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Transnormal Partial Tubes

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Abstract. We construct transnormal partial tubes about transnormal submanifolds. Particular cases are transnormal embeddings of a torus which are not product embeddings.

1. Introduction and basic definitions

Throughout this paper M will denote a compact, connected, smooth (C^∞) m-dimensional manifold without boundary and $f : M \longrightarrow \mathbf{R}^n$ will be a smooth embedding of M into Euclidean n-space. For each $p \in M$, let $\nu(p)$ denote the affine $(n-m)$-plane which is normal to $f(M)$ at $f(p)$. The total space of the normal bundle is $\mathcal{N} = \{(p, x) \in M \times \mathbf{R}^n : f(p) + x \in \nu(p)\}$ and we let Σ denote the set of singularities of the endpoint map $\eta : \mathcal{N} \longrightarrow \mathbf{R}^n$ defined by $\eta(p, x) = f(p) + x$.

We are going to consider a class of subbundles of $\mathcal{N}$ called partial tubes. Let $\mathcal{P} \subset \mathcal{N}$ be a smooth subbundle with type fibre S, where S is a compact smooth submanifold of $\mathbf{R}^{n-m}$, $\mathcal{P} \cap \Sigma = \emptyset$ and $\mathcal{P}$ is invariant under parallel transport of normals along any curve in M. Then $\mathcal{P}$ is a compact smooth manifold and $g \equiv \eta|\mathcal{P} : \mathcal{P} \longrightarrow \mathbf{R}^n$ is a smooth immersion called a *partial tube* about f [3]. Since $\mathcal{P}$ is compact, if $\mathcal{P}$ lies in a sufficiently small neighbourhood of the zero section of $\mathcal{N}$, then $g : \mathcal{P} \longrightarrow \mathbf{R}^n$ is an embedding.

The embedding $f : M \longrightarrow \mathbf{R}^n$ is said to be *transnormal* (and $f(M)$ is a transnormal submanifold of $\mathbf{R}^n$) if for all $p, q \in M$, $f(q) \in \nu(p)$ implies $\nu(q) = \nu(p)$. Then $\varphi(p) = \nu(p) \cap f(M)$ is called the *generating frame* at p. It can be shown that, for all $p, q \in M$,

$\varphi(p)$ is isometric to $\varphi(q)$ [4]. If the number of elements of $\varphi(p)$ is r, then f is said to be *r-transnormal*. The idea of transnormality was introduced by S. A. Robertson in [4]. See [5] for a survey article.

So suppose that $f : M \longrightarrow \mathbf{R}^n$ is a transnormal embedding and for $p \in M$ let $\nu^\varphi(p)$ denote the smallest affine subplane of $\nu(p)$ which contains $\varphi(p)$. Let $\mathcal{N}^\varphi = \{(p, x) \in \mathcal{N} : f(p) + x \in \nu^\varphi(p)\}$ and put $\mathcal{N}' = (\mathcal{N}^\varphi)^\perp$, the orthogonal complement of $\mathcal{N}^\varphi$ in $\mathcal{N}$. Then $\mathcal{N}^\varphi$ is invariant under the normal holonomy [6] and hence so is $\mathcal{N}'$.

We are going to consider partial tubes $g : \mathcal{P} \longrightarrow \mathbf{R}^n$ about a transnormal embedding, where $\mathcal{P}$ is a subbundle of $\mathcal{N}'$. Some examples of these are considered by B. Wegner in [7]. He started with a 2-transnormal embedding f of $\mathbf{S}^1$ in $\mathbf{R}^4$ given by $f(t) = (\cos t, \sin t, R\cos 3t, R\sin 3t)$, $0 < R < \frac{1}{3}$. The image of f lies on a sphere and the generating frame at t is $\{f(t), f(t+\pi)\}$. Then $\nu^\varphi(t)$ is the radial normal line $\{sf(t) : s \in \mathbf{R}\}$. The normal holonomy map of this curve is a rotation, by an angle depending on R, around the radial normal. If n is a normal to f, perpendicular to the radial normal and with sufficiently small length, then the orbit of n, under the normal holonomy group, is a transnormal curve parallel to f, say $h(t) = f(t) + n(t)$. For any positive integer r, R can be chosen so that h is $2r$-transnormal. This $2r$-transnormal curve can be thought of as a partial tube with type fibre $\{h(0), h(2\pi), \ldots, h(2(r-1)\pi)\}$, the set of vertices of a regular r-gon. See also [2] for descriptions of the generating frames of such curves.

The construction we describe in the next section will generalize some of these examples but we will take the type fibre of the partial tube to be a compact connected transnormal submanifold with dimension ≥ 1.

2. A construction of transnormal partial tubes

In this section $f : M \longrightarrow \mathbf{R}^n$ will be a transnormal embedding and $g : \mathcal{P} \longrightarrow \mathbf{R}^n$ will be an embedding as a partial tube about f, where $\mathcal{P}$ is a subbundle of $\mathcal{N}'$. For $p \in M$, S_p will denote the image of the fibre of $\mathcal{P}$ at p, so $S_p = g(\mathcal{P} \cap (\{p\} \times \mathbf{R}^n))$.

For $p \in M$, put $\hat{\varphi}(p) = f^{-1}(\varphi(p))$ and for $u \in \mathbf{R}^n$ let $\tau_u : \mathbf{R}^n \longrightarrow \mathbf{R}^n$ denote the translation $\tau_u(x) = x + u$. We will say that g has *translational symmetry* if $\forall p \in M, \forall q \in \hat{\varphi}(p)$, $\tau_{f(p)-f(q)}(S_q) = S_p$.

Theorem 1. *Let $f : M \longrightarrow \mathbf{R}^{m+d+k}$ be an r-transnormal embedding of a compact, connected manifold M, where $m \geq 1$, $d = \dim \nu^\varphi(p) \geq 1$ (for any $p \in M$) and $k \geq 2$. Let S be a compact connected s-transnormal submanifold of $\mathbf{R}^k$ and let $g : \mathcal{P} \longrightarrow \mathbf{R}^{m+d+k}$ be an embedding as a partial tube about f, where $\mathcal{P} \subset \mathcal{N}'$ and has type fibre S, and g has translational symmetry. Then, if $\mathcal{P}$ lies in a sufficiently small neighbourhood of the zero section of $\mathcal{N}$, g is an rs-transnormal embedding.*

We first prove a lemma.

Lemma 1. *Let $f : M \longrightarrow \mathbf{R}^n$ be a transnormal embedding and, for $\epsilon > 0$, let $\mathcal{B}_\epsilon = \{(p, x) \in \mathcal{N} : ||f(p) - x|| < \epsilon\}$. There exists $\epsilon > 0$ such that $\eta|\mathcal{B}_\epsilon : \mathcal{B}_\epsilon \longrightarrow \mathbf{R}^n$ is an embedding and $\forall p \in M$, $\forall (q, y) \in \mathcal{B}_\epsilon$, $\eta(q, y) \in \nu(p)$ iff $q \in \hat{\varphi}(p)$.*

Proof. Since M is compact we can take ρ sufficiently small so that $\eta|\mathcal{B}_\rho$ is an embedding. Then $q \in \hat{\varphi}(p)$ implies that $\nu(q) = \nu(p)$ and hence $\forall (q, y) \in \mathcal{B}_\rho$, $\eta(q, y) \in \nu(q) = \nu(p)$.

To see the converse, for each $p \in M$, let $\Pi_p : \mathbf{R}^n \longrightarrow \nu(p)$ be orthogonal projection and define $\Pi : M \times M \longrightarrow \mathcal{N}$ by $\Pi(p,q) = (p, \Pi_p(f(q)) - f(p))$. Then as Π is continuous, $\Pi^{-1}(\mathcal{B}_{\frac{\varrho}{2}})$ is an open set in $M \times M$ which contains the diagonal $\{(p,p) : p \in M\}$.

Let $p_1 \in M$ and let $\hat{\varphi}(p_1) = \{p_1, \ldots, p_r\}$. For $i = 1, \ldots, r$, let V_i be an open neighbourhood of p_i in M such that $V_i \times V_i \subset \Pi^{-1}(\mathcal{B}_{\frac{\varrho}{2}})$. Now $V_i \cap V_j = \varphi$ if $i \neq j$ since $\Pi_{p_i} = \Pi_{p_j}$ as $\nu(p_i) = \nu(p_j)$, and therefore if $p \in V_i \cap V_j$ then $||f(p_i) - f(p_j)|| \leq ||f(p_i) - \Pi_{p_i}(f(p))|| + ||\Pi_{p_j}(f(p)) - f(p_j)|| < \frac{\varrho}{2} + \frac{\varrho}{2} = \rho$. So $(p_i, f(p_j) - f(p_i)), (p_j, 0) \in \mathcal{B}_\rho$ with $\eta(p_i, f(p_j) - f(p_i)) = f(p_j) = \eta(p_j, 0)$, which contradicts $\eta|\mathcal{B}_\rho$ is injective.

Now the map $\nu : M \times M \longrightarrow H_{n-m,n}$, the Grassmannian of affine $(n-m)$-planes in $\mathbf{R}^n$, which assigns to each $p \in M$ the normal $(n-m)$-plane $\nu(p)$, is a covering map to its image [4] and hence $\nu : M \longrightarrow \nu(M)$ is open, where $\nu(M)$ is topologized as a subspace of $H_{n-m,n}$. Thus, $U = \nu^{-1}(\cap_{i=1}^r \nu(V_i))$ is an open neighbourhood of $\hat{\varphi}(p_1) = \nu^{-1}(\nu(p_1))$ in M, and for all $p \in U$, $\hat{\varphi}(p) \subset U$.

Put $V = \cup_{i=1}^r V_i$. Then $U \subset V$, for if $q \in U$, then $\nu(q) \in \nu(V_i)$, $\forall i = 1, \ldots, r$. So there exists $q_i \in V_i$ such that $\nu(q_i) = \nu(q)$. Now $q_1, \ldots, q_r$ are distinct and $q_1, \ldots, q_r \in \hat{\varphi}(q)$ which has r elements. Hence $q = q_i$ for some $i = 1, \ldots, r$ and hence $q \in V_i \subset V$. Put $U_i = U \cap V_i$. Then, for all $i = 1, \ldots, r$, U_i is an open neighbourhood of p_i in M with $U_i \subset V_i$ and $\nu^{-1}(\nu(U_i)) = U$. So, for all $p \in U$, $\hat{\varphi}(p)$ has an element in U_i, $i = 1, \ldots, r$.

Next we show that for all $p \in U$, if there exists $(q,y) \in \mathcal{B}_{\frac{\varrho}{2}}$ with $q \in V$ and $\eta(q,y) \in \nu(p)$, then $q \in \hat{\varphi}(p)$. To see this suppose that $p \in U_i$ and $q \in V_j$. Now there exists $\tilde{p} \in \hat{\varphi}(p) \cap U_j$. So $\eta(q,y) \in \nu(\tilde{p})$, $q \in V_j$ and $\tilde{p} \in U_j \subset V_j$. Therefore, $(\tilde{p}, q) \in \Pi^{-1}(\mathcal{B}_{\frac{\varrho}{2}})$ and hence $||\Pi_{\tilde{p}}(f(q)) - f(\tilde{p})|| < \frac{\varrho}{2}$. As $\eta(q,y) \in \nu(\tilde{p})$ we have $\eta(q,y) = \eta(\tilde{p}, x)$ for some $x \in \mathbf{R}^n$ such that $f(\tilde{p}) + x \in \nu(\tilde{p})$. So we have $\Pi_{\tilde{p}}(f(q)) = f(q)$ or $\Pi_{\tilde{p}}(f(q)) = f(\tilde{p}) + x = f(q) + y$ or $\Pi_{\tilde{p}}(f(q)), f(q)$ and $f(q) + y$ are the vertices of a triangle with a right angle at $\Pi_{\tilde{p}}(f(q))$. Hence $||\Pi_{\tilde{p}}(f(q)) - (f(\tilde{p}) + x)|| \leq ||f(q) - (f(q) + y)|| = ||y|| < \frac{\varrho}{2}$. Therefore $||x|| = ||f(\tilde{p}) - (f(\tilde{p}) + x)|| \leq ||f(\tilde{p}) - \Pi_{\tilde{p}}(f(q))|| + ||\Pi_{\tilde{p}}(f(q)) - (f(\tilde{p}) + x)|| < \frac{\varrho}{2} + \frac{\varrho}{2} = \rho$. Hence $(\tilde{p}, x) \in \mathcal{B}_\rho$. But $(q,y) \in \mathcal{B}_\rho$ and $\eta(\tilde{p}, x) = \eta(q,y)$. As $\eta|\mathcal{B}_\rho$ is injective it follows that $q = \tilde{p} \in \hat{\varphi}(p)$.

The next step is to take an open neighbourhood W of p_1 in M with $\overline{W} \subset U_1$. Put $C = M \setminus V$, so C is compact and $\overline{W} \cap C = \emptyset$. For all $c \in C$, let $\zeta_c = \inf\{||f(c) - \Pi_p(f(c))|| : p \in \overline{W}\} > 0$ and let $\zeta = \min\{\frac{\varrho}{2}, \inf\{\zeta_c, c \in C\}\} > 0$. We show that for all $p \in W$, if there exists $(q,y) \in \mathcal{B}_\zeta$ with $\eta(q,y) \in \nu(p)$, then $q \in \hat{\varphi}(p)$. To see this, if $q \notin V$, then $q \in C$. Now $\eta(q,y) = f(p) + x$ for some $x \in \mathbf{R}^n$ and $||f(q) - (f(p) + x)|| \geq ||f(q) - \Pi_p(f(q))|| \geq \zeta_q \geq \zeta$. But $||f(q) - (f(q) + y)|| = ||y|| < \zeta$, so we must have $q \in V$ and then, as above, $q \in \hat{\varphi}(p)$ since $\zeta \leq \frac{\varrho}{2}$.

Summarizing the situation, for all $p \in M$ we have found an open neighbourhood W_p of p in M and $\epsilon_p > 0$ such that for all $\tilde{p} \in W_p$, if $(q,y) \in \mathcal{B}_{\epsilon_p}$ with $\eta(q,y) \in \nu(\tilde{p})$, then $q \in \hat{\varphi}(\tilde{p})$. Now since M is compact, the open cover of M, $\{W_p : p \in M\}$, has a finite subcover, say $\{W_{q_1}, \ldots, W_{q_k}\}$. Taking $\epsilon = \min\{\epsilon_{q_1}, \ldots, \epsilon_{q_k}\}$ gives the required result. $\square$

Proof of Theorem 1. We will modify our notation in this proof so $\nu_f(p), \nu_g(p,x)$ will denote the normal plane to f at p, g at (p,x) respectively, and $\nu_S(p,x)$ will denote the normal to the submanifold S_p at $g(p,x)$ in $\nu_f'(p)$, the orthogonal complement of $\nu_f^\varphi(p)$ in $\nu_f(p)$ at $f(p)$. Similarly modified notation will be used for generating frames. We are assuming here that f

is 2-transnormal and for all $p \in M$, S_p is a transnormal submanifold in $\nu'_f(p)$.

From the calculations in [3], for $(p,x) \in \mathcal{P}$, $\nu_g(p,x)$ is the $(d+l)$-subplane in $\nu_f(p)$ which is normal to S_p in $\nu_f(p)$ at $g(p,x)$. It is spanned by $\nu_f^{\varphi}(p)$ and $\nu_S(p,x)$. Take $\epsilon > 0$ sufficiently small as in Lemma 1 and $\mathcal{P} \subset \mathcal{B}_\epsilon$. Then $g(q,y) \in \nu_g(p,x) \cap g(\mathcal{P})$ implies $g(q,y) \in \nu_f(p) \cap \eta(\mathcal{B}_\epsilon)$. Therefore, by Lemma 1, $q \in \hat{\varphi}_f(p)$. Since g has translational symmetry, $\tau_{f(p)-f(q)}(g(q,y)) = f(p) + y \in S_p$. Also, as $f(p), f(q) \in \nu_f^{\varphi}(p)$ and $f(q) + y \in \nu_g(p,x) \supset \nu_f^{\varphi}(p)$, it follows that $f(p) + y \in \nu_g(p,x)$. Therefore $f(p) + y \in S_p \cap \nu_g(p,x) = S_p \cap \nu_S(p,x)$, as $S_p \subset \nu'_f(p)$. But $S_p \cap \nu_S(p,x)$ is the generating frame of S_p at $g(p,x)$. Thus, $f(p)+y$ is a point of this generating frame and therefore $\nu_S(p,y) = \nu_S(p,x)$ and hence $\nu_g(p,y) = \nu_g(p,x)$. Now $\nu_g(q,y) = \nu_g(p,y)$ since $\nu_S(q,y) = \tau_{f(q)-f(p)}(\nu_S(p,y))$ and $f(p), f(q) \in \nu_f^{\varphi}(p)$. Thus, $\nu_g(q,y) = \nu_g(p,x)$, so g is transnormal. Further we have shown that $\varphi_g(p,x) = \{f(q) + y : q \in \varphi_f(p), f(p) + y \in \varphi_S(p,x)\}$, so g is rs-transnormal. □

Corollary 1. *Let $f : M \longrightarrow \mathbf{R}^{m+d+k}$ be an r-transnormal embedding of a compact, connected manifold M, where $m \geq 1$, $d = \dim \nu^{\varphi}(p) \geq 1$ (for any $p \in M$) and $k \geq 2$. Then there exists a $2r$-transnormal embedding $g : \mathcal{P} \longrightarrow \mathbf{R}^{m+d+k}$ where $\mathcal{P}$ is a $(k-1)$-sphere bundle over M.*

Proof. In Theorem 1 take the fibre of $\mathcal{P}$ to be a $(k-1)$-sphere, that is, $S_p = \{f(p) + x \in \nu'_f(p) : ||x|| = \epsilon\}$. Then g has translational symmetry since if $q \in \varphi_f(p)$, then $\nu'_f(q)$ is parallel to $\nu'_f(p)$ as both are orthogonal to $\nu_f^{\varphi}(p)$ in $\nu_f(p)$. So the conditions of Theorem 1 are satisfied and g is $2r$-transnormal. □

3. Transnormal tori

Example 1. In Section 1 we described a 2-transnormal embedding f of $\mathbf{S}^1$ in $\mathbf{R}^{1+1+2}$, $f(t) = (\cos t, \sin t, R\cos 3t, R\sin 3t)$, $0 < R < \frac{1}{3}$ [7]. Applying the construction in Corollary 1 to this embedding gives a 4-transnormal embedding of $\mathbf{S}^1 \times \mathbf{S}^1$ in $\mathbf{R}^4$ which is not the product embedding.

Example 2. Here we start with a slight modification to the f in Example 1. Take $f(t) = (\cos t, \sin t, R\cos 5t, R\sin 5t)$. An orthonormal set of normals to f is given by

$$n_1(t) = \frac{-f(t)}{\sqrt{1+R^2}},$$

$$n_2(t) = \frac{1}{\sqrt{1+25R^2}}(-5R\cos t, 5R\sin t, \cos 5t, -\sin 5t),$$

$$n_3(t) = \frac{1}{\sqrt{1+R^2}}(-R\sin t, -R\cos t, \sin 5t, \cos 5t).$$

To find the range of R for which f is 2-transnormal, let $t \in \mathbf{R}$, then

$$\langle f(t+a) - f(t), \frac{df}{dt}(t)\rangle = 0 \iff \sin a + 5R^2 \sin 5a = 0$$

$$\iff \sin a(1 + 5R^2(5 - 20\sin^2 a + 16\sin^4 a)) = 0.$$

The equation $1 + 5R^2(5 - 20\sin^2 a + 16\sin^4 a) = 0$ is quadratic in $\sin^2 a$ with discriminant $80 - \frac{64}{5R^2}$ which is negative when $0 < R < \frac{2}{5}$. Thus, if $0 < R < \frac{2}{5}$, then $\nu(t) \cap f(\mathbf{S}^1) = \{f(t), f(t+\pi)\}$ and $\nu(t+\pi) = \nu(t)$. Therefore f is 2-transnormal.

It is straightforward to calculate that the normal holonomy map of f is a rotation by an angle $\frac{10\pi\sqrt{1+R^2}}{\sqrt{1+25R^2}}$ around the radial normal. See [1] for more details.

Now consider the equation

$$\frac{10\pi\sqrt{1+R^2}}{\sqrt{1+25R^2}} = 2\pi k, \qquad k \in \mathbf{N} \tag{1}$$

For $k = 3$, equation (1) has solution $R = \frac{\sqrt{2}}{5} < \frac{2}{5}$. For this value of R, f is 2-transnormal and the normals

$$\xi_2(t) = n_2(t)\cos 3t + n_3(t)\sin 3t,$$
$$\xi_3(t) = n_2(t)\sin 3t - n_3(t)\cos 3t$$

are parallel, and $\xi_i(t+\pi) = \xi_i(t) = \xi_i(t+2\pi)$, $i = 2, 3$.

Now let $h : S \longrightarrow \mathbf{R}^l$ be an s-transnormal embedding and let $\tilde{f} : \mathbf{S}^1 \longrightarrow \mathbf{R}^{4+l}$ be defined by $\tilde{f}(t) = (f(t), \underline{0}) \in \mathbf{R}^4 \times \mathbf{R}^l$. Put $\tilde{\xi}_i(t) = (\xi_i(t), \underline{0})$,$i = 2, 3$. Then any parallel normal α to $\tilde{f}$ is a linear combination, with constant coefficients, of $\tilde{\xi}_2(t), \tilde{\xi}_3(t)$ and constant vectors in $0 \times \mathbf{R}^l \subset \mathbf{R}^4 \times \mathbf{R}^l$. So $\alpha(t+\pi) = \alpha(t) = \alpha(t+2\pi)$.

Let $\alpha_1, \ldots, \alpha_l$ be a set of orthogonal parallel normals to $\tilde{f}$ and define
$\tilde{g} : \mathbf{S}^1 \times S \longrightarrow \mathbf{R}^{4+l}$ by

$$\tilde{g}(t, x) = \tilde{f}(t) + \epsilon \sum_{i=1}^{l} h_i(x)\alpha_i(t),$$

where $h(x) = (h_1(x), \ldots, h_l(x))$. Then $\tilde{g}$ has the same image as the partial tube given by $\mathcal{P} = \{(t, \epsilon \sum_{i=1}^{l} h_i(x)\alpha_i(t)), t \in \mathbf{S}^1, x \in S\}$.

Now $\tau_{\tilde{f}(t)-\tilde{f}(t+\pi)}(\tilde{g}(t+\pi, x)) = \tilde{g}(t+x)$ so $\tilde{g}$ has translational symmetry and therefore, by Theorem 1, for ϵ sufficiently small, $\tilde{g}$ is $2s$-transnormal.

As a particular case of this take $l = 2$, $h : \mathbf{S}^1 \longrightarrow \mathbf{R}^2$ given by $h(s) = (\cos s, \sin s)$, $\alpha_1(t) = (\xi_2(t)\cos\theta, \sin\theta, 0)$, $\alpha_2(t) = (\xi_3(t)\cos\theta, 0, \sin\theta)$ for any $\theta \in \mathbf{R}$ and define $\tilde{g}_\theta : \mathbf{S}^1 \times \mathbf{S}^1 \longrightarrow \mathbf{R}^6$ by

$$\tilde{g}_\theta(t, s) = \tilde{f}(t) + \epsilon(\alpha_1(t)\cos s + \alpha_2(t)\sin s).$$

Then, for sufficiently small ϵ, $\tilde{g}_\theta$ is a 4-transnormal embedding. Now

$$\tilde{g}_0(t, s) = (f(t) + \epsilon(\xi_2(t)\cos s + \xi_3(t)\sin s), \underline{0}) \in \mathbf{R}^4 \times \mathbf{R}^2.$$

This is similar to the embedding given in Example 1 but with our modified f, and then included in $\mathbf{R}^6$. The embedding $\tilde{g}_{\frac{\pi}{2}}$ is the product embedding $f \times \epsilon h$. Therefore $H : \mathbf{S}^1 \times \mathbf{S}^1 \times [0, \frac{\pi}{2}] \longrightarrow \mathbf{R}^6$ defined by $H(t, s, \theta) = \tilde{g}_\theta(t, s)$ is an isotopy through transnormal embeddings from a non product embedding of a torus to a product embedding.

Example 3. With the same notation as in Example 2 we now solve equation (1) when $k = 4$ and get $R = \sqrt{\frac{3}{125}} < \frac{2}{5}$. For this value of R, f is 2-transnormal and the normals

$$\xi_2(t) = n_2(t)\cos 4t + n_3(t)\sin 4t,$$

$$\xi_3(t) = n_2(t)\sin 4t - n_3(t)\cos 4t$$

are parallel. But now $\xi_i(t+\pi) = -\xi_i(t)$, $\xi_i(t+2\pi) = \xi_i(t)$, $i = 2, 3$. Define $\tilde{f} : \mathbf{S}^1 \longrightarrow \mathbf{R}^4 \times \mathbf{R}$ by $\tilde{f}(t) = (f(t), 0)$. For $i = 2, 3$, $g_i : \mathbf{S}^1 \times \mathbf{S}^1 \longrightarrow \mathbf{R}^5$ defined by

$$g_i(t, s) = \tilde{f}(t) + \epsilon(\xi_i(t)\cos s, \sin s)$$

is the image of a partial tube, as in Example 2, but here

$$\tau_{\tilde{f}(t)-\tilde{f}(t+\pi)}(g_i(t+\pi, s)) = \tilde{f}(t) + \epsilon(-\xi_i(t)\cos s, \sin s) = g_i(t, \pi - s).$$

So g_i has translational symmetry and therefore, by Theorem 1, for $i = 2, 3$, g_i is a 4-transnormal embedding of $\mathbf{S}^1 \times \mathbf{S}^1$ in $\mathbf{R}^5$.

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