidx}(\mathcal{X}'')$.

Proof. Let $\mathcal{E}_2 := f(\mathcal{E}_1)$. If f is a slide then $\mathcal{X}_1, \mathcal{X}_2$ determine a wafer. Hence, $\mathcal{X}' = \emptyset$ and for every $U_1 \in \mathcal{E}_1$ there is a unique $U_2 \in \mathcal{E}_2$ such that $U_1 \sim U_2$, and consequently, by 2.13(ii) $\mathcal{E}_1, \mathcal{E}_2$ determine a wafer.

The case where f is a generalized central projection $\overset{\mathcal{X}_1}{\underset{\mathcal{X}_2}{\updownarrow\mathcal{Y}}}$ and $\overline{\mathcal{X}_1, \mathcal{X}_2}$ is a proper pencil remains. So, assume that $Z_1 = Z_2 = Z$. Then by 4.2(iv) $T_2 = (T_1 + W) \cap Y_2$, $S_2 = (S_1 + W) \cap Y_2$. Observe that $T' = T_1 \cap T_2 = T_1 \cap Y_2 \preccurlyeq T_1$, and similarly $S' = S_1 \cap S_2 = S_1 \cap Y_2 \preccurlyeq S_1$. This means that either $\mathcal{E}_1 = \mathcal{E}_2$, or $\mathcal{E}_1, \mathcal{E}_2$ span a quasi-pencil. If it is not a wafer then $T_1 \subseteq S_2$ or $T_2 \subseteq S_1$ by (W3). But then, in the first case, $T_1 \subseteq Y_2$, and hence $T_2 = T_1 + (W \cap Y_2) = T_1 + (W \cap Y_3 \cap Y_2) = T_1 + Z = T_1$ by 4.2(i) as $Y' \cap W = Z$. Analogously in the second case with respect to $\overset{\mathcal{X}_2}{\underset{\mathcal{X}_1}{\updownarrow\mathcal{Y}}}$ we get $T_1 = T_2$. The proof for $Y_1 = Y_2$ runs dually, hence, $\overline{\mathcal{E}_1, \mathcal{E}_2}$ is a pencil.

Actually, we have shown that if $\overline{\mathcal{E}_1, \mathcal{E}_2}$ is a proper pencil, then $\overline{\mathcal{X}_1, \mathcal{X}_2}$ is a proper pencil. Moreover, for $Z_1 = Z_2$ we have shown that $T_1 = T_2$, and dually for $Y_1 = Y_2$ we have $S_1 = S_2$. This suffices for the remaining justification. $\square$

Proposition 4.9. *If f is a projection of $\mathcal{X}_1$ on $\mathcal{X}_2$, and $\mathcal{E}_1$ is a segment such that $\mathcal{E}_1 \subseteq \mathcal{X}_1$, then $f|\mathcal{E}_1$ is a projection.*

Proof. Let $\mathcal{E}_2 := f(\mathcal{E}_1)$. If f is a slide, then $\mathcal{E}_1, \mathcal{E}_2$ determine a wafer, and hence, $f|\mathcal{E}_1$ is a slide. If $f = \overset{\mathcal{X}_1}{\underset{\mathcal{X}_2}{\updownarrow\mathcal{Y}}}$ is a generalized central projection, by 4.8 we have two cases. First assume that $\mathcal{E}_1, \mathcal{E}_2$ determine a wafer. Then for $U_1 \in \mathcal{E}_1$ we have $f(U_1) = \zeta^{\mathcal{E}_1}_{\mathcal{E}_2}(U_1)$ by 2.9(i). Therefore, $f|\mathcal{E}_1$ is a slide.
Now, assume that $\mathcal{E}_1, \mathcal{E}_2$ determine a proper pencil such that $T_1 = T_2 = T$. We shall find $\mathcal{Y}'$ such that $f|\mathcal{E}_1 = \overset{\mathcal{E}_1}{\underset{\mathcal{E}_2}{\updownarrow\mathcal{Y}'}}$. First, note that $Z_1 = Z_2 = Z$ by 4.8. Since $Y' \prec Y_1$ and $S_1 \subseteq Y_1$, we have $S_1 + Y' = Y_1$, for if not we would have $S_1 \subseteq Y'$ which yields $\mathcal{E}_1 \subseteq \mathcal{X}'$ and thus $\mathcal{E}_1 = \mathcal{E}_2$ by 4.2(iv). Let $\mathcal{E}_3 := [T, S' + W]_k$. Observe that $S' \preccurlyeq S' + W$ by 2.1 as $Z \prec W$ by 4.2(i) and $Z \subseteq S'$. Moreover $W \nsubseteq S'$ since otherwise $W \subseteq Y'$ which contradicts 4.2(i). Hence $S' \prec S' + W$. Note also that $S'' = S_1 + ((S_1 + W) \cap Y_2)$ by 4.2(iv), and further $S'' = (S_1 + Y_2) \cap (S_1 + W) = (S_1 + Y' + Y_2) \cap (S_1 + W) = Y'' \cap (S_1 + W) = S_1 + W$. Hence $S' + W \subseteq S''$. Consequently $\mathcal{E}_3 \in \overline{\mathcal{E}_1, \mathcal{E}_2}$. Similarly, as for S' above, we can show for T that $T \prec T + W$. It is easily seen that $\mathcal{E}' = [T, S']_k$ and $\mathcal{Y}' := [T + W, S' + W]_k$ are complementary of codimension 1 in $\mathcal{E}_3$. Therefore, $h := \overset{\mathcal{E}_1}{\underset{\mathcal{E}_2}{\updownarrow\mathcal{Y}'}}$ is a generalized central projection. Immediate consequence of formulas for f and h given by 4.2(iv) we have $f|\mathcal{E}_1 = h$. In case $S_1 = S_2$ we proceed dually. $\square$

5. Projectivities

Let $\Omega(k)$ be the set of all meaningful compositions of slides, central projections and generalized central projections in $\mathbf{P}_k(V)$. Elements of $\Omega(k)$ are called *projectivities*. Let $\Omega(k, m)$ be the class of all compositions $f_1 \circ \cdots \circ f_n$, every f_i being a shift projection and $\mathrm{dm}(f_i), \mathrm{rg}(f_i)$ being subspaces of $\mathbf{P}_k(V)$ and $\mathbf{P}_m(V)$. In $\Omega(k)$ we distinguish the subclass $\Omega_c(k)$ given by central projections and generalized central projections. Clearly, $\Omega(k)$ is a category with segment subspaces of $\mathbf{P}_k(V)$ as objects, and $\Omega(k, m)$ is a category with segments of $\mathbf{P}_k(V)$ and $\mathbf{P}_m(V)$ as objects.

In order to have a projectivity between two arbitrary segment subspaces $\mathcal{X}_1, \mathcal{X}_2$ in a space of pencils we have to guarantee that Z_1, Z_2 can be connected with a polygonal path (a sequence of subspaces of V, in which neighbour elements are adjacent), as well as Y_1, Y_2. There is no problem with Z_1, Z_2, but we have to assume in this section additionally that either

$$\dim Y_1, \dim Y_2 < \infty \text{ (i.e. } \operatorname{coidx} \mathcal{X}_i < \infty) \quad \text{or} \quad \operatorname{codim} Y_1, \operatorname{codim} Y_2 < \infty. \tag{7}$$

If f is a projection, then $\mathrm{dm}(f), \mathrm{rg}(f)$ are similar i.e. $\operatorname{pdim} \mathrm{rg}(f) = \operatorname{pdim} \mathrm{dm}(f)$. Accordingly, to any geometrical dimension δ, we associate the full subcategory $\Omega(k;\delta)$ of $\Omega(k)$, obtained by restricting the class of objects to segments $\mathcal{X}$ such that $\operatorname{pdim} \mathcal{X} = \delta$ and (7) holds. Analogously we define $\Omega(k,m;\delta)$[1].

In this terminology the class of projectivities defined in [14] is the category $\Omega_c(k;(1,1))$ as lines has pencil dimension (1,1).

Lemma 5.1. *Let $h \in \Omega(k,l;\delta)$ such that $\mathrm{dm}(h), \mathrm{rg}(h)$ are segments in $\mathbf{P}_k(V)$. If $m < l < k$ or $k < l < m$, and $\Omega(k,m;\delta) \neq \emptyset$, then $h \in \Omega(k,m;\delta)$.*

Proof. Assume that $k < l < m$. By definition, $h = g_n f_n \cdots g_1 f_1$ for some J-projections f_i and M-projections g_i. Since $\mathrm{dm}(h)$ and $\mathrm{rg}(f_1)$ are similar, there is a J-projection $t = \big\downarrow^{\mathrm{rg} f_1}_{[Z,Y]_m}$ by 3.5, such that $tf_1 = \big\downarrow^{\mathrm{dm}\, h}_{[Z,Y]_m}$ and $g_1 t^{-1} = \big\downarrow^{[Z,Y]_m}_{\mathrm{rg}\, h}$. It is easily seen that, $g_1 f_1 = (g_1 t^{-1})(t f_1) \in \Omega(k,m)$. We can repeat this reasoning for $i = 2, \dots, n$, thus we get our claim. In case $m < l < k$ we proceed analogously. □

Proposition 5.2. *If $h \in \Omega(k;\delta)$, $m \neq k$ and $\Omega(k,m;\delta) \neq \emptyset$, then $h \in \Omega(k,m;\delta)$.*

Proof. If h is a slide, then by 4.6 $h \in \Omega(k,k-1;\delta) \cap \Omega(k,k+1;\delta)$, and by 5.1 we are through. If h is a generalized central projection, then depending on $\operatorname{pdim} \mathrm{dm}(h)$ we apply either, 4.5 together with 5.1, or 4.7 and the above property of slides. □

Proposition 5.3. *For $h \in \Omega(k,m)$ there is a linear bijection φ such that $\tilde{\varphi}_k = h$.*

Proof. Immediate by 3.6 as h can be decomposed into J- and M-projections. □

In view of the above and 3.6(i) the following conclusion arises:

Corollary 5.4. *Every projection is a linear collineation.*

Lemma 5.5. *If $\mathcal{X}_a, \mathcal{X}_b$ are distinct, similar segments in $\mathbf{P}_k(V)$, there are segments $\mathcal{X}_0, \dots,$ $\mathcal{X}_r$ such that $\mathcal{X}_0 = \mathcal{X}_a$, $\mathcal{X}_r = \mathcal{X}_b$ and $\overline{\mathcal{X}_{i-1}, \mathcal{X}_i}$ is a proper pencil, $i = 1, \dots, r$.*

Proof. Let $\mathcal{X}_a = [Z_a, Y_a]_k$, $\mathcal{X}_b = [Z_b, Y_b]_k$. First consider the case where $Y_a = Y_b = Y$. Then, by connectedness of appropriate space of pencils there are $Z_0, \dots, Z_r$ such that $Z_0 = Z_a$, $Z_r = Z_b$ and Z_{i-1}, Z_i are adjacent for $i = 1, \dots, r$. The sequence $\mathcal{X}_i = [Z_i, Y]_k$, where $i = 0, \dots, r$, satisfies our claim.

For distinct Y_a, Y_b, take $Y_0, \dots, Y_s$ such that $Y_0 = Y_a$, $Y_s = Y_b$ and Y_{i-1}, Y_i are adjacent for $i = 1, \dots, s$. Then take $Z_0 := Z_a$, $Z_i \in \mathrm{Sub}_{\dim Z_0}(Y_{i-1} \cap Y_i)$, for $i = 1, \dots, s$ and $Z_{s+1} := Z_b$. Observe that for every pair $[Z_i, Y_i]_k, [Z_{i+1}, Y_i]_k$, $i = 0, \dots, s$ we can proceed as in the first considered case, hence the required sequence $\mathcal{X}_i$ now, is formed by concatenation. □

[1]Note that given $\delta = (n_1, n_2)$ the maximal m such that $\mathbf{P}_m(V)$ contains a subspace $\mathcal{X}$ with $\operatorname{pdim} \mathcal{X} = \delta$ is $\dim V - n_2$ and the minimal is n_1.

Lemma 5.6. *For similar segments* $\mathcal{X}_1, \mathcal{X}_2$ *in* $\mathbf{P}_k(V)$ *there is* $\xi \in \Omega_c(k)$ *such that* $\mathrm{dm}(\xi) = \mathcal{X}_1$ *and* $\mathrm{rg}(\xi) = \mathcal{X}_2$.

Proof. Straightforward by 5.5 as for every two neighbour segments, in the sequence joining $\mathcal{X}_1, \mathcal{X}_2$, there is a central or generalized central projection. □

Proposition 5.7. *Let* $\mathcal{X}$ *be a segment in* $\mathbf{P}_k(V)$. *If* φ *is a linear bijection such that* $\tilde{\varphi}_k \colon \mathcal{X} \longrightarrow \mathcal{X}$, *then* $\tilde{\varphi}_k \in \Omega_c(k)$.

Proof. Let us extend the map $\varphi\colon Y/Z \longrightarrow Y/Z$ to the map $\psi\colon V/Z \longrightarrow V/Z$ so that it is a linear bijection. The map ψ restricted to any subspace of V/Z can be decomposed into projections in the projective space $\mathbf{P}_1(V/Z)$. So we have projections $\xi_1, \dots, \xi_r$ in $\mathbf{P}_1(V/Z)$ such that $(\varphi)_1^* = (\psi | Y/Z)_1^* = \xi_r \circ \dots \circ \xi_1$. Every projection $\xi_i = \underset{[\Theta,Y_2]_1}{\overset{[\Theta,Y_1]_1}{\downarrow C_i}}$ induces a central projection $\underset{[\Theta,Y_2]_m}{\overset{[\Theta,Y_1]_m}{\downarrow C_i}}$, or a generalized central projection $\underset{[\Theta,Y_2]_m}{\overset{[\Theta,Y_1]_m}{\downarrow \mathcal{Y}}}$, where $\mathcal{Y} = [C_i, C_i + Y']_m$, in $\mathbf{P}_m(V/Z)$, $m = k - \dim Z$. □

Corollary 5.8. *Let* $\mathcal{X}_1$, $\mathcal{X}_2$ *be segments in* $\mathbf{P}_k(V)$. *If* φ *is a linear bijection such that* $\tilde{\varphi}_k \colon \mathcal{X}_1 \longrightarrow \mathcal{X}_2$, *then* $\tilde{\varphi}_k \in \Omega_c(k)$.

Proof. By 5.6 there is a $\xi \in \Omega_c(k)$ such that $\xi\colon \mathcal{X}_2 \longrightarrow \mathcal{X}_1$, which by 5.3 is a linear bijection ψ such that $\tilde{\psi}_k = \xi$. Hence we have a linear bijection $\varphi\psi$ such that $h = (\widetilde{\varphi\psi})_k \colon \mathcal{X}_1 \longrightarrow \mathcal{X}_1$. But $h \in \Omega_c(k)$ by 5.7. Hence $\tilde{\varphi}_k = \xi^{-1} h$. □

Corollary 5.9. *Let* h *be a map such that* $\mathrm{dm}(h), \mathrm{rg}(h)$ *are subset of points of* $\mathbf{P}_k(V)$. *The following conditions are equivalent:*

(1) $h \in \Omega_c(k)$,

(2) $h \in \Omega(k)$,

(3) $h \in \Omega(k, m)$, *provided that* $\Omega(k, m; \mathrm{pdim}(\mathrm{dm}(h))) \neq \emptyset$,

(4) $h = \tilde{\varphi}_k$, *where* φ *is a linear bijection.*

Proof. (1) implies (2) trivially.
(2) implies (3) by 5.2.
(3) implies (4) by 5.3, and finally
(4) implies (1) by 5.8. □

The equivalence (1) ≡ (4) is a direct analogue of the known characterization of projectivities in projective geometry. Note, as a tricky observation:

Corollary 5.10. *Let* h *be as in* 5.9 *and* $\delta = \mathrm{pdim}(\mathrm{dm}(h))$. *If* $\Omega(k; \delta)$ *contains at least one slide, then* $h \in \Omega(k)$ *iff* h *is a composition of slides.*

Proposition 5.11. *Let* $\mathcal{X}_i$ *be segments in* $\mathbf{P}_{k_i}(V)$, $i = 1, 2$. *If* $\xi\colon \mathcal{X}_1 \longrightarrow \mathcal{X}_2$, *then* $\xi = \tilde{\varphi}_{k_1}$ *for some linear bijection* φ *iff* $\xi \in \Omega(k_1, k_2)$.

Proof. $\Rightarrow$: By 3.5 there is $f \in \Omega(k_1, k_2)$ such that $\mathrm{dm}(f) = \mathcal{X}_2$ and $\mathrm{rg}(f) =: \mathcal{X}_1'$ is a segment in $\mathbf{P}_{k_1}(V)$. By 5.3 $f = \tilde{\psi}_{k_2}$ for some linear bijection ψ. Hence we obtain $h = f\tilde{\varphi} = \widetilde{\psi\varphi}\colon \mathcal{X}_1 \longrightarrow \mathcal{X}_1'$, by 5.8 $h \in \Omega_c(k)$, and by 5.9 we are through.

$\Leftarrow$: Follows from 5.3. □

Proposition 5.12. *If f is a collineation of $\mathbf{P}_k(V)$ which preserves stars and tops, then the following are equivalent:*

(1) $f|\mathcal{X} \in \Omega(k)$ *for some segment* $\mathcal{X}$,

(2) $f|\mathcal{X} \in \Omega(k)$ *for all segments* $\mathcal{X}$.

Proof. (1) implies (2): f is given by a bijective semi-linear map φ on V, i.e. $f = \varphi_k^*$ (cf. [1, Ch. II.10]). By (1) and 5.9 $\varphi_k^*|\mathcal{X}$ is proportional to a linear map, and φ is proportional to a linear bijection which suffices to state (2).

(2) implies (1) trivially. □

6. Projections onto pencils of segments

The principle intention of this section is to give basic, but general, description of projections onto pencils of segment subspaces.

Let G be a pencil of segment subspaces in $\mathbf{P}_k(V)$ determined by $\mathcal{X}_1, \mathcal{X}_2$. In view of 2.10(iii) whenever a line p intersects two members of G, and does not intersect $\mathcal{X}'$, then for every point U on p there is $\mathcal{X} \in \mathsf{G}$ through U. The $\mathcal{X}$ is unique since otherwise U need to lie on $\mathcal{X}'$. Conversely, for every such $\mathcal{X}$ the point U on p is unique, for if not, we would have $p \subseteq \mathcal{X}$. Eventually, there is a one-to-one correspondence $h = \xi_p^{\mathsf{G}}$ between members of G and points on p given with the following condition:

$$h(\mathcal{X}) \in \mathcal{X} \cap p, \qquad \text{for } \mathcal{X} \in \mathsf{G}. \tag{8}$$

Compositions of maps of the above form can be used to characterize wafers of lines (Proposition 6.5), and to distinguish perspectivities and characterize projectivities in terms of them (Corollary 6.8). Evidently $\xi_{\mathsf{G}}^p = (\xi_p^{\mathsf{G}})^{-1} = h^{-1}$, and $U \in h^{-1}(U) \in \mathsf{G}$ for $U \in p$.

Proposition 6.1. *Let G be a proper pencil of segments and p a line such that ξ_{G}^p is a reasonable projection.*

(i) *If $Z' = Z'' = Z$, then $\xi_{\mathsf{G}}^p(U) = [Z, f(U)]_k$ for $U \in p$, where $f = \downarrow^p_{\mathbf{p}(Y',Y'')}$.*

(ii) *If $Y' = Y'' = Y$, then $\xi_{\mathsf{G}}^p(U) = [g(U), Y]_k$ for $U \in p$, where $g = \downarrow^p_{\mathbf{p}(Z',Z'')}$.*

Proof. (i) Let $U \in p$ and $p = \mathbf{p}(H, B)$. Clearly $H \prec U$, by 2.8 $H \subseteq Y'$, and $U \nsubseteq Y'$ since otherwise $U \in \mathcal{X}'$. Hence $Y' \prec U + Y'$ by 2.1. Therefore $U \in [Z, U + Y']_k \in \mathsf{G}$, and $\xi_{\mathsf{G}}^p(U) = [Z, f(U)]_k$, where $f = U \rightsquigarrow U + Y'$.

(ii) is dual to (i). □

Proposition 6.2. *Let G be a wafer and p a line crossing two members of G. Then $\xi_{\mathsf{G}}^p(U) = [g(U), f(U)]_k$ for $U \in p$, where $f = \downarrow^p_{\mathbf{p}(Y',Y'')}$, $g = \downarrow^p_{\mathbf{p}(Z',Z'')}$.*

Proof. Let $U \in p$. By 2.9(ii) we find that $U \in [U \cap Z'', U + Y']_k \in \mathsf{G}$, and hence $\xi^p_{\mathsf{G}}(U) = [g(U), f(U)]_k$, where $f = U \rightsquigarrow U + Y'$ and $g = U \rightsquigarrow U \cap Z''$ are required maps. □

This is 2.9(ii) expressed in terms of projections. In view of 3.6 a projection ξ^p_{G} in 6.1 induces a linear map. Note that in 6.2 for a wafer G we have the shift $h = \downarrow^{\mathbf{P}(Z',Z'')}_{\mathbf{P}(Y',Y'')}$ given, therefore $f = hg$ and in consequence f, g are mutually definable. It justifies to state that projection ξ^p_{G} induces a linear map in that case too.

We say that a pencil G' extends G if for every $\mathcal{X} \in \mathsf{G}$ there is a unique $\mathcal{Y} \in \mathsf{G}'$ with $\mathcal{X} \subseteq \mathcal{Y}$, and for every $\mathcal{Y} \in \mathsf{G}'$ there is a unique $\mathcal{X} \in \mathsf{G}$ with $\mathcal{X} \subseteq \mathcal{Y}$.

Lemma 6.3. *If G is a pencil of segments of dimension δ, and there exists a pencil of segments of dimension δ', with $\delta \le \delta'$, then there is a pencil G' of the same type as G such that G' extends G.*

Proof. Let $\delta = (k - k_1, k_2 - k)$ and $\delta' = (k - m_1, m_2 - k)$. Then $m_1 \le k_1$ and $k_2 \le m_2$. Consider Q, R such that $Z' + Q = Z'' + Q$, $Y' \cap R = Y'' \cap R$ and $\dim Z \cap Q = m_1$ for $Z \in [Z', Z'']_{k_1}$, and $\dim Y + R = m_2$ for $Y \in [Y', Y'']_{k_2}$. By 3.1 maps $f = Z \rightsquigarrow Z \cap Q$ with $\mathrm{dm}(f) = [Z', Z'']_{k_1}$, and $g = Y \rightsquigarrow Y + R$ with $\mathrm{dm}(g) = [Y', Y'']_{k_2}$ form a bijection $h \colon \mathsf{G} \longrightarrow \mathsf{G}'$ such that $h([Z, Y]_k) = [f(Z), g(Y)]_k$. The pencil G' has the same type as G since $\mathrm{dm}(f), \mathrm{rg}(f)$, and $\mathrm{dm}(g), \mathrm{rg}(g)$ are similar. □

Lemma 6.4. *Let p_1, p_2 be adjacent lines such that $\mathsf{G} = \overline{p_1, p_2}$ is a wafer. If q_1, q_2 are lines such that q_i crosses p_j for $i, j = 1, 2$, then q_1, q_2 determine a wafer.*

Proof. By 2.10(iii) and 2.8 we find that $q_1 \lhd\!|\!\rhd q_2$, then by 2.6 q_1, q_2 are adjacent since if they lie on a strong subspace, lines p_1, p_2 would also lie on a strong subspace.

Now, suppose that $W_1 \in q_1$, $W_2, U_2 \in q_2$, $W_2 \ne U_2$ and $W_1 \sim W_2, U_2$. We can assume that $p_1' := \overline{W_1, W_2} \in \mathsf{G}$. Through U_2 there goes a line $p_2' \in \mathsf{G}$ by 2.10(iii). Note that $\mathsf{G} = \overline{p_1', p_2'}$. Since $U_2 \sim W_1, W_2$ the contradiction arises with 2.9(i). □

In projective geometry Steiner's construction of a quadric is well known. The same idea we can utilize to construct wafers.

Proposition 6.5. *Let p_1, p_2 be adjacent lines such that $\mathsf{G} = \overline{p_1, p_2}$ is a wafer. Then stars $S_i = \boldsymbol{S}(p_i)$ as well as tops $T_i = \boldsymbol{T}(p_i)$ determine proper pencils S, T respectively, and*

$$\overline{p_1, p_2} = \{S \cap f(S) \colon S \in \mathsf{S}\},$$

where $f = \xi^q_{\mathsf{T}} \circ \xi^{\mathsf{S}}_q$ for some line q that crosses S and T.

Proof. Stars S_1, S_2 are adjacent and tops T_1, T_2 are adjacent since lines p_1, p_2 are adjacent. This suffices to have proper pencils S, T respectively.

Let $U_1 \in p_1$. By 2.9(i) there is a unique $U_2 \in p_2$ collinear with U_1. Set $q = \overline{U_1, U_2}$. Consider a line q' through $U_1' \in p_1$, $U_1' \ne U_1$, taken analogously as q. Note that q, as well as q', crosses S and T properly, hence by 6.1 there are suitable projections ξ^q_{T}, ξ^{S}_q. Their composition $f = \xi^q_{\mathsf{T}} \circ \xi^{\mathsf{S}}_q$ is meaningful and $p_i = S_i \cap f(S_i)$. Moreover, since G is a wafer and q crosses G there is a projection ξ^{G}_q by 6.2.

Let $p \in \mathsf{G}$. Consider $U = \xi^{\mathsf{G}}_q(p)$ and take $S = \xi^q_{\mathsf{S}}(U)$, $T = \xi^q_{\mathsf{T}}(U)$. By 2.8 q' crosses S in U'. Then $U \sim U'$ and by 6.4 q, q' determine a wafer, hence $U' \in p$. In consequence $p \subseteq S$. Similarly $p \subseteq T$, hence $S \cap T = p$. Clearly $T = f(S)$. □

Following another idea of projective geometry we introduce a concept of *perspectivity* to be a map $\xi^{\mathsf{G}}_q \circ \xi^p_{\mathsf{G}}$, where p, q are lines and G a pencil of segment subspaces in $\mathbf{P}_k(V)$. Compositions of such perspectivities are already known projectivities.

Lemma 6.6. *If G is a pencil of segments and p, q are lines such that ξ^{G}_q, ξ^p_{G} are reasonable projections, then $\xi^{\mathsf{G}}_q \circ \xi^p_{\mathsf{G}} \in \Omega(k)$.*

Proof. Let $U \in p$ and $\mathcal{X} = [Z, Y]_k \in \mathsf{G}$. Assume that G is a proper pencil with $Z' = Z'' = Z$. Then $\xi^p_{\mathsf{G}}(U) = [Z, U + Y']_k$ and $\xi^{\mathsf{G}}_q(\mathcal{X}) = Y \cap B$, where $q = \mathbf{p}(H, B)$, by 6.1(i). Hence $\xi^{\mathsf{G}}_q \circ \xi^p_{\mathsf{G}}(U) = (U + Y') \cap B$.

Now, assume that G is a wafer. By 6.2 we have $\xi^p_{\mathsf{G}}(U) = [U \cap Z'', U + Y']_k$ and $\xi^{\mathsf{G}}_q(\mathcal{X}) = Z + H = Y \cap B$. Hence, $\xi^{\mathsf{G}}_q \circ \xi^p_{\mathsf{G}}(U) = (U \cap Z'') + H = (U + Y') \cap B$.

In both cases $\xi^{\mathsf{G}}_q \circ \xi^p_{\mathsf{G}}$ is a composition of shift projections. □

Lemma 6.7. *For every slide $f = \overset{p_1}{\underset{p_2}{\updownarrow}}$ or a central projection $f = \overset{p_1}{\underset{p_2}{\updownarrow}}{}_C$, and arbitrary δ such that $\mathbf{P}_k(V)$ contains segments of dimension δ, there is a pencil G of segments $\mathcal{X}$ with $\operatorname{pdim}\mathcal{X} = \delta$ such that $f = \xi^{\mathsf{G}}_{p_2} \circ \xi^{p_1}_{\mathsf{G}}$.*

Proof. Let U_1, U_2 be distinct points on p_1, and W_1, W_2 points on p_2 such that $U_i \sim W_i$. We consider lines $q_i = \overline{U_i, W_i}$.

In case where f is a slide, p_1, p_2 determine a wafer and by 6.4 lines q_i determine a wafer, which can be extended to a wafer G of segments of dimension δ by 6.3.

If f is a central projection, then p_1, p_2 determine a proper pencil and lines q_1, q_2 can be extended to some proper pencil of segments of dimension δ by 6.3.

It is seen that $\xi^{\mathsf{G}}_{p_2}, \xi^{p_1}_{\mathsf{G}}$ are well defined in both cases and $f = \xi^{\mathsf{G}}_{p_2} \circ \xi^{p_1}_{\mathsf{G}}$. □

Proposition 6.8. *$h \in \Omega(k; (1, 1))$ iff h is a composition of perspectivities.*

Proof. $\Rightarrow$: by 6.7 and 5.9 as ξ is a composition of slides and central projections.
$\Leftarrow$: by 6.6. □

Proposition 6.9. *Let p_1, p_2 be lines of $\mathbf{P}_k(V)$ and $h\colon p_1 \longrightarrow p_2$ such that $h = gf$ for some shifts f, g. Then h is a perspectivity.*

Proof. Let us drop the trivial case where $f = g = \mathrm{id}$. Then either, f is a J-projection and g is an M-projection, or conversely, as $k < m$ or $m < k$. Both cases are mutually dual so, we investigate the first one. Let $p_i = \mathbf{p}(H_i, B_i)$ and $f = \Big\downarrow^{t}_{p_2}$, $g = \Big\downarrow^{p_1}_{t}$ for some line t of $\mathbf{P}_m(V)$. Assume that $f = U \leadsto U + Q$ and $g = U \leadsto U \cap R$. Then $H_2 = (H_1 + Q) \cap R$ and $B_2 = (B_1 + Q) \cap R$. Consider a pencil G of segments $[\Theta, Y]_k$ where $Y \in t$. Observe that $Y = U + Q$ for some $U \in p_1$. It is seen that lines p_i cross all elements of G. Hence we have projections $\xi^{\mathsf{G}}_{p_2}, \xi^{p_1}_{\mathsf{G}}$ such that $\xi^{p_i}_{\mathsf{G}}(U_i) = [\Theta, U_1 + Q]_k$, for $U_i \in p_i$, which proves our claim. □

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