Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 46 (2005), No. 2, 581-586.

On p-hyperelliptic Involutions of Riemann Surfaces

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Abstract. A compact Riemann surface X of genus $g > 1$ is said to be p-hyperelliptic if X admits a conformal involution ρ, called a p-hyperelliptic involution, for which X/ρ is an orbifold of genus p. Here we give a new proof of the well known fact that for $g > 4p+1$, ρ is unique and central in the group of all automorphisms of X. Moreover we prove that every two p-hyperelliptic involutions commute for $3p+2 \leq g \leq 4p+1$ and X admits at most two such involutions if $g > 3p+2$. We also find some bounds for the number of commuting p-hyperelliptic involutions and general bound for the number of central p-hyperelliptic involutions.

Keywords: p-hyperelliptic Riemann surfaces, automorphisms of Riemann surfaces, fixed points of automorphisms

1. Introduction

A Riemann surface $X = \mathcal{H}/\Gamma$ of genus $g \geq 2$ is said to be *p-hyperelliptic* if X admits a conformal involution ρ, called a *p-hyperelliptic involution*, such that X/ρ is an orbifold of genus p. This notion has been introduced by H. Farkas and I. Kra in [1] where they also proved that for $g > 4p+1$, p-hyperelliptic involution is unique and central in the group of all automorphisms of X. We prove these facts in a combinatorial way using the Hurwitz-Riemann formula and certain theorem of Macbeath [2] about fixed points of an automorphism of X; the Hurwitz-Riemann formula asserts that a p-hyperelliptic involution has $2g+2-4p$ fixed points. The advantage of our approach is that it allows us to study of p-hyperelliptic involutions in case $g \leq 4p+1$ also. First we show that for g in range $3p+2 \leq g \leq 4p+1$, every two p-hyperelliptic involutions commute and afterwards we argue that X admits at most two such involutions for $3p+2 < g \leq 4p+1$ and at most 6 for $g = 3p+2$. Finally we find some bounds for the number of commuting p-hyperelliptic involutions and general bound for the number of central p-hyperelliptic involutions.

2. Preliminaries

We shall approach the problem using Riemann uniformization theorem by which each compact Riemann surface X of genus $g \geq 2$ can be represented as the orbit space of the hyperbolic plane $\mathcal{H}$ under the action of some Fuchsian surface group Γ. Furthermore a group G of automorphisms of a surface $X = \mathcal{H}/\Gamma$ can be represented as $G = \Lambda/\Gamma$ for another Fuchsian group Λ. Each Fuchsian group Λ is given a signature $\sigma(\Lambda) = (g; m_1, \ldots, m_r)$, where g, m_i are integers verifying $g \geq 0, m_i \geq 2$. The signature determines the presentation of Λ:

generators: $x_1, \ldots, x_r, a_1, b_1, \ldots, a_g, b_g$,
relations : $x_1^{m_1} = \cdots = x_r^{m_r} = x_1 \cdots x_r [a_1, b_1] \cdots [a_g, b_g] = 1.$

Such set of generators is called the *canonical set of generators* and often, by abuse of language, the set of *canonical generators*. Geometrically x_i are elliptic elements which correspond to hyperbolic rotations and the remaining generators are hyperbolic translations. The integers $m_1, m_2, \ldots, m_r$ are called the *periods* of Λ and g is the genus of the orbit space $\mathcal{H}/\Lambda$. Fuchsian groups with signatures $(g; -)$ are called *surface groups* and they are characterized among Fuchsian groups as these ones which are torsion free.

The group Λ has associated to it a fundamental region whose area $\mu(\Lambda)$, called the *area of the group*, is:

$$\mu(\Lambda) = 2\pi \left(2g - 2 + \sum_{i=1}^{r} (1 - 1/m_i) \right). \tag{1}$$

If Γ is a subgroup of finite index in Λ, then we have the *Riemann-Hurwitz formula* which says that

$$[\Lambda : \Gamma] = \frac{\mu(\Gamma)}{\mu(\Lambda)}. \tag{2}$$

The points of $\mathcal{H}$ with non-trivial stabilizers in Λ fall into r Λ-orbits $o_1, \ldots, o_r$ such that every point belonging to o_i has a stabilizer which is a cyclic group of order m_i. The points of X with non-trivial stabilizers fall into r G-orbits $O_1, \ldots, O_r$, where $O_i = \pi(o_i)$ and $\pi : \mathcal{H} \to X$ is a projection map. Furthermore a homomorphism $\theta : \Lambda \to G$ induces an isomorphism between stabilizers and so the stabilizer of $y \in O_i$ is cyclic of order m_i. The number of fixed points of an automorphism of X can be calculated by the following theorem of Macbeath [2].

Theorem 2.1. *Let $X = H/\Gamma$ be a Riemann surface with the automorphism group $G = \Lambda/\Gamma$ and let $x_1, \ldots, x_r$ be elliptic canonical generators of Λ with periods $m_1, \ldots, m_r$ respectively. Let $\theta : \Lambda \to G$ be the canonical epimorphism and for $1 \neq g \in G$ let $\varepsilon_i(g)$ be 1 or 0 according as g is or is not conjugate to a power of $\theta(x_i)$. Then the number $F(g)$ of points of X fixed by g is given by the formula*

$$F(g) = |N_G(\langle g \rangle)| \sum_{i=1}^{r} \varepsilon_i(g)/m_i. \tag{3}$$

3. On p-hyperelliptic involutions of Riemann surfaces

Here we deal with the number of p-hyperelliptic involutions which a Riemann surface can admit. Along the chapter X is a p-hyperelliptic Riemann surface of genus $g \geq 2$ and we call its p-hyperelliptic involutions briefly by *p-involutions*. First we give a new proof of the well known result of H. Farkas and I. Kra.

Theorem 3.1. *A p-involution of a surface X of genus $g > 4p+1$ is unique and central in the full automorphism group of X.*

Proof. Suppose that a Riemann surface $X = \mathcal{H}/\Gamma$ admits two distinct p-involutions ρ and ρ'. Then they generate a dihedral group G, say of order $2n$ and there exist a Fuchsian group Λ and an epimorphism $\theta : \Lambda \to G$ with the kernel Γ. If x_i is a canonical elliptic generator of Λ corresponding to some period $m_i > 2$ then $\theta(x_i) \in \langle \rho\rho' \rangle$. But none conjugation of ρ nor of ρ' belongs to $\langle \rho\rho' \rangle$ and so in terms of Macbeath's theorem $\varepsilon_i(\rho) = \varepsilon_i(\rho') = 0$.

Now if n is odd then $|N_G(\langle \rho \rangle)| = 2$ and $F(\rho) = 2g+2-4p$ implies that Λ has $2g+2-4p$ periods equal to 2. If n is even then $|N_G(\langle \rho \rangle)| = 4$ and so $g+1-2p$ canonical elliptic generators are mapped by θ onto conjugates of ρ. Similarly another $g+1-2p$ canonical elliptic generators are mapped by θ onto conjugates of ρ'. So in both cases $\sigma(\Lambda) = (\gamma; 2, \overset{s}{\ldots}, 2, m_{s+1}, \ldots, m_r)$, for $s = 2g+2-4p$ and some integer $r \geq s$. Now applying the Hurwitz-Riemann formula for (Λ, Γ), we obtain $2g-2 = 2n(2\gamma - 2 + g + 1 - 2p + \sum_{i=s+1}^{r}(1 - 1/m_i))$ which implies

$$g - 1 \geq n(g - 1 - 2p). \tag{4}$$

Since $n \geq 2$, it follows that $g \leq 4p+1$. Thus for $g > 4p+1$ a p-involution is unique.

Now given $g \in G$, $g\rho g^{-1}$ has the same number of fixed points as ρ. So by the Hurwitz-Riemann formula it is also a p-involution which implies that $g\rho g^{-1} = \rho$ for $g > 4p+1$. □

Theorem 3.2. *Every two p-involutions of a Riemann surface X of genus $3p+2 \leq g \leq 4p+1$ commute. Moreover for $3p+2 < g \leq 4p+1$, X can admit two and no more such involutions.*

Proof. Let X be a Riemann surface of genus $3p+2 \leq g \leq 4p+1$. If X admits two p-involutions then they generate the group $\mathrm{D}_n = \Lambda/\Gamma$ for some n satisfying the inequality (4), which implies

$$n \leq 1 + \frac{2p}{g - 1 - 2p}. \tag{5}$$

Thus $n = 2$ and so every two p-involutions of X commute. Moreover their product cannot be a p-involution. Otherwise, by Theorem 2.1, Λ would have the signature $(\gamma; 2, \overset{3(g+1-2p)}{\ldots}, 2)$ and applying the Hurwitz-Riemann formula for (Λ, Γ) we would obtain $2\gamma = 3p - g$ and consequently $g \leq 3p$, a contradiction. So if X admits three p-involutions ρ_1, ρ_2, ρ_3 then they generate the group $G = Z_2 \oplus Z_2 \oplus Z_2$ which can be identified with Δ/Γ for some Fuchsian group Δ with a signature $(\delta; 2, \overset{r}{\ldots}, 2)$. Let $\theta : \Delta \to G$ be the canonical epimorphism and let s_k denote the number of elliptic generators of Δ which are transformed by θ onto ρ_k, for $k = 1, 2, 3$. Then by Theorem 2.1, $s_k = (g+1-2p)/2$ for $k = 1, 2, 3$ and so applying the Hurwitz-Riemann formula for (Δ, Γ) we obtain $2g - 2 = 8(2\delta - 2 + 3(g+1-2p)/4 + t/2)$, where $t = r - 3(g+1-2p)/2$. Thus $\delta = (2+3p-g-t)/4 \geq 0$ if and only if $g \leq 3p+2$. Consequently a surface X of genus $3p+2 < g \leq 4p+1$ admits at most two p-involutions.

Now we shall prove that Riemann surfaces of such genera with two p-involutions actually exist. For, let Δ be a Fuchsian group with the signature $(0; 2, \overset{r}{\ldots}, 2)$, where $r = g+3$ and let us define an epimorphism $\theta : \Delta \to Z_2 \oplus Z_2 = \langle \rho \rangle \oplus \langle \rho' \rangle$ by the assignment $\theta(x_1) = \cdots = \theta(x_s) = \rho, \theta(x_{s+1}) = \cdots = \theta(x_{2s}) = \rho', \theta(x_{2s+1}) = \cdots = \theta(x_r) = \rho\rho'$, where $s = g+1-2p$. Since s and $r-2s$ have the same parities, it follows that the relation $\theta(x_1)\cdots\theta(x_r) = 1$ holds. Moreover by Theorem 2.1, $F(\rho) = F(\rho') = 2g+2-4p$ and so by the Hurwitz-Riemann formula, ρ and ρ' are two commuting p-involutions. □

Proposition 3.3. *Let $\rho_1, \ldots, \rho_l$ be pairwise commuting p-involutions of a surface X of genus g and let they generate the group $G_k = Z_2 \oplus \overset{k}{\ldots} \oplus Z_2$, where $l \geq k$. Then*

(i) $g \equiv 1\,(2^{k-2})$ *and* $p \equiv 1\,(2^{k-3})$,

(ii) *the integers k and l are limited in the following cases:*

$$\begin{array}{llll} k \leq 2 & \textit{and} \;\; l \leq 3 & \textit{if} & g \equiv 0\,(2) \\ k \leq 3 & \textit{and} \;\; l \leq 4 & \textit{if} & p \equiv 0\,(2) \\ k \leq 3 & \textit{and} \;\; l \leq 7 & \textit{if} & g \equiv 3\,(4) \\ k \leq 4 & \textit{and} \;\; l \leq 15 & \textit{if} & p \equiv 3\,(4). \end{array}$$

Proof. (i) Suppose that pairwise commuting p-involutions of a Riemann surface X generate a group $G_k = Z_2 \oplus \overset{k}{\ldots} \oplus Z_2$. Then G_k can be identified with Δ/Γ for a Fuchsian group Δ with the signature $(\gamma; 2, \overset{r}{\ldots}, 2)$. Applying the Hurwitz-Riemann formula for (Δ, Γ) we obtain $g - 1 = 2^{k-2}(4\gamma - 4 + r)$ which implies that $g \equiv 1\,(2^{k-2})$. Furthermore, by Theorem 2.1, a p-involution $\rho \in G_k$ admits fixed points in $(g+1-2p)/2^{k-2}$ orbits and so in particular $g+1-2p \equiv 0\,(2^{k-2})$. Consequently $p \equiv 1\,(2^{k-3})$.

(ii) The restrictions for k are direct consequence of the conditions from (i). We need only to show that for even p, the group G_3 can admit at most 4 p-involutions. For, let us suppose that the product of two p-involution $\rho_1, \rho_2 \in G_3$ is a p-involution. Then they generate the group G_2 isomorphic with Λ/Γ, where Λ is a Fuchsian group with the signature $(\delta; 2, \overset{3(g+1-2p)}{\ldots}, 2)$. Thus $\delta = (3p-g)/2$ and so $3p - g \equiv 0\,(2)$. However p is even and g is odd which implies that $3p - g$ is odd, a contradiction. Consequently in this case G_3 may admit only one more p-involution, namely $\rho_1\rho_2\rho_3$ and so $l \leq 4$. □

By Proposition 3.3, the number of pairwise commuting p-involutions corresponding to given p is limited for $p \equiv 0\,(2)$ or $p \equiv 3\,(4)$. The next proposition give a bound for such number for $p \equiv 1\,(4)$.

Proposition 3.4. *Let $p = 1 + 2^m\alpha$, where α is odd and $m \geq 2$. Then the number of pairwise commuting p-involutions of a Riemann surface X of genus $g \neq 2p-1$ does not exceed $2^n\alpha + 5$, where n is the least integer in range $0 \leq n \leq m+2$ such that $2^n\alpha \geq m - n - 1$.*

Proof. Given such p, let X be a Riemann surface whose pairwise commuting p-involutions generate $G_k = Z_2 \oplus \overset{k}{\ldots} \oplus Z_2$. Then by Proposition 3.3, $k \leq m+3$. So let us write $k = m+3-n$ for some integer n in range $0 \leq n \leq m+2$ and let $G_k = \Delta/\Gamma$ for a Fuchsian group Δ with a signature $(\gamma; 2, \overset{r}{\ldots}, 2)$. Since no single G_k-orbit contains fixed points of two different p-involutions, it follows that $r \geq ks$, where s is the number of G_k-orbits containing fixed points of a single p-involution. In order to check the greatest value of k, we consider the minimum value of s and the maximum value of r. Thus we take $s = 1$ and $\gamma = 0$. By Theorem 2.1, $s = (g + 1 - 2p)/2^{k-2}$ and so $s = 1$ for $g = 1 + 2^{m+1-n} + 2^{m+1}\alpha$. But the Hurwitz-Riemann formula for such g and $\gamma = 0$ gives $r = 2^n\alpha + 5$ which clearly limits the number of p-involutions in G_k . Since for $s = 1$, the epimorphism $\theta : \Delta \to G_k$ cannot be defined for $r < k + 1$, it follows that n is the least integer satisfying the inequality $2^n\alpha \geq m - n - 1$. □

Proposition 3.5. *Let X be a p-hyperelliptic Riemann surface of genus $g = 3p+2$. Then X admits at most 2 p-involutions if $p \equiv 0\,(2)$ or $p \equiv 3\,(4)$ and at most 3 if $p \equiv 1\,(4)$ and $p > 5$. For $p = 1$ or $p = 5$, X can admit 5 or 6 and no more p-involutions respectively.*

Proof. By Theorem 3.2, all p-involutions of a Riemann surface of genus $g = 3p + 2$ commute one to each other and so they generate the group $G_k = Z_2 \oplus \overset{k}{\ldots} \oplus Z_2$ for some k. Let $G_k = \Delta/\Gamma$ for some Fuchsian group Δ, say with a signature $(\gamma; 2, \overset{r}{\ldots}, 2)$. Denote by s_k the number of G_k-orbits containing the fixed points of a single p-involution from G_k. By Theorem 2.1, $s_k = (g + 1 - 2p)/2^{k-2} = (p + 3)/2^{k-2}$. Thus $k \leq 2$ for p even and $k \leq 3$ and s_k is odd for $p \equiv 3\,(4)$. However, by the Hurwitz-Riemann formula for $k = 3$ and (Δ, Γ), we have $2\gamma + r - 3s_3 = 0$, which implies $\gamma = 0$ and $r = 3s_3$ in virtue of obvious $r \geq 3s_3$. Therefore, for $p \equiv 3\,(4)$, an epimorphism $\theta : \Delta \to G_3$ actually can not exist. Consequently $k \leq 2$ if $p \equiv 0\,(2)$ or $p \equiv 3\,(4)$. Furthermore X admits at most 2 p-involutions in these cases since, by the proof of the Theorem 3.2, a product of two p-involutions cannot be a p-involution for $g > 3p$.

Now let $p \equiv 1\,(4)$. First we shall show that $k \leq 5$ and that surfaces whose p-involutions generate G_4 or G_5 exist only for $p \leq 5$. For, let us write $p = 4\alpha + 1$ for some integer α. Then $g = 1 + 4(1 + 3\alpha)$ and $s_k = (\alpha + 1)/2^{k-4}$. Let n and m be the greatest integers such that $g \equiv 1\,(2^n)$ and $p \equiv 1\,(2^m)$. Then for even α, we have $n = 2$ which by (i) of the Proposition 3.3 implies $k \leq 4$ and for odd α, $m = 2$ and consequently $k \leq 5$.

Now let $t = r - ks_k$. Applying the Hurwitz-Riemann formula for (Δ, Γ) and $k = 4$, we obtain $1 = 4\gamma + \alpha + t$. Thus $\gamma = 0$ and either $\alpha = 1, r = 4s_4$ or $\alpha = 0, r = 4s_4 + 1$. Consequently $p = 5$, $s_4 = 2$ and $\sigma(\Delta) = (0; 2, 2, 2, 2, 2, 2, 2, 2)$ or $p = 1$, $s_4 = 1$ and $\sigma(\Delta) = (0; 2, 2, 2, 2, 2)$. So there exists exactly one possible epimorphism $\theta : \Delta \to G_4$ whose image is generated by p-involutions and it is given by the assignment

$$\theta(x_i) = \rho_j \text{ for } 1 \leq j \leq k, (j - 1)s_k < i \leq js_k, \tag{6}$$

in the first case and by the assignment

$$\theta(x_i) = \rho_j,\ \theta(x_{ks_k+1}) = \rho_1 \cdots \rho_k \text{ for } 1 \leq j \leq k, (j - 1)s_k < i \leq js_k \tag{7}$$

in the second one, where $k = 4$. Thus the surface whose p-involutions generate G_4 exists only for $p = 1$ or $p = 5$ and the corresponding group G_4 admits exactly five or four p-involutions respectively.

Similarly for $k = 5$ we obtain $4\gamma+\alpha+t = 2$. Since for even α we have $k \leq 4$, it follows that $\alpha = 1$, $\gamma = 0$ and $r = 5s_5 + 1$. Thus $p = 5$, $s_5 = 1$ and Δ has the signature $(0; 2, 2, 2, 2, 2, 2)$. Now the assignment (7) defines the only possible epimorphism onto G_5. Thus the surface whose p-involutions generate G_5 exists only for $p = 5$ and the corresponding group G_5 admits exactly six 5-involutions.

Summing up, for $p > 5$ and $p \equiv 1\,(4)$ we have $k \leq 3$. However, from the first part of the proof s_3 is even and Δ has the signature $(0; 2, \overset{3s_3}{\ldots}, 2)$. Thus the assignment (6) for $k = 3$, defines the only possible epimorphism $\Delta \to G_3$ whose image is generated by p-involutions and so the group G_3 contains exactly 3 p-involutions. $\square$

Let us notice that for arbitrary positive integer $k \geq 5$, we can find integers p and g such that there exists a Riemann surface of genus g admitting k pairwise commuting p-involutions. Indeed for $g = 1+(k-4)2^{k-3}$ and $p = 1+(k-5)2^{k-4}$ we can take a Fuchsian group Δ with the signature $(0; 2, \overset{k}{\ldots}, 2)$ and define an epimorphism $\theta : \Delta \to Z_2 \oplus \overset{k-1}{\ldots} \oplus Z_2 = \langle\rho_1\rangle \oplus \cdots \oplus \langle\rho_{k-1}\rangle$ by the assignment $\theta(x_i) = \rho_i$ for $i = 1, \ldots, k-1$ and $\theta(x_k) = \rho_1 \cdots \rho_{k-1}$. Then $\Gamma = \ker\theta$ is a surface Fuchsian group of orbit genus g and ρ_i are p-involutions of a Riemann surface $X = \mathcal{H}/\Gamma$.

At the end of the paper we give a bound for the number of all central p-involutions of a surface X.

Theorem 3.6. *Let X be a p-hyperelliptic Riemann surface of genus $g \geq 2$ and let G be its automorphism group of order $2N$. Assume that the canonical projection $X \to X/G$ is ramified at r points with multiplicities $m_1, \ldots, m_r$. Then for $g \neq 2p-1$, the number of central p-involutions of X does not exceed*

$$(N \sum_{i=1}^{r} 1/m_i)/(g + 1 - 2p).$$

Proof. Here $X = \mathcal{H}/\Gamma$ for some Fuchsian surface group Γ with the signature $(g; -)$ and $G = \Delta/\Gamma$ for some Fuchsian group Δ with the signature $(\delta; m_1, \ldots, m_r)$. Let $x_1, \ldots x_r$ be canonical elliptic generators of Δ and let $\theta : \Delta \to G$ be the canonical epimorphism. Assume that X admits a central p-involution ρ. If $g \neq 2p-1$ then ρ has fixed points and so it is conjugate to $\theta(x_i)^{m_i/2}$ for some x_i corresponding to an even period m_i. However since ρ is central, it follows that actually $\rho = \theta(x_i)^{m_i/2}$. In particular for distinct p-involutions ρ and ρ', $\varepsilon_i(\rho) \neq \varepsilon_i(\rho')$. Moreover by Theorem 2.1, $N\sum_{i=1}^{r} \varepsilon_i(\rho)/m_i = g+1-2p = N\sum_{i=1}^{r} \varepsilon_i(\rho')/m_i$. Thus if n is the number of all p-involutions of X then $n(g+1-2p) \leq N\sum_{i=1}^{r} 1/m_i$ and so the theorem is proved. $\square$

Acknowledgement. The author wishes to thank the referee for his comments and suggestion.

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Received August 2, 2004