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An Application of Type Sequences to the Blowing-up

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Abstract. Let I be an $\mathfrak{m}$-primary ideal of a one-dimensional, analytically irreducible and residually rational local Noetherian domain R. Given the blowing-up of R along I, we establish connections between the type-sequence of R and classical invariants like multiplicity, genus and reduction exponent of I.

1. Introduction

Let $(R, \mathfrak{m}, k)$ be a one-dimensional local Noetherian domain which is analytically irreducible and residually rational. In this paper we deal with the blowing-up $\Lambda := \Lambda(I) = \bigcup_{n\geq 0} I^n : I^n$ along a not principal $\mathfrak{m}$-primary ideal I of R.
The problem of finding relations involving the multiplicity $e := e(I)$, the genus $\rho := \rho(I) = l_R(\Lambda/R)$ and the reduction exponent $\nu := \nu(I)$, was first studied for $I = \mathfrak{m}$ by Northcott in the 1950s and later by Matlis (see [8]), Kirby (see [5]), Lipman (see [6]) and many others.

In this note we show that it is possible to describe the difference $2\rho - e\nu$ in terms of the type sequence $[r_1, \dots, r_n]$ of R (r_1 is the Cohen-Macaulay type). Our main result is the formula of Theorem 4.7 in Section 4:

$$2\rho = e\nu + \sum_{i\notin\Gamma} (r_i - 1) - d(R : \Lambda) - l_R(\Lambda^{**}/\Lambda) - l_R(R : \Lambda/I^\nu).$$

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Afterwards we use this statement to improve classical results concerning the equality

$$R : \Lambda = I^\nu$$

which has been studied by several authors under the hypothesis that R is Gorenstein. Starting from a theorem of Matlis valid for $\Lambda(\mathfrak{m})$ ([7], Theorem 13.4), Orecchia and Ramella ([14], Theorem 2.6) proved that if the associated graded ring $G(\mathfrak{m}) = \bigoplus_{n\geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is Gorenstein, then $R : \Lambda = \mathfrak{m}^\nu$. Successively Ooishi, in the case of the blowing-up along an ideal I, proved that $2\rho \leq e\nu$ and that equality holds if and only if $R : \Lambda = I^\nu$ ([12], Theorem 3).

In Section 5, we consider the rings having type sequence $[r_1, 1, \dots, 1]$ which are called almost Gorenstein. For these rings we prove that the Ooishi's inequality $2\rho \leq e\nu$ becomes $2\rho \leq \nu e + r_1 - 1$ and that equality holds if and only if $R : \Lambda = I^\nu$ (Theorem 5.3).

In Section 6 we consider the case of the blowing-up along $\mathfrak{m}$. The study of the conductor $R : \Lambda$ provides some useful remarks when $e = \mu + 1$ (μ is the embedding dimension of R) and when the reduction exponent is 2 or 3.

2. Notations and preliminaries

Throughout this paper $(R, \mathfrak{m})$ denotes a one-dimensional local Noetherian domain with residue field k. For simplicity, we assume that k is an infinite field. Let $\overline{R}$ be the integral closure of R in its quotient field K; we suppose that $\overline{R}$ is a finite R-module and a DVR with a uniformizing parameter t, which means that R is analytically irreducible. We also suppose R to be residually rational, i.e., $k \simeq \overline{R}/t\overline{R}$. We denote the usual valuation associated to $\overline{R}$ by

$$v : K \longrightarrow \mathbb{Z} \cup \infty, \qquad v(t) = 1.$$

2.1. Under our hypotheses, for any fractional ideals $I \supseteq J$ the length of the R-module I/J can be computed by means of valuations (see [8], Proposition 1):

$$l_R(I/J) = \#(v(I) \setminus v(J)).$$

Given two fractional ideals I, J we define $I : J = \{x \in K \mid xJ \subseteq I\}$.

2.2. In the sequel we shall consider an $\mathfrak{m}$-primary ideal I of R which is not principal. The *Hilbert function* and the *Hilbert-Poincaré series* of I are respectively

$$H_I(n) = l_R(I^n/I^{n+1}), \quad n \geq 0, \qquad P_I(z) = \sum_{n\geq 0} H_I(n) z^n.$$

It is well-known that the power series $P_I(z)$ is rational:

$$P_I(z) = \frac{h_I(z)}{1-z}, \quad \text{where } h_I(z) = h_0 + h_1 z + h_2 z^2 + \cdots + h_\nu z^\nu \in \mathbb{Z}[z],$$

$h_0 = l_R(R/I), \quad h_i = l_R(I^i/I^{i+1}) - l_R(I^{i-1}/I^i)$, for all $i, \quad 1 \leq i \leq \nu$.

The polynomial $h_I(z)$ is called the *h-polynomial* of I; moreover,

$e(I) := h_I(1)$ is the *multiplicity* of I,
$\rho(I) := h'_I(1)$ is called *genus* of I, or *reduction number* of R if $I = \mathfrak{m}$.
We shall say that $h_I(z)$ is *symmetric* if $h_i = h_{\nu-i}$ for all i, $0 \le i \le \nu$.

The *blowing-up of R along I* is defined by

$$\Lambda := \Lambda(I) = \bigcup_{n\ge 0} I^n : I^n \qquad (cf.\ [6]).$$

Let $x \in I$ denote an element (called *a minimal reduction of I*) such that $I^{n+1} = xI^n$ for $n \gg 0$. Then (see [6], 1):

(1) $x^n\Lambda = I^n\Lambda, \quad \forall n \ge 0$.
(2) $e(I) = l_R(R/xR) = v(x) \ge H_I(n)$, for every $n \ge 0$.
(3) The least integer $\nu := \nu(I)$ such that $I^{n+1} = xI^n \quad \forall n \ge \nu$, is called the *reduction exponent* of I. It is known that $\nu(I) \le e(I) - 1$ and that the following equalities hold:
$$\nu(I) = \deg h_I(z) = \min\{n \mid l_R(I^n/I^{n+1}) = e(I)\}$$
$$= \min\{n \mid \Lambda = I^n : I^n\} = \min\{n \mid I^n\Lambda = I^n\}.$$
(4) $\rho(I) = l_R(\Lambda/R)$. Hence
$$l_R(R/I^n) = e(I)n - \rho(I), \qquad \forall n \ge \nu.$$
(5) If $h_I(z)$ is symmetric, then $l_R(R/I^\nu) = \dfrac{e(I)\nu}{2}$.
This follows immediately from the fact that, if $h_I(z)$ is symmetric, then $2\rho(I) = e(I)\,\nu$ (see the proof of Lemma 3.3, [13]).
(6) The inclusion $R : \Lambda \supseteq I^\nu$ always holds and the equality $R : \Lambda = I^n$ implies that $n = \nu$ ([12], Proposition 1, [14], Lemma 1.5).

2.3. We shall consider also:

$v(R) := \{v(x),\ x \in R,\ x \ne 0\} \subseteq \mathbb{N}$, the *numerical semigroup* of R.
$\gamma_R := R : \overline{R}$, the *conductor ideal* of R.
$c := l_R(\overline{R}/\gamma_R)$, the *conductor* of $v(R)$, such that $\gamma_R = t^c\overline{R}$.
$\delta := l_R(\overline{R}/R)$, the *singularity degree of R*.
$n := c - \delta = l_R(R/\gamma_R)$.

2.4. In our hypotheses R has a *canonical module* ω, unique up to isomorphism.

We list below some well-known properties of ω, useful in the sequel (see [4]). We always assume that $R \subseteq \omega \subset \overline{R}$.

(1) $\omega : \omega = R$ and $\omega : (\omega : I) = I$ for every fractional ideal I.
(2) If $I \supseteq J$, then $l_R(I/J) = l_R(\omega : J/\omega : I)$.
(3) $v(\omega) = \{j \in \mathbb{Z} \mid c - 1 - j \notin v(R)\}$, hence $c - 1 \notin v(\omega)$ and $c + \mathbb{N} \subseteq v(\omega)$.
(4) R is Gorenstein if and only if $\omega = R$ if and only if $R : \omega = R$.
Otherwise $\gamma_R \subseteq R : \omega \subseteq m$.

(5) (see [9], Lemma 2.3) For every fractional ideal I,

$$s \in v(I\omega) \quad \text{if and only if } c - 1 - s \notin v(R : I).$$

2.5. We recall the notion of *type sequence* given for rings by Matsuoka in 1971, recently revisited in [2] and extended to modules in [10].
Let $n := c - \delta$, and let $s_0 = 0 < s_1 < \cdots < s_n = c$ be the first $n+1$ elements of $v(R)$. For each $i = 1, \dots, n$, define the ideal $R_i := \{x \in R \; : \; v(x) \geq s_i\}$ and consider the chains:

$$R = R_0 \supset R_1 = m \supset R_2 \supset \ldots \supset R_n = \gamma_R$$

$$R = R \; : \; R_0 \subset R \; : \; m \subset R \; : R_2 \subset \ldots \subset R \; : R_n = \overline{R}$$

For every $i = 1, \ldots, n$, put $r_i := l_R(R : R_i / R : R_{i-1}) = l_R(\omega R_{i-1} / \omega R_i)$.
The *type sequence* of R, denoted by $t.s.(R)$, is the sequence $[r_1, \dots, r_n]$.
We list some properties of type sequences useful in the sequel (see [2]):

(1) $r := r_1$ is the Cohen-Macaulay type of R.
(2) For every $i = 1, \dots, n$, we have $1 \leq r_i \leq r_1$.
(3) $\delta = \sum_1^n r_i$, and $2\delta - c = l_R(\omega/R) = \sum_1^n (r_i - 1)$.
(4) If $s_i \in v(R : \omega)$, then the correspondent r_{i+1} is 1 (see [9], Prop.3.4).

2.6. We recall that ring R is called *almost Gorenstein* if it satisfies the equivalent conditions

(1) $m = m\,\omega$.
(2) $r_1 - 1 = 2\delta - c$.
(3) $R : \omega \supseteq m$.

By the above property 2.5,(3), it is clear that R is *almost Gorenstein* if and only if $t.s.(R) = [r_1, 1, \dots, 1]$ and that *Gorenstein* means *almost Gorenstein* with $r_1 = 1$.

2.7. For any fractional ideal I of R we set $I^* := R : I$. Notice that:

$$I \subseteq I^{**} \subseteq I\omega.$$

In fact, $I^{**} = R : (R : I) \subseteq \omega : (R : I) = I\omega$.

2.8. We recall that the *integral closure* of an ideal I of R is $\overline{I} := I\overline{R} \cap R$ and that I is said to be *integrally closed* if $I = \overline{I}$.
In [11] Ooishi characterizes curve singularities which can be normalized by the first blowing-up along the ideal I in terms of integral closures:

$$\Lambda = \overline{R} \quad \text{if and only if } I^n = \overline{I^n} \quad \text{for all } n \geq \nu. \qquad (*)$$

We introduce a weaker notion of closure, namely the *canonical closure* of I as $\widetilde{I} := I\omega \cap R$. We'll see that this notion is particularly meaningful for almost Gorenstein rings. Recalling 2.7, we can easily see that $I \subseteq I^{**} \subseteq \widetilde{I} \subseteq \overline{I}$, so

$$I = \overline{I} \quad \text{implies that } I = I^{**} = \widetilde{I}.$$

For the canonical closure the analogue of statement $(*)$ is:

$$\Lambda = \omega\Lambda \quad \text{if and only if } I^n = \widetilde{I^n} \quad \text{for all } n \geq \nu.$$

This fact is shown in the next proposition.

Proposition 2.9. *Let $\Lambda := \Lambda(I)$ be as above. We have the following groups of equivalent conditions:*

(A) (A$_1$) $\omega \subseteq \Lambda$;

(A$_2$) $\omega\Lambda = \Lambda$;

(A$_3$) $\omega : \Lambda = R : \Lambda$;

(A$_4$) $I^n = \widetilde{I^n} \quad \forall n \geq \nu$;

(A$_5$) $\omega I^n = I^n \quad \forall n \geq \nu$;

(A$_6$) *there exists $n > 0$ such that $\omega I^n = I^n$.*

(B) (B$_1$) $\Lambda = \Lambda^{**}$;

(B$_2$) $\omega : \Lambda = \omega(R : \Lambda)$.

Moreover, the following facts are equivalent

(1) *Conditions* (A) *hold.*

(2) *Conditions* (B) *hold and $R : \Lambda \subseteq R : \omega$.*

Proof. Let's begin to prove that the equalities $I^n = \widetilde{I^n} \quad \forall n \geq \nu$ imply that $\omega \subseteq \Lambda$. Let k_1 be the minimal exponent such that $I^{k_1} \subseteq R : \omega$ (k_1 exists since $R : \omega \supseteq \gamma_R$). If $k_1 \geq \nu$, then $I^{k_1} = \widetilde{I^{k_1}} = \omega I^{k_1} \cap R = \omega I^{k_1}$ and this yields $\omega \subseteq I^{k_1} : I^{k_1} = I^\nu : I^\nu = \Lambda$. If $k_1 < \nu$, then $I^\nu \subseteq I^{k_1} \subseteq R : \omega$, hence $\omega I^\nu \subseteq R$. Thus, $I^\nu = \widetilde{I^\nu} = I^\nu\omega \cap R = I^\nu\omega$, which means $\omega \subseteq \Lambda$.

All the other implications in group (A) and also that ones in group (B) hold by the properties of the canonical module.

To prove (A) implies (B), note that by 2.7 $\Lambda^{**} \subseteq \omega\Lambda = \Lambda$.

Moreover, if (A) holds, then $(R : \Lambda)\omega \subseteq (R : \Lambda)\Lambda \subseteq R$, hence $R : \Lambda \subseteq R : \omega$.

Under the further assumption $R : \Lambda \subseteq R : \omega$, we can prove (B) implies (A) because the fact $\omega : \Lambda = \omega(R : \Lambda) \subseteq \omega(R : \omega) \subseteq R$ leads to $\Lambda \supseteq \omega$.

Remark 2.10. (1) If R is almost Gorenstein, then $R : \Lambda \subseteq R : \omega$, hence conditions (A) and (B) above are equivalent.

(2) If I is a canonical ideal, i.e., $I \simeq \omega$, then conditions (A) and (B) hold, because Λ is reflexive and $R : \Lambda \subseteq R : \omega$ (see [9], Remark 2.5).

3. The first formula

In the following we use the notation introduced in Section 2.
$\Lambda := \Lambda(I) = \bigcup_{n>0} I^n : I^n$ is the blowing-up of R in an m-primary ideal I which is not principal and $e := e(I),\ \nu := \nu(I),\ \rho := \rho(I)$ are respectively the multiplicity, the reduction exponent and the genus of I.
Moreover, we consider

$$\begin{aligned} \gamma_R &:= R : \overline{R}, & \delta &:= l_R(\overline{R}/R), & c &:= l_R(\overline{R}/\gamma_R),\\ \gamma_\Lambda &:= \Lambda : \overline{R}, & \delta_\Lambda &:= l_R(\overline{R}/\Lambda), & c_\Lambda &:= l_R(\overline{R}/\gamma_\Lambda). \end{aligned}$$

Finally, $x \in I$ denotes a minimal reduction of I.

3.1. We begin with a few remarks involving the conductor ideals respect to the canonical inclusions $R \subseteq \Lambda \subseteq \overline{R}$. We have the following diagram:

$$\begin{array}{ccc} \gamma_\Lambda & & \\ \cup\!\shortmid & & \\ \gamma_R & \subseteq & R : \Lambda \\ \cup\!\shortmid & & \cup\!\shortmid \\ (R : \Lambda)\ \gamma_\Lambda & & I^\nu \\ \cup\!\shortmid & & \cup\!\shortmid \\ x^\nu\ \gamma_\Lambda & = & I^\nu : \overline{R} \end{array}$$

Proposition 3.2.
(1) $c - c_\Lambda \le e\nu$.
(2) $l_R(R/\gamma_R) - l_R(\Lambda/\gamma_\Lambda) = c - c_\Lambda - \rho = e\nu - \rho - l_R(\gamma_R/x^\nu\gamma_\Lambda) \le l_R(R/I^\nu)$.
(3) *The following facts are equivalent:*

(a) $c - c_\Lambda = e\nu$.
(b) $\gamma_R = x^\nu\gamma_\Lambda$.
(c) $\gamma_R \subseteq I^\nu$.
(d) $l_R(R/\gamma_R) - l_R(\Lambda/\gamma_\Lambda) = l_R(R/I^\nu)$.

Proof. (1) Considering the diagram in 3.1 we see that: $c - c_\Lambda = l_R(\gamma_\Lambda/\gamma_R) = l_R(\gamma_\Lambda/x^\nu\gamma_\Lambda) - l_R(\gamma_R/x^\nu\gamma_\Lambda) = e\nu - l_R(\gamma_R/x^\nu\gamma_\Lambda)$.
(2) Since $\rho = \delta - \delta_\Lambda$, using part (1) of the proof we obtain:

$$\begin{aligned} l_R(R/\gamma_R) - l_R(\Lambda/\gamma_\Lambda) &= (c-\delta) - (c_\Lambda - \delta_\Lambda) = c - c_\Lambda - \rho\\ &= e\nu - \rho - l_R(\gamma_R/x^\nu\gamma_\Lambda) \le l_R(R/I^\nu). \end{aligned}$$

(3) Equivalences (a) if and only if (b) and (b) if and only if (d) are immediate by item (2). To prove (b) implies (c), we note that $\gamma_R = x^\nu\gamma_\Lambda = I^\nu : \overline{R} \subseteq I^\nu$. Conversely, assumption (c) implies that $\gamma_R = \gamma_R : \overline{R} \subseteq I^\nu : \overline{R} \subseteq \gamma_R$, hence $\gamma_R = I^\nu : \overline{R} = x^\nu\gamma_\Lambda$.

Remark 3.3. (1) In view of item (2) of the above proposition we have the inequality

$$l_R(R/\gamma_R) - l_R(\Lambda/\gamma_\Lambda) \geq -\rho$$

and, in the case $I = m$,

$$l_R(R/\gamma_R) - l_R(\Lambda/\gamma_\Lambda) \geq e - \rho.$$

In Example 7.1 we show that both these minimal values can be reached.
(2) Conditions (3) of 3.2 imply that $R : I^\nu \subseteq \overline{R}$, but if this inclusion holds we need not have the above equivalent conditions (see Example 7.2).
(3) Conditions (3) of 3.2 imply the conductors transitivity formula:

$$\gamma_R = (R : \Lambda)\gamma_\Lambda.$$

Example 7.3 shows that the converse does not hold.
(4) Conditions (3) of 3.2 do not imply that $R : \Lambda = I^\nu$. This can be seen in Example 7.4; however next lemma shows that the converse is true.

Lemma 3.4. *If $R : \Lambda = I^\nu$, then we have:*

(1) *The equivalent conditions of Proposition* 3.2, (3) *hold.*

(2) $\Lambda^{**} = \Lambda$.

Proof. (1) It is clear considering the diagram in 3.1.
To prove part (2), observe that condition $R : \Lambda = I^\nu = \Lambda I^\nu$ implies $\Lambda^{**} = R : \Lambda I^\nu = I^\nu : I^\nu = \Lambda$.

From the above considerations we obtain a first formula connecting the invariants ρ, e, ν associated to the ideal I with the invariants c, δ of R by means of the length of the quotient $R : \Lambda/I^\nu$. This formula will be successively improved in Theorem 4.7 by using type sequences.

Proposition 3.5. (1) $2\rho = e\nu + (2\delta - c) - l_R(R : \Lambda/I^\nu) - l_R(\omega\Lambda/\Lambda)$.

(2) *The following facts are equivalent:*

(a) $2\rho = e\nu + (2\delta - c)$.

(b) Λ *is Gorenstein and* $c - c_\Lambda = e\nu$.

(c) $R : \Lambda = I^\nu$ *and* $\omega\Lambda = \Lambda$.

(d) $R : \Lambda = I^\nu \subseteq R : \omega$.

Proof. From $2\rho = 2\delta - 2\delta_\Lambda = 2\delta - c - (2\delta_\Lambda - c_\Lambda) + c - c_\Lambda + e\nu - e\nu$, we get

$$2\rho = e\nu + (2\delta - c) - (2\delta_\Lambda - c_\Lambda) - (e\nu - c + c_\Lambda) \qquad (*)$$

Hence the equivalence (a) if and only if (b) of (2) is clear.
Since $I^\nu \subseteq R : \Lambda \subseteq R$, we have

$$l_R(R/R : \Lambda) = l_R(R/I^\nu) - l_R(R : \Lambda/I^\nu) = e\nu - \rho - l_R(R : \Lambda/I^\nu). \qquad (**)$$

From the inclusions $R \subseteq \Lambda \subseteq \omega\Lambda$ and $R \subseteq \omega \subseteq \omega\Lambda$, we obtain that

$$l_R(R/R : \Lambda) = l_R(\omega\Lambda/\omega) = l_R(\omega\Lambda/\Lambda) + \rho - (2\delta - c).$$

Substituting this in the first member of $(**)$ we get the first formula and also the equivalence (a) if and only if (c).
Finally, (c) if and only if (d) follows by using Proposition 2.9.

4. Formulas involving type sequences

We keep the notation of the above section. We have seen in 3.5 that

$$2\rho \leq e\nu + (2\delta - c).$$

Using the notion of type sequence we insert a new term in this inequality (see Theorem 4.7):

$$2\rho \leq e\nu + \sum_{i \notin \Gamma} (r_i - 1) \leq e\nu + (2\delta - c).$$

We study also conditions to have equalities. To do this we introduce the positive invariant $d(R:\Lambda)$, which plays a crucial role in this context.

Definition 4.1. *Let, as above,* $s_0 = 0, s_1, \ldots, s_n = c$ *be the first* $n+1$ *elements of* $v(R)$, $n = c - \delta$. *Let* $t.s.(R) = [r_1, \ldots, r_n]$ *be the type sequence of* R. *We call* $d(R:\Lambda)$ *the number*

$$d(R:\Lambda) := l_R(\overline{R}/\Lambda^{**}) - \sum_{i \in \Gamma} r_i$$

where Γ *denotes the numerical set* $\Gamma := \{i \in \{1,..,n\} \mid s_{i-1} \in v(R:\Lambda)\}$.

Note that

$$\#\Gamma = l_R(R:\Lambda/\gamma) = l_R(\overline{R}/\omega\Lambda)$$

The following proposition ensures that $d(R:\Lambda) \geq 0$.

Proposition 4.2. *We have*

$$l_R(\overline{R}/\omega\Lambda) \leq \sum_{i \in \Gamma} r_i \leq l_R(\overline{R}/\Lambda^{**}).$$

Proof. The first inequality is obvious since $r_i \geq 1 \ \ \forall i$.
For the second one we shall use property (5) of 2.4 with $I = R:\Lambda$,

$s \in v(I\omega)$ if and only if $c - 1 - s \notin v(\Lambda^{**})$.

If $x_{i-1} \in I$ is such that $v(x_{i-1}) = s_{i-1}$, then by definition

$$r_i = l_R(\omega R_{i-1}/\omega R_i) = l_R(x_{i-1}\omega + \omega R_i/\omega R_i) = \#\{v(x_{i-1}\omega + \omega R_i) \setminus v(\omega R_i)\}.$$

Since $v(x_{i-1}\omega) \subseteq v(I\omega)$, the assignment $y \to c - 1 - y$ defines an injective map

$$\bigcup_{i \in \Gamma} \{v(x_{i-1}\omega + \omega R_i) \setminus v(\omega R_i)\} \longrightarrow \mathbb{N} \setminus v(\Lambda^{**}).$$

From the fact that the numerical sets

$$\{v(x_{i-1}\omega + \omega R_i) \setminus v(\omega R_i)\}, \quad i \in \{1, \ldots, n\},$$

are disjoint by construction we deduce that

$$\sum_{i \in \Gamma} r_i \leq l_R(\overline{R}/\Lambda^{**}).$$

The next proposition collects some useful properties of the invariant $d(R:\Lambda)$ and allows us to find sufficient conditions to have $d(R:\Lambda) = 0$.

Proposition 4.3. *Let $i_o \in \mathbb{N}$ be such that $e(R:\Lambda) = s_{i_0}$. Then*

(1) $d(R:\Lambda) = l_R(\omega\Lambda/\Lambda^{**}) - \sum_{i\in\Gamma}(r_i - 1)$.

(2) *If $\omega \subseteq \Lambda^{**}$, i.e., $R:\Lambda \subseteq R:\omega$, then $d(R:\Lambda) = 0$.*

(3) $d(R:\Lambda) = \sum_{i>i_0,\ i\notin\Gamma} r_i - l_R(\Lambda^{**}/R^*_{i_0})$.

(4) *If $R:\Lambda$ is integrally closed, then $d(R:\Lambda) = 0$.*

Proof. (1) $d(R:\Lambda) = l_R(\overline{R}/\Lambda^{**}) - \sum_{i\in\Gamma} r_i = l_R(\omega\Lambda/\Lambda^{**}) - (\sum_{i\in\Gamma} r_i - l_R(\overline{R}/\omega\Lambda))$.

(2) The inclusion $\omega \subseteq \Lambda^{**}$ implies that $\omega\Lambda = \Lambda^{**}$, hence the thesis by (1), recalling that $d(R:\Lambda) \geq 0$.

(3) After writing $l_R(\overline{R}/\Lambda^{**}) = l_R(\overline{R}/R^*_{i_0}) - l_R(\Lambda^{**}/R^*_{i_0})$, the thesis is clear since

$$l_R(\overline{R}/R^*_{i_0}) = \sum_{i>i_0} r_i.$$

(4) This results from the above item, because the fact that $R:\Lambda$ is integrally closed means that $R:\Lambda = R_{i_0}$.

The next theorem provides a link between the type sequence of R and the genus ρ of the ideal I.

Theorem 4.4.

(1) $\rho = \sum_{i\notin\Gamma} r_i - l_R(\Lambda^{**}/\Lambda) - d(R:\Lambda) \leq r\ l_R(R/R:\Lambda)$.

(2) *Let $i_o \in \mathbb{N}$ be such that $e(R:\Lambda) = s_{i_0}$. Then*
$\rho = \sum_{i\leq i_0} r_i - l_R(\Lambda^{**}/\Lambda) + l_R(\Lambda^{**}/R^*_{i_0})$.

Proof. (1) From the inclusions $R \subseteq \Lambda \subseteq \Lambda^{**} \subseteq \overline{R}$ we obtain

$$\rho = l_R(\Lambda/R) = \delta - l_R(\Lambda^{**}/\Lambda) - l_R(\overline{R}/\Lambda^{**}) = \delta - l_R(\Lambda^{**}/\Lambda) - d(R:\Lambda) - \sum_{i\in\Gamma} r_i.$$

Thus the first equality is clear since $\delta - \sum_{i\in\Gamma} r_i = \sum_{i\notin\Gamma} r_i$.

The inequality follows immediately, recalling that $r_i \leq r \ \ \forall\ i$ and that

$$l_R(R/R:\Lambda) = \#\big(\{1,\dots,n\} \setminus \Gamma\big).$$

(2) By substituting formula (3) of 4.3 in formula (1) above, we obtain

$$\rho = \sum_{i\notin\Gamma} r_i - l_R(\Lambda^{**}/\Lambda) - \sum_{i>i_0, i\notin\Gamma} r_i + l_R(\Lambda^{**}/R^*_{i_0}) = \sum_{i\leq i_0} r_i - l_R(\Lambda^{**}/\Lambda) + l_R(\Lambda^{**}/R^*_{i_0}).$$

Remark 4.5. In the case $\Lambda = \overline{R}$ the inequality $\rho \leq r\ l_R(R/R:\Lambda)$ of Theorem 4.4 gives the well-known relation $\delta \leq r\ (c-\delta)$ ([8], Theorem 2).

The maximal value $\rho = r\ l_R(R/R:\Lambda)$ is achieved if and only if $r_i = r$ for all $i \notin \Gamma$, $\Lambda = \Lambda^{**}$ and $d(R:\Lambda) = 0$; this happens for instance if $I = m$ and $e = \mu$ (see 6.2), or if R is Gorenstein.

Corollary 4.6.

(1) $e\nu + rl_R(R:\Lambda/I^\nu) \leq (r+1)\ l_R(R/I^\nu)$.

(2) *If the h-polynomial is symmetric, then* $l_R(R:\Lambda/I^\nu) \leq \dfrac{r-1}{r} \cdot \dfrac{e\nu}{2}$

Proof. (1) From the first item of the theorem we have:

$$\rho = e\nu - l_R(R/I^\nu) \leq rl_R(R/R:\Lambda) = rl_R(R/I^\nu) - rl_R(R:\Lambda/I^\nu).$$

The thesis follows.

(2) By property (5) of 2.2 it suffices to substitute $l_R(R/I^\nu) = \dfrac{e\nu}{2}$ in (1).

Theorem 4.7.

(1) $2\rho = e\nu + \sum_{i\notin\Gamma} (r_i - 1) - d(R:\Lambda) - l_R(\Lambda^{**}/\Lambda) - l_R(R:\Lambda/I^\nu)$.

(2) *The following facts are equivalent:*

(a) $2\rho = e\nu + \sum_{i\notin\Gamma} (r_i - 1)$,

(b) $R:\Lambda = I^\nu$ *and* $d(R:\Lambda) = 0$.

Proof. (1) We can rewrite formula (1) of Proposition 3.5 as:

$$2\rho = e\nu + \sum_{i\notin\Gamma} (r_i - 1) + \sum_{i\in\Gamma} (r_i - 1) - l_R(R:\Lambda/I^\nu) - l_R(\Lambda^{**}/\Lambda) - l_R(\omega\Lambda/\Lambda^{**}).$$

So using item (1) of Proposition 4.3, we obtain part (1).

(2) follows from part (1) by virtue of Lemma 3.4 recalling that $d(R:\Lambda) \geq 0$.

We remark that the equality $R:\Lambda = I^\nu$ does not ensure that $d(R:\Lambda) = 0$ (see Example 7.5).

5. Almost Gorenstein rings.

In this section we deal with almost Gorenstein rings. The notations will be the same as in the preceding sections.

Under the hypothesis R almost Gorenstein, the formulas in 3.5, 4.4 and 4.7 involving the genus $\rho(I)$ are considerably simplified and allow us to extend some well-known results concerning the equality $R:\Lambda = I^\nu$. Recently Barucci and Fröberg stated the equivalence (a) if and only if (c) of next Theorem 5.3 in the case R almost Gorenstein and $\Lambda = \Lambda(m)$ (see [3], Proposition 26).

First, inspired by the famous result of Bass: *A one-dimensional Noetherian local domain R is Gorenstein if and only if each nonzero fractional ideal of R is reflexive* (see [1], Theorem 6.3), we notice that:

Proposition 5.1. *R is almost Gorenstein if and only if $\omega J = J^{**}$ for every not principal fractional ideal J.*

Proof. Suppose R almost Gorenstein. By 2.7 it suffices to prove that $\omega J \subseteq J^{**}$. Since $R : J = m : J$, we have $(R : J)J\omega \subseteq m\omega = m$, hence $\omega J \subseteq J^{**}$.
The opposite implication follows immediately by taking $J = m$.

Corollary 5.2. *If R is an almost Gorenstein ring, then*

(1) $\Lambda^{**} = \omega\Lambda$ *and* $d(R : \Lambda) = 0$,
(2) $\rho = r - 1 + l_R(R/R : \Lambda) - l_R(\Lambda^{**}/\Lambda)$.

Proof. (1) The second equality follows from Proposition 4.3,(1).
(2) Apply formula (1) of Theorem 4.4, observing that in the almost Gorenstein case

$$\sum_{i \notin \Gamma} r_i = r - 1 + l_R(R/R : \Lambda).$$

Under the assumption R almost Gorenstein, since

$$\omega\Lambda = \Lambda^{**}, \quad \sum_{i \notin \Gamma}(r_i - 1) = r - 1 = 2\delta - c \quad \text{and} \quad d(R : \Lambda) = 0$$

both Proposition 3.5 and Theorem 4.7 give the next theorem.

Theorem 5.3. *Assume that R is an almost Gorenstein ring and let $\Lambda = \Lambda(I)$. Then:*

(1) $2\rho = e\nu + r - 1 - l_R(R : \Lambda/I^\nu) - l_R(\Lambda^{**}/\Lambda)$.
(2) *The following conditions are equivalent:*
   (a) $2\rho = e\nu + r - 1$.
   (b) Λ *is Gorenstein and* $c - c_\Lambda = e\nu$.
   (c) $R : \Lambda = I^\nu$.
   (d) $\omega : \Lambda = I^\nu$.

In this case the equivalent conditions (A) *of Proposition* 2.9 *hold.*

Proof. We have only to prove (c) if and only if (d).
(c) implies (d). By Lemma 3.4 we have $\Lambda = \Lambda^{**} = \omega\Lambda$. Hence $\omega : \Lambda = R : \Lambda$ by duality.
To prove (d) implies (c), we notice that $I^\nu \subseteq R : \Lambda \subseteq \omega : \Lambda$.

Corollary 5.4. *If R is an almost Gorenstein ring and the h-polynomial is symmetric, then $l_R(R : \Lambda/I^\nu) \leq r - 1$ and the equality holds if and only if $\Lambda = \Lambda^{**}$.*

Proof. The symmetry of the h-polynomial gives $2\rho = e\nu$ (see 2.2(5)), hence it suffices to substitute this in formula (1) of the theorem.
We note that the condition $l_R(R : \Lambda/I^\nu) = r - 1$ does not imply that the h-polynomial is symmetric: see for instance Example 7.7, where R is almost Gorenstein with $r(R) > 1$ and Example 7.6, where R is Gorenstein. Example 7.6 shows also that the hypotheses R Gorenstein and $2\rho = e\nu$ do not give the symmetry of the h-polynomial.

The following statement of Ooishi (see [12], Corollary 6) can be obtained as a direct consequence of our preceding results.

Corollary 5.5. *If R is Gorenstein and the h-polynomial is symmetric, then the equivalent conditions* (2) *of Theorem* 5.3 *hold.*

Another immediate consequence of Theorem 5.3 is the natural generalization of Theorem 10 of [12] to the almost Gorenstein case.

Corollary 5.6. *Suppose R almost Gorenstein. The equality $\gamma = I^\nu$ holds if and only if $\Lambda = \overline{R}$ and $2\delta = e\nu + r - 1$.*

Formula (1) of Theorem 5.3 is very useful in applications, especially when $\Lambda = \Lambda^{**}$. In the next theorem we prove that in the almost Gorenstein case the blowing-up along a reflexive ideal I is reflexive; this is not always true (see Example 7.9). Nevertheless, in Example 7.4 we have R almost Gorenstein, Λ reflexive, but I not reflexive.

First we recall the following property (see [10], Corollary 3.15).

5.7. *Let R be almost Gorenstein and let J be a fractional ideal not isomorphic to R, then*

$$J \text{ is reflexive if and only if } J : J \supseteq R : m.$$

Theorem 5.8. *Suppose R almost Gorenstein and let $\Lambda = \Lambda(I)$. Then*

(1) *The equivalent conditions of the groups* (A), (B) *of Proposition* 2.9 *are equivalent to the following ones:*

(C) (C$_1$) $\Lambda \supseteq R : m$. (C$_2$) *I^ν is reflexive.*
(C$_3$) *I^n is reflexive $\forall n \geq \nu$.* (C$_4$) *I^n is reflexive for some $n \geq \nu$.*

(2) *If I is reflexive, then the equivalent conditions* (A), (B), (C) *hold, in particular Λ is reflexive.*

Proof. (1) The equivalence of conditions (C) is immediately achieved by using 5.7.

In order to prove the equivalence (A) if and only if (C), we note that $\omega I^n = (I^n)^{**}$ by Proposition 5.1, hence

$$\omega I^n = I^n \quad \text{if and only if} \quad I^n = (I^n)^{**}.$$

(2) By applying as before Proposition 5.1 we deduce that $I = I^{**} = \omega I$; but this is equivalent to $\omega \subseteq I : I \subseteq \Lambda$.

6. Blowing-up along the maximal ideal

Our purpose is now to consider the special case $I = m$. We denote by $\Lambda := \Lambda(m)$ the blowing-up of R along the maximal ideal, e the multiplicity, $\mu := l_R(m/m^2)$ the embedding dimension, r the Cohen-Macaulay type of R; $x \in m$ is a minimal reduction of m.

When $e = \mu$, namely m is stable, we can prove that the Gorensteiness of the blowing-up Λ is equivalent to the almost-Gorensteiness of the ring R.

When $e = \mu + 1$, we get an explicit formula for the length of the module $R : \Lambda/m^\nu$. It turns out that this length is zero if and only if R is Gorenstein and $\nu = 2$.

In the cases $\nu = 2$ and $\nu = 3$ we state formulas involving the conductor $R : \Lambda$ which extend some results of Ooishi valid for Gorenstein rings (see [12]).

We begin with two simple remarks, useful in the sequel.

Remark 6.1. (1) $l_R(R/R:\Lambda) = l_R(x(R:m)/R:\Lambda) + (e-r)$.

(2) *If R is almost Gorenstein, then $\Lambda = \Lambda^{**}$.*

Proof. (1) We know that $x\Lambda = m\Lambda$ by property (1) of 2.2. Therefore the inclusion $m \subseteq x\Lambda$ implies that $R:\Lambda \subseteq x(R:m) \subseteq R \subseteq R:m$. From this chain we get the thesis.

(2) This is true by Theorem 5.8, since $I = m$ is reflexive.

6.2. Case $e = \mu$. We recall that m is said to be *stable* if $\Lambda = m:m$. We have the following well-known equivalent conditions for the stability of m (see [7], Theorem 12.15):

(1) m is stable

(2) $e = \mu$

(3) $\rho = e-1$

(4) $r = e-1$.

Proposition 6.3. *If m is stable, then the following facts are equivalent:*

(1) *R is almost Gorenstein,*

(2) *Λ is Gorenstein.*

Proof. By hypothesis $R:\Lambda = m$ and $\nu = 1$. Hence if R is almost Gorenstein, then Λ is Gorenstein by Theorem 5.3. Vice versa, the hypothesis $\Lambda = m:m$ implies that $c_\Lambda = c-e$. Thus if Λ is Gorenstein, then condition (2),(b) of Proposition 3.5 is satisfied and R is almost Gorenstein because

$$2\delta - c = 2\rho - e = e - 2 = r - 1.$$

6.4. Case $e = \mu + 1$. If $e = \mu+1$, the structure of R is quite well understood, see e.g. [15]. From the form of the h-polynomial $h(z) = 1 + (\mu-1)z + z^\nu$, one can infer that $\rho = \mu - 1 + \nu$. Moreover, there are two possibilities depending on the Cohen-Macaulay type r:

(A) If $r < e-2$, then $\nu = 2$;

(B) If $r = e-2$, then $m^2 = xm + (w^2)R$, with $w \in m \setminus m^2$ and $m^3 \subset xm$ (see [15], Prop. 5.1).

We begin with a technical lemma.

Lemma 6.5. *Assume that $r = e-2$. Then there exists an element $w \in m$ with $v(w) - e \notin v(m:m)$ such that:*

(1) *$m = x(m:m) + wR$ and $wm \subset x(m:m)$.*

(2) *$m^j = xm^{j-1} + w^jR = x^{j-1}m + x^{j-2}w^2R + \cdots + xw^{j-1}R + w^jR, \forall j = 2,\dots,\nu$.*

(3) *$m^3 \subseteq xm$.*

(4) *For every element $s \in m:m$ such that $v(s) > 0$ we have $sw^j \in x^{j-1}m,\ \forall j = 2,\dots,\nu$.*

Proof. (1) The assumption $r = e-2$ means that $l_R(m/x(m:m)) = 1$, hence by 2.1 there exists an element $w \in m$ such that $v(w) - e \notin v(m:m)$ and $m = x(m:m) + wR$. To prove the inclusion $wm \subseteq x(m:m)$ it suffices to consider the chain $x(m:m) \subseteq x(m:m) + wm \subset m$.

(2) We prove our claim by induction on j. Suppose $j = 2$. From (1) we have that $m^2 \subseteq xm + wm = xm + w(x(m:m) + wR) \subseteq xm + w^2R \subseteq m^2$. Suppose now the assertion true

for j. Claim: $m^{j+1} = xm^j + w^{j+1}R = x^j m + x^{j-1}w^2R + \cdots + xw^jR + w^{j+1}R$.
By using repeatedly the inductive hypothesis we get
$m^{j+1} = xm^j + w^j m \subseteq xm^j + wm^j = xm^j + w(xm^{j-1} + w^jR) \subseteq xm^j + w^{j+1}R \subseteq m^{j+1}$.
We are left to prove the second equality of the claim. We have:
$m^{j+1} = x^{j-1}m^2 + x^{j-2}w^2m + \cdots + xw^{j-1}m + w^jm \subseteq x^{j-1}m^2 + x^{j-2}wm^2 + \cdots + xw^{j-2}m^2 + w^{j-1}m^2 = x^{j-1}(xm+w^2R)+x^{j-2}w(xm+w^2R)+\cdots+xw^{j-2}(xm+w^2R)+ w^{j-1}(xm+w^2R) \subseteq x^jm + x^{j-1}w^2R + \cdots + x^2w^{j-1}R + xw^jR + w^{j+1}R \subseteq m^{j+1}$.
For the last but one inclusion we have used the fact that
$x^{j-1}wm+\cdots+x^2w^{j-2}m+xw^{j-1}m = x^{j-1}w(x(m:m)+wR)+\cdots+xw^{j-1}(x(m:m)+wR) \subseteq x^jm + x^{j-1}w^2R + \cdots + x^2w^{j-1}R + xw^jR$.
(3) As seen in the proof of item (2), $m^2 = xm+wm$. Hence $m^3 = xm^2+wm^2 \subseteq xm$, because $wm^2 \subseteq xm$ by item (1).
(4) Let $s \in m : m$ be such that $v(s) > 0$. We proceed by induction on j. Suppose $j = 2$. By item (1) there exist $y \in m : m$ and $a \in R$ such that $sw = xy + aw$. If $v(a) = 0$, then $v(s-a) = 0$, contradicting the fact that $v(w) - e \notin v(m : m)$. Hence $a \in m$. Thus $sw^2 = xyw + aw^2 \in xm$, because $m^3 \subseteq xm$. Assume now the inductive hypothesis $\frac{sw^j}{x^{j-1}} \in m$, then $\frac{sw^j}{x^{j-1}} = xz + bw$, with $z \in m : m$, $b \in R$, i.e., $(\frac{sw^{j-1}}{x^{j-1}} - b)w = xz$. Since the element $\frac{sw^{j-1}}{x^{j-1}}$ has a positive valuation, by the same reasoning as above we conclude that $b \in m$. Therefore $\frac{sw^{j+1}}{x^{j-1}} = xzw + bw^2 \in xm$, which is our thesis.

Proposition 6.6. (1) *If $r = e-2$, then $e = \mu + 1$.*
(2) *If $e = \mu+1$, then $l_R(x(R:m)/R:\Lambda) = 1$.*

Proof. (1) By Lemma 6.5,(2), $m^2 = xm + w^2R$. Hence

$$l_R(m/m^2) = l_R(m/xm) - l_R(m^2/xm) = e - 1.$$

(2) We shall prove that the R-module

$$R : m/x^{\nu-1}(R : m^\nu) \simeq x(R : m)/R : \Lambda$$

is monogenous generated by $\overline{1}$. We divide the proof in two parts, following cases: (A) $r < e-2$ and (B) $r = e-2$ above.
Case (A) $\nu = 2$. Since $m^2 = xm + (a)R$, $a \notin xm$, we have that $x(R : m^2) = (R : m) \cap \left(\frac{x}{a}\right)R$. If $y \in R : m$, then $ya \in m^2$ and we can write $ya = xr + as$, with $r \in m$, $s \in R$, namely $y = \frac{x}{a}r + s$, so $\overline{y} = \overline{1}s$.
Case (B). We want to prove that if $s \in m : m$ has a positive valuation, then $s \in x^{\nu-1}(R : m^\nu)$. By item (2) and (4) of Lemma 6.5 we have

$$\frac{sm^\nu}{x^{\nu-1}} \in sm + s\frac{w^2}{x}R + \cdots + s\frac{w^{\nu-1}}{x^{\nu-2}}R + s\frac{w^\nu}{x^{\nu-1}}R \subseteq m.$$

Theorem 6.7. *Let* $e = \mu + 1$. *Then*

$$l_R(R : \Lambda/m^\nu) = r - 1 + (e - 1)(\nu - 2).$$

Proof. We have to compute the difference $l_R(R/m^\nu) - l_R(R/R : \Lambda)$. As recalled in 6.4 $e\nu - l_R(R/m^\nu) = \rho = e - 2 + \nu$. Combining the above results 6.1 and 6.6 we obtain $l_R(R/R : \Lambda) = e - r + 1$. The conclusion follows.

Corollary 6.8. *Let* $e = \mu + 1$. *Then*
(1) $R : \Lambda = m^\nu$ *if and only if* R *is Gorenstein and* $\nu = 2$.
(2) $\sum_{i \notin \Gamma, i \neq 1} (r_i - 1) = d(R : \Lambda) + l_R(\Lambda^{**}/\Lambda) + (\nu - 2)$.
(3) R *is almost Gorenstein if and only if* $\nu = 2$ *and* $\omega\Lambda = \Lambda$.

Proof. (1) It follows directly from Theorem 6.7.
(2) As recalled in 6.4 $\rho = \mu - 1 + \nu$, then the formula of Theorem 4.7 gives:

$$l_R(R : \Lambda/m^\nu) = (\mu + 1)\nu - 2(\mu - 1 + \nu) + \sum_{i \notin \Gamma} (r_i - 1) - d(R : \Lambda) - l_R(\Lambda^{**}/\Lambda).$$

By comparing with Theorem 6.7 the thesis follows.
(3) This is clear after observing that item (2) combined with equality (1) of Proposition 4.3 becomes:

$$(2\delta - c) - (r - 1) = l_R(\omega\Lambda/\Lambda) + (\nu - 2).$$

Corollary 6.9. $r = e - 2$ *and* $R : \Lambda = m^\nu$ *if and only if* R *is Gorenstein with* $e = 3$.

6.10. Case $\nu = 2$. We recall that in this case the invariants ρ, e, μ are related by the equality:

$$\rho = 2e - \mu - 1.$$

Proposition 6.11. *Assume* $\nu = 2$.
(1) $2e + rl_R(R : \Lambda/m^2) \leq (r + 1)(\mu + 1)$.
(2) *If* R *is almost Gorenstein, then* $e - (\mu + 1) = \dfrac{r - 1}{2} - \dfrac{1}{2} l_R(R : \Lambda/m^2)$.
In particular:
if R *is Gorenstein, then* $e = \mu + 1$ *and* $R : \Lambda = m^2$;
if R *is a Kunz ring (namely almost Gorenstein of type* 2*), then*

$$e = \mu + 1 \text{ and } l_R(R : \Lambda/m^2) = 1.$$

Proof. (1) The inequality follows directly from Corollary 4.6.
(2) This is Theorem 5.3 with $\nu = 2$, $\rho = 2e - \mu - 1$ and $\Lambda = \Lambda^{**}$.
We deduce from Proposition 6.11 that if R is almost Gorenstein, then:

$$R : \Lambda = m^2 \text{ if and only if } e - (\mu + 1) = \frac{r - 1}{2} \text{ and } \nu = 2. \qquad (*)$$

This equivalence was already known for Gorenstein rings: assertion $(*)$ in the case $r = 1$ is exactly Corollary 7 of [12].

We remark that there exist almost Gorenstein rings satisfying the condition

$$e - (\mu + 1) = \frac{r-1}{2}$$

with $\nu \neq 2$ (see Example 7.3). The next corollary shows that this cannot happen when R is Gorenstein.

Corollary 6.12. *Let R be a Gorenstein ring. Then the following conditions are equivalent:*

(1) $e = \mu + 1$.

(2) $\nu = 2$

(3) $R : \Lambda = m^2$.

Proof. If $e = \mu + 1$, we have that $\nu = 2$ by Corollary 6.8. To conclude the proof it suffices to apply Proposition 6.11.

In the case R Gorenstein, Corollary 6.8 combined with Corollary 6.12 gives Proposition 12 of [12]: $\gamma_R = m^2$ if and only if $\Lambda = \overline{R}$ and $e = \mu + 1$.

The following proposition states a more general relation between e and $\mu + 1$.

Proposition 6.13. *The following conditions are equivalent:*

(1) $\gamma_R = m^2$,

(2) $\Lambda = \overline{R},\ e - (\mu + 1) = \frac{1}{2}(2\delta - c)$ *and* $\nu = 2$.

Proof. If $\gamma_R = m^2$, then we have $m^2 : m^2 = \overline{R} = \Lambda$ and $\nu = 2$; hence we get

$$c = 2e \text{ and } 2\rho = 2e + 2\delta - c,$$

because condition (c) of Proposition 3.5 is verified. On the other hand, since $\nu = 2$, $\rho = 2e - \mu - 1$. By comparing the two equalities we see that (2) holds.

Conversely, $\gamma_R = R : \Lambda$ and $\delta = \rho = 2e - \mu - 1$ imply that

$$2\delta = 4e - 2\mu - 2 = 2e - 2(\mu + 1) + c.$$

Hence $c = 2e$, and again by Proposition 3.5 we obtain that $\gamma_R = R : \Lambda = m^2$.

6.14. Case $\nu = 3$. This case has been considered by Ooishi in [12]. Statement (3) of the next proposition extends to almost Gorenstein rings Proposition 8 of his quoted paper, valid in the case $r = 1$.

Proposition 6.15. *Assume that $\nu = 3$.*

(1) *If $r = 2$, then* $e - (\mu + 1) + \frac{2}{3} l_R(R : \Lambda / m^2) \leq l_R(m^2/m^3)$.

(2) *If the h-polynomial is symmetric, then*

$$l_R(R : \Lambda/m^3) \leq 3\mu\frac{(r-1)}{r}.$$

(3) *If R is almost Gorenstein, then the h-polynomial is symmetric if and only if $l_R(R : \Lambda/m^3) = r - 1$ and $e = 2\mu$.*

Proof. (1) It suffices to apply Corollary 4.6.

(2) Of course, $h_m(z) = 1 + (\mu - 1)z + a_2z^2 + a_3z^3$ is symmetric if and only if $a_3 = 1$ and $a_2 = \mu - 1$. In this case we obtain $e = 2\mu$ and $\rho = 3\mu$. The inequality follows from Corollary 4.6.

(3) This comes directly from the main formula of Theorem 5.3.

Using again the formula of Theorem 5.3 we get immediately the next result.

Corollary 6.16. *Suppose R almost Gorenstein. If $e = 2\mu$, then*

$$R : \Lambda = m^\nu \text{ if and only if } \rho = \nu\ \mu + \frac{r-1}{2}.$$

We notice that in the case $r = 1$ and $\nu = 3$ result 6.16 gives again Proposition 8 of [12]; however there exist almost Gorenstein, not Gorenstein, rings satisfying the equivalent conditions of 6.16 (see Example 7.10).

7. Examples

In all examples listed below we suppose that $R = \mathbb{C}[[t^h]],\ h \in v(R)$, is a semigroup ring and that $\Lambda = \Lambda(I)$ is the blowing-up of R along the specified ideal I.

Notation $< \cdots >$ means "the semigroup generated by $\cdots$". Notation $a - b$ in the semigroup means "all the integers between a and b". Notation $a \rightarrow$ means "all the integers $\geq a$".

Example 7.1. (See the first remark in 3.3.) Let $v(R) = \{0, 10, 12, 20 \rightarrow\}$.

(1) If $I = (t^{10}, t^{12})$, then $v(\Lambda) =< 2, 21 >$, hence $c = c_\Lambda = 20$ and $l_R(R/\gamma_R) - l_R(\Lambda/\gamma_\Lambda) = -\rho$.

(2) If $I = m$, then $v(\Lambda) =< 2, 11 >$, hence $c - c_\Lambda = 10 = e$ and $l_R(R/\gamma_R) - l_R(\Lambda/\gamma_\Lambda) = e - \rho = -2$.

Example 7.2. (See the second remark in 3.3.) Let $v(R) = \{0, 5, 10, 11, 12, 15, 16, 17, 19 \rightarrow\}$, i.e., $v(m) =< 5, 11, 12, 19 >$, and let $I = m$. We have $\nu = 2$ and $v(\Lambda) = \{0, 5, 6, 7, 10 \rightarrow\}$. Moreover: $v(R : m^2) = \{0, 5, 6, 7, 9 \rightarrow\}$, hence $R : m^2 \subset \overline{R}$, but $c - c_\Lambda = 9 < e\nu = 10$.

Example 7.3. (See the third remark in 3.3 and also the remark after 6.11.) Let $v(R) = \{0, 7, 8, 12, 13, 14, 15, 16, 18 \rightarrow\}$, i.e., $v(m) =< 7, 8, 12, 13, 18 >$. R is almost Gorenstein and its h-polynomial is $h(z) = 1 + 4z + z^2 + z^4$. We have $m^4 = t^{28}\overline{R}$, hence $\Lambda = \overline{R}$ and $m^4 \subset R : \Lambda = \gamma_R = t^{18}\overline{R}$. In this case formula $\gamma_R = (R : \Lambda)\gamma_\Lambda$ holds, but $c - c_\Lambda = 18 < e\nu = 28$. Since $e = 7, \mu = 5, r = 3$, condition $e - (\mu + 1) = \dfrac{r-1}{2}$ is satisfied, but $\nu = 4$.

Example 7.4. (See the fourth remark in 3.3 and also the remark after 5.6.) Let $v(R) = \{0, 5, 10, 15, 20, 21, 25, 26, 30-32, 35-37, 40-42, 45-48, 50-53, 55-58, 60 \rightarrow\}$, i.e., $v(m) =< 5, 21, 32, 48 >$. R is an almost Gorenstein ring with Cohen-Macaulay type $r = 3$ and $e = \mu+1$.

(1) If $I=m$, then $v(\Lambda)=\{0, 5, 10, 15, 16, 20, 21, 25-27, 30-32, 35-37, 40-43, 45-48, 50\rightarrow\}$ and $\nu = 2$, hence $c - c_\Lambda = 10 = e\nu$ and Λ is reflexive, but $m^2 \subset R : \Lambda$.

(2) If $I = (t^{31}, t^{32}, t^{40})$, then I is not reflexive, whereas $\Lambda = I^4 : I^4 = \overline{R}$ is reflexive.

Example 7.5. (See the remark after 4.7.) Let $v(R) = \{0, 10, 20, 21, 25, 26, 30 - 36, 40 - 47, 50\rightarrow\}$, i.e., $v(m)=< 10, 21, 25, 26, 32, 33, 34 >$, and let $I=m$. Here $v(\Lambda)=\{0, 10, 11, 15, 16, 20 - 27, 30 \rightarrow\}$, $\Lambda = \Lambda^{**} = m^2 : m^2$ and $R : \Lambda = m^2$. Since the type sequence of R is $[3, 2, 1, 2, 1, 3, 1, 1, 1, 1, 1, 1, 2, 1, 1, 1, 1, 1, 1, 1, 2]$, $v(R : \Lambda) = \{20, 30, 31, 35, 36, 40-47, 50\rightarrow\}$, $\Gamma = \{3, 7, 8, 12 \rightarrow\}$, we have that

$$\sum_{i\in\Gamma} r_i = 15 \quad \text{and} \quad d(R : \Lambda) = l_R(\overline{R}/\Lambda) - \sum_{i\in\Gamma} r_i = 17 - 15 = 2.$$

Example 7.6. (See the remark after 5.4.) Let $v(R) = \{0, 11, 12, 15, 22 - 27, 29, 30, 33 - 42, 44 \rightarrow\}$, i.e., $v(m) =< 11, 12, 15, 25, 29 >$, and let $I = m$. R is a Gorenstein ring and its h-polynomial $h_I(z) = 1 + 4z + 2z^2 + 2z^3 + 2z^4$ is not symmetric. We have $\nu = 4$ and $\rho = 22$, hence $2\rho = e\nu$.

Example 7.7. (See the remark after 5.4.) Let R be such that $v(m) :=< 10, 23, 55, 58, 82 >$ and let $I = m$. R is almost Gorenstein with Cohen-Macaulay type $r = 3$ and its h-polynomial $h_I(z) = 1 + 4z + z^2 + 2z^3 + 2z^4$ is not symmetric. We have $\rho = 20$, $\nu = 4$. Hence R verifies the condition $2\rho = e\nu$.

Example 7.8. In this example R is an almost Gorenstein ring with Cohen-Macaulay type $r = 3$, verifying the equivalent conditions of Theorem 5.3. Let $v(m) :=< 10, 16, 95, 99 >$ and let $I = m$. Its h-polynomial is $h_I(z) = 1+3z+2z^2+2z^3+2z^4$ and $c = 124$. Since $\rho = 21$ and $\nu = 4$, we have $2\rho = e\nu + r - 1$. It follows that $\Lambda = \Lambda^{**}$ is Gorenstein and $c_\Lambda = c - e\nu = 84$.

Example 7.9. (See the remark after 5.6.) Let R be such that $v(m) :=< 6, 11, 16, 20, 25 >$ and let $I = m$. The blowing-up $\Lambda = m^2 : m^2$ is not reflexive.

Example 7.10. (See the remark after 6.16.) Let $v(R) = \{0, 8, 10, 13, 15, 16, 18, 20, 21, 23 - 26, 28 \rightarrow\}$, i.e., $v(m) =< 8, 10, 13, 15 >$, and let $I = m$. R is an almost Gorenstein ring with Cohen-Macaulay type $r = 3$, verifying the conditions of Corollary 6.16. In fact, $e = 2\mu$ and the h-polynomial is $h_I(z) = 1 + 3z + 2z^2 + 2z^3$, hence $\rho = 13 = \nu\mu + (r - 1)/2$. Thus $R : \Lambda = m^3$ and by 5.3 $c_\Lambda = c - e\nu = 4$.

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