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Conic Sections in Space Defined by Intersection Conditions

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Abstract. We investigate and visualize the set of planes in complex projective three-space $\mathbb{P}^3$ that intersect m conics C_i and $n = 6 - 2m$ straight lines L_j in a total of six points of a conic. The solution manifold $\mathcal{S}_m$ is algebraic and of class $8 - m$. It contains the pencils of planes through L_j with multiplicity two and the planes of the conics C_i with multiplicity three.

1. Introduction

This text generalizes and extends results on a certain class of incidence problems related to conic sections. We consider the set of planes in complex projective three-space $\mathbb{P}^3$ that intersect $m \leq 3$ conic sections C_i and $n = 6 - 2m$ straight lines L_j in six points of a conic section. The case of $m = 0$ has been treated in [7] while $m = 1$ is the topic of [8]. In this paper we also consider $m = 2$ and $m = 3$.

Dual to the set of solution planes is the vertex locus of those quadratic cones that share two tangent planes with m given quadratic cones and have n given straight lines as tangents. This viewpoint relates this article to a number of publications during the last twenty years. In [5, 6, 9, 10, 11, 4, 12] similar problems were considered, usually with additional metric constraints while the purely projective viewpoint (that will also be taken in this text) dates back to the 19th century ([2]).

Of course, we also may consider $\mathbb{P}^3$ as projective extension of a euclidean space with the base conic C_0 as absolute circle. Doing so, we contribute to the task of finding circles that intersect $m - 1$ conic sections in two points and n straight lines in one point. This approach allowed the advantageous use of the geometry of circles in space ([1]) in [8] but seems inappropriate for the other cases.

In general, we can expect a two-parametric set $\mathcal{S}_m$ of solution planes. For $m \in \{0, 1\}$, the following facts have been shown in [7] and [8]:

1. The solution manifold $\mathcal{S}_m$ is algebraic and of class $8 - m$.
2. The base lines L_j are double lines, the planes γ_i of the base conics C_i are triple planes of $\mathcal{S}_m$.

In this paper, we want to extend these results to $m \in \{2, 3\}$ while at the same time attaching importance to a consistent treatment of all four cases. Thus we obtain further insight into the general problem and new proofs for the known results.

After introducing a few basic notions and facts in Section 2, we dedicate Section 3 to the computation of a reduced algebraic equation of $\mathcal{S}_m$. We do this in several steps: At first, we compute a non-algebraic equation in a very straightforward way. A closer inspection will suggest a modification that yields an algebraic but still reducible equation. We eliminate the unwanted components, compute a reduced algebraic equation and determine the class of $\mathcal{S}_m$.

In Section 4 we compute the multiplicity of the pencils of planes through the base lines L_j and the base conic planes γ_i. Using ideas from the preceding section, this turns out to be quite simple and straightforward. Finally, we present visualizations of the dual solution manifold.

2. Basic notions and facts

Given are $m \le 3$ conic sections $C_0, \ldots, C_{m-1}$ and $n = 6 - 2m$ straight lines $L_m, \ldots, L_{n-1}$ in complex projective three space $\mathbb{P}^3$. They will be referred to as *base conics* and *base lines*, respectively. A plane ε is called *solution plane* if it contains a conic C that intersects all base lines in at least one and all base conics in at least two, possibly coinciding, points. Obviously, the supporting planes γ_i of the base conics and the planes in the pencils through the base lines are solution planes.

The union $\mathcal{S}_m$ of all solution planes will be called the *solution manifold*. In general, it is a two-parameter variety of planes. There are many ways of seeing that it is algebraic (one of them will be presented in this text). For the time being, we take this for granted and try to identify those singular configurations, where *all* planes of $\mathbb{P}^3$ are solution planes.

We consider the set of straight lines $\mathcal{L}$ that contain at least three points on base conics or base lines. If two base conics or one base conic and two base lines or four base lines are co-planar, we already have a singular configuration. Otherwise, the line-set $\mathcal{L}$ is one-parametric and generates a (reducible) algebraic ruled surface Φ. By assumption, a generic tangent plane of Φ contains two rulings, i.e., it is a double plane of Φ. This is only possible, if Φ is of degree two and we have:

Theorem 1. *All planes of $\mathbb{P}^3$ are solution planes, if two base conics, one base conic and two base lines or four base lines are co-planar or if all base conics and lines lie on a common quadric.*

We exclude singular configurations from now on. For the rest of this paper, the solution manifold $\mathcal{S}_m$ is always assumed to be a one-dimensional surface in dual space.

3. The solution manifold's equation

We want to find an algebraic equation $G_m = 0$ that describes the solution manifold $\mathcal{S}_m$ in homogeneous plane coordinates. A non-algebraic equation $\hat{G}_m = 0$ of a super-manifold $\hat{\mathcal{S}}_m$ of the actual solution manifold $\mathcal{S}_m$ can be found by straightforward computation. A closer investigation of $\hat{G}_m$ will yield the desired algebraic equation and provide tools for further investigations.

3.1. A non-algebraic equation

We start with a yet undetermined plane

$$\varepsilon\colon u_0x_0 + u_1x_1 + u_2x_2 + u_3x_3 = 0$$

in $\mathbb{P}^3$. Its intersection points with the base conics and base lines be $\mathbf{c}_0, \dots, \mathbf{c}_5$. We choose the indices so that the two intersection points of ε and the base conic C_i are $\mathbf{c}_{2i}$ and $\mathbf{c}_{2i+1}$. The intersection point of ε and the base line L_j be $\mathbf{c}_j$. In order to apply a handy conic criterion to these points, we cancel their last coordinate. This is equivalent to projecting them from the center $\mathbf{z}$ with homogeneous coordinate vector $[0 : 0 : 0 : 1]$ onto the image plane π with equation $x_3 = 0$ and identifying π with $\mathbb{P}^2$. Doing so, we implicitly assume that both $\mathbf{z}$ and π have a generic position with respect to the base conics and base lines. Of course, this is no loss of generality.

The projected points be denoted by $\mathbf{c}_i'$, their homogeneous coordinates be $[c_{i0} : c_{i1} : c_{i2}]$. If the points $\mathbf{c}_i'$ lie on a conic section, the determinant $\hat{G}_m$ of the matrix $M = (\mathbf{m}_0^{\mathrm{T}}, \dots, \mathbf{m}_5^{\mathrm{T}})$ with

$$\mathbf{m}_i = (c_{i0}^2, c_{i1}^2, c_{i2}^2, 2c_{i0}c_{01}, 2c_{i0}c_{02}, 2c_{i1}c_{02}) \tag{1}$$

necessarily vanishes. This is easy to see and also referenced, for example in [3]. Unfortunately, this criterion is not sufficient. It fails precisely if the plane ε

- contains the projection center $\mathbf{z}$ or
- is a tangent plane of a base conic.

Therefore, the manifold of planes $\hat{\mathcal{S}}_m$ described by the equation $\hat{G}_m = 0$ consists of

1. the solution manifold $\mathcal{S}_m$,
2. the bundle of planes $\mathbf{z}(\varepsilon)$ through $\mathbf{z}$ and
3. the tangent planes of the base conics.

Furthermore, $\hat{G}_m$ is not algebraic, at least not for $m > 0$. In order to find an algebraic equation, we have a closer look at the matrix M whose columns are given by (1).

3.2. The star product

We define a bilinear composition ("star product") on the vector space $\mathbb{C}^3$:

$$\begin{pmatrix} p_0 \\ p_1 \\ p_2 \end{pmatrix} \star \begin{pmatrix} q_0 \\ q_1 \\ q_2 \end{pmatrix} := \begin{pmatrix} p_0q_0 \\ p_1q_1 \\ p_2q_2 \\ p_0q_1 + p_1q_0 \\ p_0q_2 + p_2q_0 \\ p_1q_2 + p_2q_1 \end{pmatrix}. \tag{2}$$

It induces a binary composition in the complex projective plane that associates a point $\mathbf{r} \in \mathbb{P}^5$ to two points $\mathbf{p}, \mathbf{q} \in \mathbb{P}^2$. For reasons of simplicity, we will denote this composition by the same symbol "$\star$". With the help of this star product, we can write the equation of $\hat{\mathcal{S}}_m$ as

$$\hat{G}_m = \det(\mathbf{c}_0' \star \mathbf{c}_0', \ldots, \mathbf{c}_5' \star \mathbf{c}_5'). \tag{3}$$

The geometric meaning of the star product can be revealed if we identify $\mathbb{P}^5$ with the projective space of dual conics in $\mathbb{P}^2$ via the usual embedding that maps the dual conic with equation

$$C^\star \colon d_0u_0^2 + d_1u_1^2 + d_2u_2^2 + 2d_3u_0u_1 + 2d_4u_0u_2 + 2d_5u_1u_2 = 0$$

to the point with homogeneous coordinates $[d_0 : \cdots : d_5]$. Since the entries of the star product $(p_0, p_1, p_2)^{\mathrm{T}} \star (q_0, q_1, q_2)^{\mathrm{T}}$ are the coefficients of the polynomial

$$(p_0u_0 + p_1u_1 + p_2u_2)(q_0u_0 + q_1u_1 + q_2u_2)$$

with respect to the monomial basis $\{u_iu_j\}$, the point $\mathbf{p} \star \mathbf{q}$ represents the singular dual conic consisting of the two pencils of lines $\mathbf{p}(P)$ and $\mathbf{q}(Q)$ through $\mathbf{p}$ and $\mathbf{q}$. The star product $\mathbf{r} \star \mathbf{r}$ of a point with itself is a pencil of lines that is counted with multiplicity two ("double point").

When successively expanding (3) according to Laplace's theorem by the first and second column, wedge products of the form

$$(\mathbf{c}_{2i}' \star \mathbf{c}_{2i}') \wedge (\mathbf{c}_{2i+1}' \star \mathbf{c}_{2i+1}')$$

arise. They describe pencils of dual conics in $\mathbb{P}^5$ that are spanned by the double points $\mathbf{c}_{2i}' \star \mathbf{c}_{2i}'$ and $\mathbf{c}_{2i+1}' \star \mathbf{c}_{2i+1}'$. All dual conics of these pencils are singular and contain the span E_i of $\mathbf{c}_{2i}'$ and $\mathbf{c}_{2i+1}'$. Furthermore, the following simple lemma from elementary projective geometry holds:

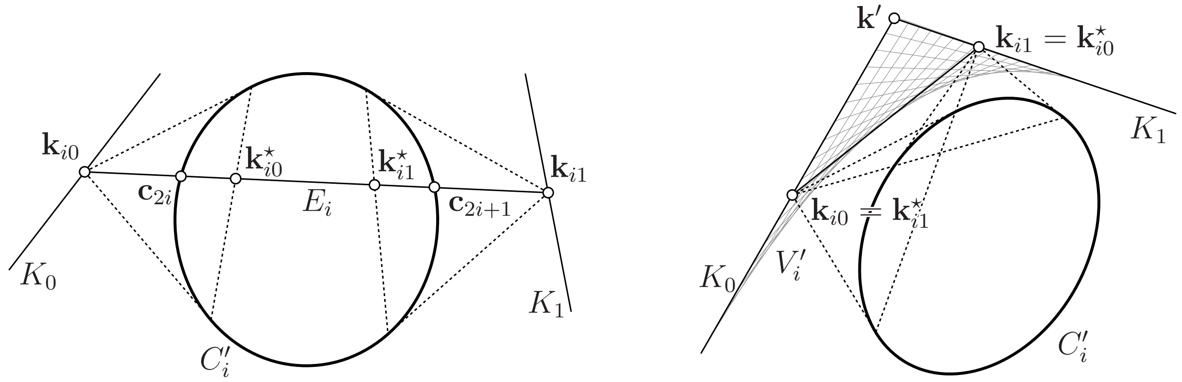
Lemma 1. *Let $\mathbf{p} \star \mathbf{p}$ and $\mathbf{q} \star \mathbf{q}$ be two double points in $\mathbb{P}^2$. The singular dual conic $\mathbf{r} \star \mathbf{s}$ lies in the pencil of dual conics spanned by $\mathbf{p} \star \mathbf{p}$ and $\mathbf{q} \star \mathbf{q}$ iff the quadruple $(\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s})$ is harmonic.*

3.3. An algebraic equation

According to Lemma 1, the mixed star product $\mathbf{a} \star \mathbf{b}$ can replace $\mathbf{c}'_{2i} \star \mathbf{c}'_{2i}$ or $\mathbf{c}'_{2i+1} \star \mathbf{c}'_{2i+1}$ in the Laplace expansion of (3), if the quadruple

$$(\mathbf{c}'_{2i}, \mathbf{c}'_{2i+1}, \mathbf{a}, \mathbf{b})$$

is harmonic. This provides the key for finding an algebraic equation of $\mathcal{S}_m$.



(a) (b)

Figure 1. The construction of the points $\mathbf{k}_{ij}$ and $\mathbf{k}^\star_{ij}$ (a) and its singularities (b).

We slightly alter the expansion of the solution manifold's equation (3). Let K_0 and K_1 be two arbitrary straight lines in $\mathbb{P}^2$. For $i \le m$ we denote the intersection point of K_j with the straight line E_i through $\mathbf{c}'_{2i}$ and $\mathbf{c}'_{2i+1}$ by $\mathbf{k}_{ij}$. The conjugate point of $\mathbf{k}_{ij}$ on E_i with respect to the projection C'_i of the base conic C_i be $\mathbf{k}^\star_{ij}$ (Figure 1a). Since the quadruple

$$(\mathbf{c}'_{2i}, \mathbf{c}'_{2i+1}, \mathbf{k}_{ij}, \mathbf{k}^\star_{ij})$$

is harmonic, we can simultaneously replace the entry $\mathbf{c}'_{2i} \star \mathbf{c}'_{2i}$ in (3) by $\mathbf{k}_{i0} \star \mathbf{k}^\star_{i0}$ and $\mathbf{c}'_{2i+1} \star \mathbf{c}'_{2i+1}$ by $\mathbf{k}_{i1} \star \mathbf{k}^\star_{i1}$. We do this for all values $i < m$ and denote the determinant of the resulting matrix by $\tilde{G}_m$. The point $\mathbf{k}_{ij}$ is linear in the coordinates u_i of the plane ε and $\mathbf{k}^\star_{ij}$ is quadratic. Therefore, the equation $\tilde{G}_m$ is algebraic and of degree $12 + 2m$. It describes a super-manifold $\tilde{\mathcal{S}}_m$ of the solution manifold $\mathcal{S}_m$ and the bundle of planes $\mathbf{z}(\varepsilon)$.

In order to find the unwanted components of $\tilde{\mathcal{S}}_m$, we have to study the singularities of the above construction. In contrast to Section 3.1, the tangent planes of C_i are not among them: For coinciding points $\mathbf{c}'_{2i}$ and $\mathbf{c}'_{2i+1}$ we can replace E_i by the appropriate tangent of C'_i. Our method fails precisely if $\mathbf{k}_{i0} = \mathbf{k}_{i1}$ or $\mathbf{k}_{i0} = \mathbf{k}^\star_{i1}$ (which implies $\mathbf{k}^\star_{i1} = \mathbf{k}_{i0}$, Figure 1b). The first instance occurs if E_i contains the intersection point $\mathbf{k}'$ of K_0 and K_1, the latter, if E_i is generated by the projectivity ν between K_0 and K_1 that is induced by the polar system of C'_i. The union of these straight lines is a dual conic V'_i. In Figure 1b it is visualized as hull-curve. All in all, $\tilde{\mathcal{S}}_m$ consists of

1. the solution manifold $\mathcal{S}_m$,
2. the bundle of planes through $\mathbf{z}$,

3. the bundles of planes through the back-projection of $\mathbf{k}'$ into the base conic plane γ_i and
4. the tangent planes of the back-projection V_i of V_i' into the base conic plane γ_i.

The unwanted components of $\tilde{G}_m$ with exception of $\mathbf{z}(\varepsilon)$ have a total multiplicity of $3m$. Splitting them off, we obtain an algebraic equation $\bar{G}_m$ of degree $12 - m$ that describes $\mathcal{S}_m$ and, with yet unknown multiplicity, the bundle of planes $\mathbf{z}(\varepsilon)$. This multiplicity has to be determined in the final step.

3.4. The solution manifold's class

We consider a generic straight line $E \subset \mathbb{P}^3$. The pencil of planes $E(\varepsilon)$ through E be parameterized linearly by $\varepsilon = \varepsilon(t)$ such that $\varepsilon(0)$ describes the span of $\mathbf{z}$ and E. The parameterization $\varepsilon(t)$ induces linear parameterizations of the points $\mathbf{c}_j$, linear parameterizations of the points $\mathbf{k}_{ik}$ and quadratic parameterizations of $\mathbf{k}^\star_{ik}$ $(i \le m,\ j > 2m,\ k = 0, 1)$. These points define a polynomial equation $\bar{G}_m(t)$ of degree $12 - m$. For $t = 0$, all points $\mathbf{c}_j(t)$ and $\mathbf{k}_{ik}(t)$ are collinear and $t = 0$ is a zero of $\bar{G}_m$. We are done, if we can show that its multiplicity is four. This will be a consequence of the following lemma.

Lemma 2. *For $i \in \{0, \dots, 5\}$ let $\mathbf{a}_i(t)$ be a rational parameterized equation of degree one with coefficients $\mathbf{a}_{ij} \in \mathbb{C}^3$ and $\mathbf{b}_i(t)$ a like parameterization of degree two with coefficients $\mathbf{b}_{ij}$. If all points $\mathbf{a}_{i0} = \mathbf{a}_i(0)$ and $\mathbf{b}_{i0} = \mathbf{b}_i(0)$ lie on a straight line E, the parameter value $t = 0$ is a zero of multiplicity four of the polynomial*

$$D(t) := \det\bigl(\mathbf{a}_0 \star \mathbf{b}_0, \dots, \mathbf{a}_5 \star \mathbf{b}_5\bigr). \tag{4}$$

Proof. Because of the star product's bilinearity, the determinant (4) can be expanded to a polynomial of degree 18 in t where the coefficient to t^i is

$$D_i = \sum \det(\mathbf{a}_{0k_0} \star \mathbf{b}_{0l_0}, \dots, \mathbf{a}_{5k_5} \star \mathbf{b}_{5l_5}) \tag{5}$$

and the sum ranges over all indices k_j and l_j that add up to i. If for any $j \in \{0, \dots, 5\}$ either k_j or l_j vanishes, the singular dual conics described by the star products $\mathbf{a}_{ik_j} \star \mathbf{b}_{il_j}$ lie in a four-dimensional subspace of $\mathbb{P}^5$ (the subspace of all dual conics through the straight line E). The same is true, if at least four index pairs (k_j, l_j) vanish simultaneously. In this case, the four corresponding dual conics lie in a plane of $\mathbb{P}^5$. It is easy to see that either the first or second instance occurs for all summands of D_i if $i \le 3$. Thus, $t = 0$ is really a zero of multiplicity four of $D(t)$. $\square$

Using Lemma 2 with $\mathbf{a}_i = \mathbf{k}_{i0}$ and $\mathbf{b}_i = \mathbf{k}^\star_{i0}$ for $i < m$ and $\mathbf{a}_i = \mathbf{b}_i = \mathbf{c}_i$ (implying $\mathbf{b}_{i2} = \mathbf{o}$) for $i \ge m$ we see that the multiplicity of the bundle of planes $\mathbf{z}(\varepsilon)$ as component of $\bar{G}_m$ is four. An algebraic equation that describes precisely the planes of $\mathcal{S}_m$ is given as $G_m = \bar{G}_m \cdot u_3^{-4}$. This is the desired reduced algebraic equation of $\mathcal{S}_m$. It allows us to state

Theorem 2. *The solution manifold $\mathcal{S}_m$ is algebraic and of class $8 - m$.*

The fact that increasing m yields solution manifolds of lower class is perhaps a surprise. One might assume that the presence of conic sections increases the problem's complexity.

A glance of the above calculations shows, however, that this is not true. For $i < m$ (base conics), the wedge product

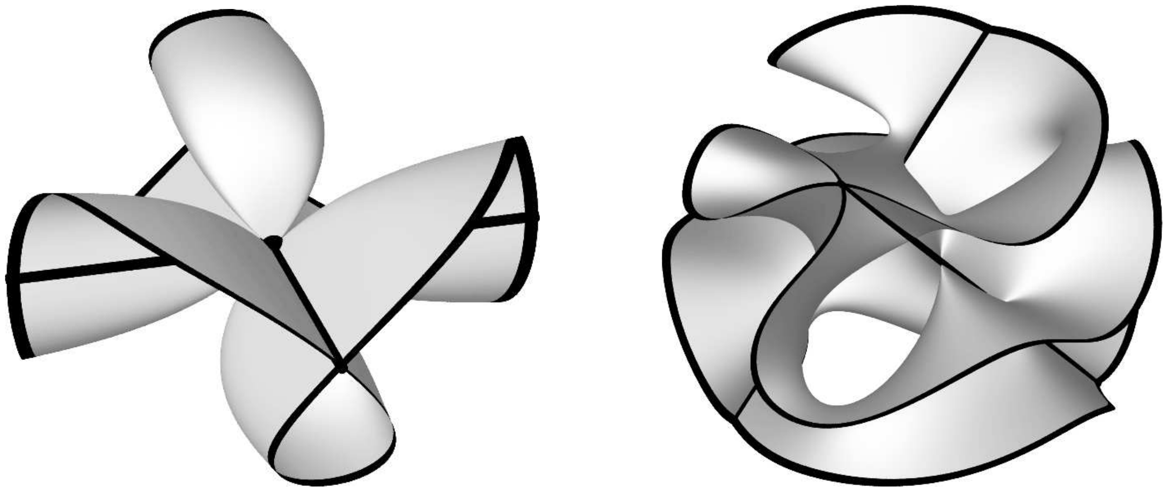
$$(\mathbf{k}_{i0} \star \mathbf{k}_{i0}^{\star}) \wedge (\mathbf{k}_{i1} \star \mathbf{k}_{i1}^{\star})$$

occurs in the Laplace expansion of (4). It is of degree six but only a cubic factor is relevant. The wedge product to a base line is of the shape

$$(\mathbf{c}_j \star \mathbf{c}_j) \wedge (\mathbf{c}_{j+1} \star \mathbf{c}_{j+1}).$$

It is of degree four and produces no unwanted components. Thus, replacing two base lines by one conic section reduces the class of the solution manifold by one.

4. Special lines and planes



(a) $m = 3$ (b) $m = 2$

Figure 2. Dual solution surfaces

Now we turn to the investigation of special lines and planes in $\mathcal{S}_m$. After our considerations in the preceding section, this turns out to be quite easy. We already mentioned that the base lines are double lines of $\mathcal{S}_0$ and $\mathcal{S}_1$ and the base conic plane γ_0 is a triple plane of $\mathcal{S}_1$ ([7, 8]). In order to verify these results and to extend them to the cases of $m = 2$ and $m = 3$, we proceed similar to Section 3.3.

We consider a generic transversal line E of the base line L_j. The pencil of planes $E(\varepsilon)$ be parameterized linearly according to $\varepsilon = \varepsilon(t)$ so that the span of E and L_j belongs to the parameter value $t = \infty$. As in Section 3.3, we compute $\tilde{G}_m(t)$ – the algebraic equation of $\mathcal{S}_m$ plus certain additional components. The straight line L_j is a double line, if $\tilde{G}_m$ is of degree 16 and not, as in the generic case, of degree 18. In order to prove this, we may also consider the situation of Lemma 2 with the additional constraint that $\mathbf{a}_5(t)$ and hence also $\mathbf{b}_5(t)$ are constant (i.e., $\mathbf{a}_{51} = \mathbf{b}_{51} = \mathbf{b}_{52} = \mathbf{o}$). It is easy to see that this implies $D_{17} = D_{18} = 0$ which is exactly what has to be shown.

Now we turn to the base conic plane γ_i. We consider a straight line $E \subset \gamma_i$ and the pencil of planes $\varepsilon = \varepsilon(t)$ through E. The plane γ_i shall be assigned the parameter value $t = \infty$. Since the intersection points of $\varepsilon(t)$ with the base conic C_i remain fixed, we can consider the situation of Lemma 2 with

$$\mathbf{a}_{41} = \mathbf{b}_{41} = \mathbf{b}_{42} = \mathbf{a}_{51} = \mathbf{b}_{51} = \mathbf{b}_{52} = \mathbf{o}.$$

Again, it is easy to see that this implies a degree reduction of $D(t)$ by four. Therefore we have

Theorem 3. *The base lines are double lines, the base conic planes are triple planes of the solution manifold* $\mathcal{S}_m$.

The contents of Theorem 3 are illustrated in Figure 2. There, we present visualizations of solution manifolds, obtained by dualization at the quadric with equation $x_0^2+x_1^2+x_2^2+x_3^2 = 0$. The resulting surfaces contain m triple points and n double lines that stem from the base conic planes and base lines.

We restrict ourselves to visualizing the solution manifold in case of $m = 2$ and $m = 3$. Similar images of the two remaining cases can be found in [7] and [8].

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