Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 46 (2005), No. 2, 447-466.

Generalized GCD Modules

Majid M. Ali **David J. Smith**

Department of Mathematics, Sultan Qaboos University
e-mail: mali@squ.edu.om

Department of Mathematics, University of Auckland
e-mail: smith@math.auckland.ac.nz

Abstract. In recent work we called a ring R a GGCD ring if the semigroup of finitely generated faithful multiplication ideals of R is closed under intersection. In this paper we introduce the concept of generalized GCD modules. An R-module M is a GGCD module if M is multiplication and the set of finitely generated faithful multiplication submodules of M is closed under intersection. We show that a ring R is a GGCD ring if and only if some R-module M is a GGCD module.
Glaz defined a p.p. ring to be a GGCD ring if the semigroup of finitely generated projective (flat) ideals of R is closed under intersection. As a generalization of a Glaz GGCD ring we say that an R-module M is a Glaz GGCD module if M is finitely generated faithful multiplication, every cyclic submodule of M is projective, and the set of finitely generated projective (flat) submodules of M is closed under intersection. Various properties and characterizations of GGCD modules and Glaz GGCD modules are considered.

MSC 2000: 13C13, 13A15
Keywords: Multiplication module, projective module, flat module, invertible ideal, p.p. ring, greatest common divisor, least common multiple

0. Introduction

Let R be a ring and M an R-module. For submodules K and L of M, $[K : L]$ is defined as $\{x \in R : xL \subseteq K\}$. The *annihilator* of L is $\mathrm{ann}L = [0 : L]$. L is *faithful* if $\mathrm{ann}L = 0$. M is *multiplication* if each submodule N of M has the form IM for some ideal I of R, [13].

Equivalently, M is multiplication if and only if for all submodules N of M, $N = [N : M]M$. A submodule L of M is multiplication if and only if $K \cap L = [K : L]L$ for each submodule K of M, [19, Lemma 3.1]. M is a *cancellation* module if for all ideals I and J of R, $IM \subseteq JM$ implies that $I \subseteq J$, [7] and [9]. Finitely generated faithful multiplication modules are cancellation modules, [22, Corollary of Theorem 9]. Using this fact it is easy to see that if M is a finitely generated faithful multiplication module then $I[N : M] = [IN : M]$ for each submodule N of M and each ideal I of R.

Let M be an R-module. Anderson, [5] and [6], defined $\theta(M) = \sum_{m \in M} [Rm : M]$. He proved [6, Proposition 1 and Theorem 1] that if M is a multiplication module then $M = \theta(M)M$, and $\theta(M) = R$ if and only if M is also finitely generated. Let M be a multiplication module and N a submodule of M. Then $N = IM$ for some ideal I of R, and hence

$$N = I\theta(M)M = \theta(M)(IM) = \theta(M)N.$$

If N is finitely generated then $R = \theta(M) + \mathrm{ann}N$, [16, Theorem 76]. If moreover N is faithful then $R = \theta(M)$, and hence M is finitely generated. Also, M is faithful since $\mathrm{ann}M \subseteq \mathrm{ann}N = 0$. Thus modules that contain finitely generated faithful submodules are always finitely generated and faithful, and hence they are cancellation modules. Multiplication modules have received considerable attention in recent years, see for example [3], [5]–[10], [18], and [22].

In [4] and [2] we studied the greatest common divisor and least common multiple of finitely generated faithful multiplication and finitely generated projective ideals. The main purpose of the present paper is to extend and generalize those results to finitely generated faithful multiplication and finitely generated projective submodules of multiplication modules.

Let M be a multiplication module and $S(M)$ the set of finitely generated faithful multiplication submodules of M. In Section 2 we investigate GCD and LCM of elements of $S(M)$. Let $N, K \in S(M)$. If $\mathrm{GCD}(N, K)$ (resp. $\mathrm{LCM}(N, K)$) exists then it is unique and is in $S(M)$. We show in Proposition 2.1 that if $\mathrm{LCM}(N, K)$ exists then so too does $\mathrm{GCD}(N, K)$, and in this case

$$\begin{aligned}[\mathrm{GCD}(N, K) : M]\mathrm{LCM}(N, K) &= [\mathrm{LCM}(N, K) : M]\mathrm{GCD}(N, K)\\ &= [N : M]K = [K : M]N.\end{aligned}$$

Proposition 2.4 extends Euclid's Lemma to submodules. The relationships between GCD and LCM of submodules of $S(M)$ and ideals of $S(R)$ are investigated in Proposition 2.5. We prove that $\mathrm{GCD}(N, K)$ (resp. $\mathrm{LCM}(N, K)$) exists for all $N, K \in S(M)$ if and only if $\gcd(I, J)$ (resp. $\mathrm{lcm}(I, J)$) exists for all $I, J \in S(R)$. We also show that $\mathrm{GCD}(N, K)$ exists for all $N, K \in S(M)$ if and only if $\mathrm{LCM}(N, K)$ exists for all $N, K \in S(M)$.

In [4] we called a ring R a generalized GCD ring (GGCD ring) if $S(R)$ is closed under intersection. We extend this to modules: A *generalized* GCD *module* (GGCD *module*) M is a multiplication module such that $S(M)$ is closed under intersection. Proposition 2.5 shows that R is a GGCD ring if and only if some R-module is a GGCD module. Several characterizations of GGCD module are given in Theorem 2.6.

For a GGCD module M and for $N, K \in S(M)$, we define

$$\Phi^M_{N,K} = \Phi_{N,K} = \{T \;:\; T \text{ is a submodule of } M,\ T|N,\ \mathrm{GCD}(T, K) = M\}.$$

In Section 3 we give several properties of this lattice of submodules of $S(M)$. We show in Theorem 3.4 that $T \in \Phi_{N,K}$ is the smallest element if and only if the only submodule dividing $[N : T]M$ and M-coprime to K (i.e. $\mathrm{GCD}(T, K) = M$) is M. We also prove that if M is a GGCD module and $N, K \in S(M)$ and $G = \mathrm{GCD}(N, K)$, then $\Phi_{N,K} = \Phi_{[N:G]M,G}$, from which we derive several consequences, see Corollary 3.7 and Theorem 3.8.

Let M be a finitely generated faithful multiplication module, and let $S^*(M)$ be the set of finitely generated projective submodules of M. In Section 4 we study the GCD and LCM of elements of $S^*(M)$. We relate the existence of GCD and LCM of elements of $S^*(M)$ to one another and to the existence of the gcd and lcm of certain annihilator ideals. We also establish arithmetic relationships between them.

Glaz, [14] and [15], defined a GGCD ring to be a p.p. ring R such that $S^*(R)$ is closed under intersection. In this paper we generalize this to modules: An R-module M is a *Glaz* GGCD *module* if M is a finitely generated faithful multiplication module, every cyclic submodule of M is projective, and $S^*(M)$ is closed under intersection. Theorem 4.5 lists twenty conditions equivalent to this.

All rings considered in this paper are commutative with identity, and all modules are unital. For the basic concepts used, see [12], [13], [16], [17].

1. Preliminaries

Our first result collects several properties and characterizations of submodules of a finitely generated faithful multiplication module from [1, Propositions 2.3 and 3.7] and [18, Lemma 1.4].

Lemma 1.1. *Let R be a ring and N a submodule of a finitely generated faithful multiplication module M.*

(1) *N is finitely generated if and only if $[N : M]$ is a finitely generated ideal of R.*
(2) *N is multiplication if and only if $[N : M]$ is a multiplication ideal of R.*
(3) *N is flat if and only if $[N : M]$ is a flat ideal of R.*
(4) *If N is finitely generated then N is projective if and only if $[N : M]$ is a projective ideal of R.*
(5) *If N is finitely generated then N is projective if and only if N is multiplication and* $\mathrm{ann}N = Re$ *for some idempotent e.*
(6) *If N is finitely generated then N is flat if and only if N is multiplication and* $\mathrm{ann}N$ *is a pure ideal of R.*
(7) $\mathrm{ann}N = \mathrm{ann}[N : M]$. *In particular, N is faithful if and only if $[N : M]$ is faithful.*

It is evident from the above lemma that N and $[N : M]$ are closely related. They are locally isomorphic, but they need not to be isomorphic. In fact even if N is a faithful multiplication module, N need not embed in R (see [21] for an example).

Corollary 1.2. *Let R be a ring and M a finitely generated faithful multiplication R-module. Let N, K be finitely generated submodules of M.*

(1) *If K is a faithful multiplication then $N \cap K$ is a finitely generated faithful multiplication submodule if and only if $[N : K]$ is a finitely generated faithful multiplication ideal of R.*

(2) *If N and K are projective then $N \cap K$ is a finitely generated projective submodule of M if and only if $[N : K]$ is a finitely generated projective ideal of R.*

Proof. (1) Suppose $N \cap K$ is a finitely generated faithful multiplication submodule of M. By Lemma 1.1, $[N : K] = [(N \cap K) : K]$ is a finitely generated faithful multiplication ideal. The converse follows by [10, Corollary 1.4] since $N \cap K = [N : K]K$.

(2) Let $N \cap K$ be a finitely generated projective submodule of M. By Lemma 1.1, $N \cap K$ is multiplication. Moreover K is finitely generated, multiplication and $\mathrm{ann}K = Re$ for some idempotent e. It follows by [22, Corollary 2 to Theorem 11] that $[N : K] = [N \cap K : K]$ is multiplication. Now $[N : M] \subseteq [N : K]$, and hence $\mathrm{ann}[N : K] \subseteq \mathrm{ann}[N : M] = \mathrm{ann}N$. Also, $\mathrm{ann}K \subseteq [N : K]$, and $\mathrm{ann}[N : K] \subseteq \mathrm{ann}(\mathrm{ann}K)$, and therefore $\mathrm{ann}[N : K] \subseteq \mathrm{ann}N \cap \mathrm{ann}(\mathrm{ann}K) = \mathrm{ann}(N + \mathrm{ann}K)$. The reverse inclusion is also true (see the proof of [2, Theorem 2.2(1)]). Hence, $\mathrm{ann}[N : K] = \mathrm{ann}(N + \mathrm{ann}K)$. It follows by [18, Corollary 1 to Lemma 1.5] that $[N : K]$ is finitely generated. By [20, Theorem 2.1], $[N : K]$ is projective. The converse is true since

$$N \cap K = [N : K]K \cong [N : K] \otimes K,$$

see for example [12, Corollary 11.16]. □

We note that the above corollary may also be proved using Lemma 1.1, [4, Theorem 3.1], [2, Theorem 2.3] and the fact that $[N : K] = [[N : M] : [K : M]]$.

Let N, K be submodules of an R-module M. We say that N *divides* K, denoted $N|K$, if there exists an ideal I of R such that $K = IN$. It is clear that if $N|K$ then $K \subseteq N$, and if N is multiplication the converse is also true. A common divisor of N and K which is divisible by every common divisor of N and K (which is clearly unique if it exists) is denoted by $\mathrm{GCD}(N, K)$, and LCM is defined analogously. The existence and arithmetic properties of these in the case of finitely generated faithful multiplication and finitely generated projective ideals are investigated in [4] and [2] respectively.

Lemma 1.3. *Let R be a ring and N, K submodules of a finitely generated faithful multiplication R-module M.*

(1) *If $\mathrm{GCD}(N, K)$ exists then so too does $\mathrm{gcd}([N : M], [K : M])$, and in this case* $[\mathrm{GCD}(N, K) : M] = \mathrm{gcd}([N : M], [K : M])$.

(2) $\mathrm{LCM}(N, K)$ *exists if and only if* $\mathrm{lcm}([N : M], [K : M])$ *exists, and in this case* $[\mathrm{LCM}(N, K) : M] = \mathrm{lcm}([N : M], [K : M])$.

Proof. (1) Let $G = \mathrm{GCD}(N, K)$. Then $G|N$ and $G|K$ and hence $[G : M]|[N : M]$ and $[G : M]|[K : M]$, because M is finitely generated faithful and multiplication. Suppose that I is any ideal of R such that $I|[N : M]$ and $I|[K : M]$. Then $IM|N$ and $IM|K$. Hence $IM|G$. It follows that $I|[G : M]$, and hence $[G : M] = \mathrm{gcd}([N : M], [K : M])$.

(2) Let $L = \mathrm{LCM}(N, K)$. Then $N|L$ and $K|L$. It follows that $[N : M]|[L : M]$ and $[K : M]|[L : M]$. Suppose that I is an ideal of R such that $[N : M]|I$ and $[K : M]|I$. Then $N|IM$ and $K|IM$, and hence $IM|L$. This implies that $I|[L : M]$, and hence $[L : M] = \mathrm{lcm}([N : M], [L : M])$. Conversely, let $J = \mathrm{lcm}([N : M], [K : M])$. Then $[N : M]|J$ and $[K : M]|J$, and hence $N|JM$ and $K|JM$. Assume that H is a submodule of M such that $N|H$ and $K|H$. Then $[N : M]|[H : M]$ and $[K : M]|[H : M]$, and hence $[H : M]|J$. This gives that $H|JM$ and hence $JM = \mathrm{LCM}(N, K)$. This also shows that $\mathrm{lcm}([N : M], [K : M]) = [\mathrm{LCM}(N, K) : M]$. □

The converse of Part (1) above is not true. Let $R = k[x^2, x^3]$, k a field. Let $N = x^5R$, $K = x^6R$ and $M = x^3R$. Then $\gcd([N : M], [K : M]) = \gcd(x^2R, x^3R)$ exists, but $\mathrm{GCD}(N, K)$ does not.

2. GCD and LCM of multiplication modules

Let R be a ring and M a multiplication module. Define $S(M)$ to be the set of finitely generated faithful multiplication submodules of M. As mentioned earlier, if $S(M)$ is non-empty then $M \in S(M)$. In this section we investigate the GCD and LCM of modules in $S(M)$, generalizing our results in [4].

If $N \in S(M)$ and G is any submodule of M such that $G|N$, then $G \in S(M)$. For if $G|N$ then $[G : M]|[N : M]$. By Lemma 1.1, $[N : M] \in S(R)$, so that by [4, Lemma 1.4], $[G : M] \in S(R)$, and hence $G = [G : M]M \in S(M)$. In particular, if $N, K \in S(M)$ and $\mathrm{GCD}(N, K)$ exists then it is in $S(M)$. On the other hand, if $\mathrm{LCM}(N, K)$ exists then by Lemma 1.3 $\mathrm{lcm}([N : M], [K : M])$ exists, and hence by [4, Lemma 1.5] it is in $S(R)$. Hence $\mathrm{LCM}(N, K) = \mathrm{lcm}([N : M], [K : M])M \in S(M)$.

It is easily verified that for all $N, K \in S(M)$, $\mathrm{LCM}(N, K)$ exists (and hence is in $S(M)$) if and only if $N \cap K \in S(M)$, and in this case $\mathrm{LCM}(N, K) = N \cap K$. It follows by Lemma 1.3 that for all $N, K \in S(M)$, the following are equivalent:

$$\mathrm{LCM}(N, K)\text{exists}, \quad N \cap K \in S(M), \quad [N : K] \in S(R).$$

Compare the next result with [4, Theorem 2.1].

Proposition 2.1. *Let R be a ring and M a multiplication R-module. For all submodules $N, K \in S(M)$, if $\mathrm{LCM}(N, K)$ exists then so too does $\mathrm{GCD}(N, K)$, and in this case*

$$\begin{aligned}[][\mathrm{GCD}(N, K) : M]\mathrm{LCM}(N, K) &= [\mathrm{LCM}(N, K) : M]\mathrm{GCD}(N, K) \\ &= [N : M]K = [K : M]N.\end{aligned}$$

Proof. By Lemma 1.3, $\mathrm{lcm}([N : M], [K : M])$ exists, and by [4, Theorem 2.1], $\gcd([N : M], [K : M])$ exists and

$$[N : M][K : M] = \gcd([N : M], [K : M])\mathrm{lcm}([N : M], [K : M]).$$

Let $L = \mathrm{lcm}([N : M], [K : M])$. Then $L|[N : M][K : M]$, and hence there exists an ideal G such that $[N : M][K : M] = LG$. Hence $[N : M]K = [K : M]N = LGM$. We show that $GM = \mathrm{GCD}(N, K)$. Now $[N : M]K|LK$ since $[N : M]|L$, and hence $LGM|LK$. It follows that $GM|K$. Similarly, $GM|N$. Suppose that G' is any common divisor of N and K. Then $G' \in S(M)$ and $[K : M]G'|[K : M]N$. Hence $[K : M]N \subseteq [K : M]G'$, and hence $[[K : M]N : G'] \subseteq [[K : M]G' : G'] = [K : M]$. Similarly, $[[N : M]K : G'] \subseteq [N : M]$, and hence $[[K : M]N : G'] = [[N : M]K : G']$ is a common multiple of $[N : M]$ and $[K : M]$. This gives that $L|[[K : M]N : G']$, and hence there exists an ideal H of R such that $[[K : M]N : G'] = HL$. As $[K : M]N \subseteq G'$, we infer that $[K : M]N = HLG'$, and hence

$$LG = [N : M][K : M] = [[K : M]N : M] = [HLG' : M] = L[HG' : M].$$

It follows that $G = [HG' : M]$, and hence $GM = HG'$. This shows that $G'|GM$, and $GM = \mathrm{GCD}(N, K)$. Finally, by Lemma 1.3, $G = \gcd([N : M], [K : M])$ and $[\mathrm{LCM}(N, K) : M] = \mathrm{lcm}([N : M], [K : M])$, and the second assertion follows. □

Let M be a multiplication module and $N, K \in S(M)$ such that $N + K \in S(M)$. By [4, Theorem 3.6] and [22, Lemma 7], $N \cap K \in S(M)$, and hence $\mathrm{LCM}(N, K)$ exists and equals $N \cap K$. By Proposition 2.1, $\mathrm{GCD}(N, K)$ exists, and hence by a remark made after [4, Corollary 1.2],

$$[\mathrm{GCD}(N, K) : M] = [[N : M]K : K \cap N] = [[N : M]K : K] + [[N : M]K : N]$$

$$= [[N : M]K : K] + [[K : M]N : N] = [N : M] + [K : M] = [N + K : M].$$

Hence $\mathrm{GCD}(N, K) = N + K$.

Compare the next result with [4, Theorem 2.2].

Theorem 2.2. *Let R be a ring and M a multiplication R-module. Let $N, K \in S(M)$ and $I, J \in S(R)$.*

1. $\mathrm{LCM}(N, K)$ *exists if and only if* $\mathrm{LCM}(IN, IK)$ *exists, and in this case* $\mathrm{LCM}(IN, IK) = I\mathrm{LCM}(N, K)$.
2. $\mathrm{lcm}(I, J)$ *exists if and only if* $\mathrm{LCM}(IN, JN)$ *exists, and in this case* $\mathrm{LCM}(IN, JN) = \mathrm{lcm}(I, J)N$.
3. *If* $\mathrm{GCD}(IN, IK)$ *exists then so too does* $\mathrm{GCD}(N, K)$*, and in this case* $\mathrm{GCD}(IN, IK) = I\mathrm{GCD}(N, K)$.
4. *If* $\mathrm{GCD}(IN, JN)$ *exists then so too does* $\gcd(I, J)$*, and in this case* $\mathrm{GCD}(IN, JN) = \gcd(I, J)N$.

Proof. (1) By Lemma 1.3, if $\mathrm{LCM}(IN, IK)$ exists then $\mathrm{lcm}([I[N : M], I[K : M])$ exists, and by [4, Theorem 2.2] $\mathrm{lcm}([N : M], [K : M])$ exists. Again by Lemma 1.3, $\mathrm{LCM}(N, K)$ exists. Next we infer from Lemma 1.3 and [4, Theorem 2.2] that

$$I[\mathrm{LCM}(N, K) : M] = I\mathrm{lcm}([N : M], [K : M]) = \mathrm{lcm}(I[N : M], I[K : M])$$

$$= \mathrm{lcm}([IN : M], [IK : M]) = [\mathrm{LCM}(IN, IK) : M].$$

Hence, $I\mathrm{LCM}(N, K) = \mathrm{LCM}(IN, IK)$. The converse is now clear by Lemma 1.3 and [4, Theorem 2.2].

(2) Assume that $\mathrm{lcm}(I,J)$ exists. Then $I\cap J\in S(R)$, and by [10, Theorem 1.6] and [1, Theorem 2.1] we obtain that $IN\cap JN=(I\cap J)N\in S(M)$. Hence $\mathrm{LCM}(IN,JN)$ exists and $\mathrm{LCM}(IN,JN)=\mathrm{lcm}(I,J)N$. Conversely, if $\mathrm{LCM}(IN,JN)$ exists then $(I\cap J)N=IN\cap JN=\mathrm{LCM}(IN,JN)\in S(M)$, and hence $I\cap J=[(I\cap J)N:N]\in S(R)$. Hence $\mathrm{lcm}(I,J)$ exists and $\mathrm{LCM}(IN,JN)=\mathrm{lcm}(I,J)N$.

(3) Suppose $G=\mathrm{GCD}(IN,IK)$. Then $\gcd(I[N:M],I[K:M])=\gcd([IN:M],[IK:M])$ exists and equals $[G:M]$. As $I[N:M]\subseteq[G:M]$, we obtain that $[N:M]\subseteq[[G:M]:I]\subseteq[G:IM]$, and hence $N\subseteq[G:IM]M$. Similarly, $K\subseteq[G:IM]M$. Since $IM|IN$ and $IM|IK$, we have $IM|G$, and hence $G\subseteq IM$, $[G:IM]\in S(R)$, and $[G:IM]M\in S(M)$. This shows that $[G:IM]M$ is a common divisor of N and K. If D is any common divisor of N and K, then $ID|IN$ and $ID|IK$, and hence $ID|G$, (in other words $G\subseteq ID$). It follows that $[G:IM]\subseteq[ID:IM]=[D:M]$. Next, $D\in S(M)$, and hence $[D:M]\in S(R)$. Therefore $[D:M]|[G:IM]$, and hence $D|[G:IM]M$. This shows that $[G:IM]M=\mathrm{GCD}(N,K)$, and hence $G=[G:IM]IM=I\mathrm{GCD}(N,K)$.

(4) Suppose $\mathrm{GCD}(IN,JN)$ exists. By Lemma 1.3, $\gcd([IN:M],[JN:M])=\gcd(I[N:M],J[N:M])$ exists, and by [4, Theorem 2.2], $\gcd(I,J)$ exists and

$$[\mathrm{GCD}(IN,JN):M]=\gcd(I[N:M],J[N:M])=\gcd(I,J)[N:M],$$

and hence $\mathrm{GCD}(IN,JN)=\gcd(I,J)N$. □

Proposition 2.3. *Let M be a multiplication R-module and $N,K\in S(M)$.*

1. *If $G=\mathrm{GCD}(N,K)$ exists then $R=\gcd([N:G],[K:G])$.*
2. *If $L=\mathrm{LCM}(N,K)$ exists (whence also $G=\mathrm{GCD}(N,K)$ exists) then $[N:G]=[N:K]$, and*
$$R=\gcd([L:K],[L:N])=\gcd([N:K],[K:N]).$$
3. *If $L=\mathrm{LCM}(N,K)$ exists then for all integers $r\geq 1$,*
   (i) $[L:M]^r=\mathrm{lcm}([N:M]^r,[K:M]^r)$,
   (ii) $[G:M]^r=\gcd([N:M]^r,[K:M]^r)$.
   (iii) $[N:K]^r=[[N:M]^r:[K:M]^r]]$.

Proof. (1) If $G=\mathrm{GCD}(N,K)$ exists then by Theorem 2.2

$$G=\mathrm{GCD}([N:G]G,[K:G]G)=\gcd([N:G],[K:G])G.$$

Hence $\gcd([N:G],[K:G])=R$.

(2) If $L=\mathrm{LCM}(N,K)$ exists then by Proposition 2.1, $G=\mathrm{GCD}(N,K)$ exists and

$$[L:M]G=[K:M]N=[N:M]K.$$

Hence

$$[N:G]=[[L:M]:[K:M]]=[L:K]=[N\cap K:K]=[N:K].$$

Similarly $[K:G]=[K:N]$. The remaining assertions are now clear.

(3) These results generalize [4, Lemma 2.4, Theorem 2.6 and some facts on p. 225], and the proofs are similar. □

Let M be a multiplication R-module and $N, K \in S(M)$. We say that N and K are *M-coprime* if $\mathrm{GCD}(N, K) = M$. Compare the following generalization of Euclid's Lemma with [4, Proposition 2.3].

Proposition 2.4. *Let M be a multiplication R-module. Let $N, K \in S(M)$ and $I, J \in S(R)$.*

(1) *If N and K are M-coprime and $\mathrm{GCD}(IN, IK)$ exists then $\mathrm{GCD}(N, IM) = \mathrm{GCD}(N, IK)$.*
(2) *If I and J are relatively prime and $\mathrm{GCD}(IN, JN)$ exists then $\mathrm{GCD}(IM, N) = \mathrm{GCD}(IM, JN)$.*

Proof. (1) By Theorem 2.2, $\mathrm{GCD}(IN, IK) = I\mathrm{GCD}(N, K) = IM$. It follows that

$$\mathrm{GCD}(N, IM) = \mathrm{GCD}(N, \mathrm{GCD}(IN, IK)) = \mathrm{GCD}(\mathrm{GCD}(N, IN), IK) = \mathrm{GCD}(N, IK).$$

(2) Again by Theorem 2.2, $\mathrm{GCD}(IN, JN) = \gcd(I, J)N = N$, and hence

$$\mathrm{GCD}(IM, N) = \mathrm{GCD}(IM, \mathrm{GCD}(IN, JN)) = \mathrm{GCD}(\mathrm{GCD}(IM, IN), JN) = \mathrm{GCD}(IM, JN),$$

as required. □

The third part of the next result generalizes [4, Theorem 2.5].

Proposition 2.5. *Let M be a multiplication R-module.*

(1) *$\mathrm{LCM}(N, K)$ exists for all $N, K \in S(M)$ if and only if $\mathrm{lcm}(I, J)$ exists for all $I, J \in S(R)$.*
(2) *If $\mathrm{GCD}(N, K)$ exists for all $N, K \in S(M)$ then $\gcd(I, J)$ exists for all $I, J \in S(R)$.*
(3) *$\mathrm{GCD}(N, K)$ exists for all $N, K \in S(M)$ if and only if $\mathrm{LCM}(N, K)$ exists for all $N, K \in S(M)$.*
(4) *If $\gcd(I, J)$ exists for all $I, J \in S(R)$ then $\mathrm{GCD}(N, K)$ exists for all $N, K \in S(M)$.*

Proof. (1) Suppose that $\mathrm{LCM}(N, K)$ exists for all $N, K \in S(M)$, and let $I, J \in S(R)$. Then $IM, JM \in S(M)$, and hence $\mathrm{LCM}(IM, JM)$ exists. By Theorem 2.2(1), the result follows. Conversely assume that $\mathrm{lcm}(I, J)$ exists for all $I, J \in S(R)$, and let $N, K \in S(M)$. Then $[N : M], [K : M] \in S(R)$, and hence $\mathrm{lcm}([N : M], [K : M])$ exists and by Lemma 1.3(2) the result follows.

(2) Suppose that $\mathrm{GCD}(N, K)$ exists for all $N, K \in S(M)$, and let $I, J \in S(R)$. Then $\mathrm{GCD}(IM, JM)$ exists and by Theorem 2.2(3) we obtain the existence of $\gcd(I, J)$.

(3) This follows by (1) and (2) above, [4, Theorem 2.5], and Proposition 2.1 above.

(4) If $\gcd(I, J)$ exists for all $I, J \in S(R)$ then $\mathrm{lcm}(I, J)$ exists for all $I, J \in S(R)$ [4, Theorem 2.5], and by (1) above $\mathrm{LCM}(N, K)$ exists for all $N, K \in S(M)$, and the conclusion follows from Proposition 2.1. □

In [4] we called a ring R a generalized GCD ring (GGCD ring) if $S(R)$ is closed under intersection. We extend this to modules: An R-module M is a *generalized* GCD *module* (GGCD *module*) if M is multiplication and $S(M)$ is closed under intersection. It is clear from Proposition 2.5 that R is a GGCD ring if and only if some R-module M is a GGCD module. By Proposition 2.5, any multiplication module over a principal ideal ring, Bezout ring, arithmetical ring, Prüfer domain, von Neumann regular ring or GGCD domain (an integral domain in which the intersection of two invertible ideals is an invertible ideal, [8]) is a GGCD module since each of these rings is a GGCD ring.

The next result summarizes several equivalent conditions for a multiplication module to be a GGCD module by combining Proposition 2.5 and [4, Theorems 2.5 and 3.1].

Theorem 2.6. *Let M be a multiplication R-module. The following conditions are equivalent.*

(1) *M is a* GGCD *module.*
(2) *R is a* GGCD *ring.*
(3) $\mathrm{GCD}(N, K)$ *exists for all* $N, K \in S(M)$.
(4) $\mathrm{LCM}(N, K)$ *exists for all* $N, K \in S(M)$.
(5) $[N : K] \in S(R)$ *for all* $N, K \in S(M)$.
(6) $\mathrm{lcm}(I, J)$ *exists for all* $I, J \in S(R)$.
(7) $\mathrm{gcd}(I, J)$ *exists for all* $I, J \in S(R)$.
(8) $[I : J] \in S(R)$ *for all* $I, J \in S(R)$.

The next result generalizes [4, Corollaries 3.2 and 3.4].

Corollary 2.7. *Let M be a* GGCD *R-module. Then for all $N, K, L \in S(M)$:*

(1) $[\mathrm{GCD}(N, K) : L] = \mathrm{gcd}([N : L], [K : L])$.
(2) $[L : \mathrm{LCM}(N, K)] = \mathrm{gcd}([L : N], [L : K])$.
(3) $\mathrm{LCM}(\mathrm{GCD}(N, K), L) = \mathrm{GCD}(\mathrm{LCM}(N, L), \mathrm{LCM}(K, L))$.
(4) $\mathrm{GCD}(\mathrm{LCM}(N, K), L) = \mathrm{LCM}(\mathrm{GCD}(N, L), \mathrm{GCD}(K, L))$.

Proof. (1) By Corollary 1.2, $[N : L], [K : L] \in S(R)$, hence by Theorem 2.6, $\mathrm{gcd}([N : L], [K : L])$ exists. Using [4, Corollary 3.2(1)] and Lemma 1.3 we get that

$$\begin{aligned}[\mathrm{GCD}(N, K) : L] &= [[\mathrm{GCD}(N, K) : M] : [L : M]] \\ &= [\mathrm{gcd}([N : M], [K : M]) : [L : M]] \\ &= \mathrm{gcd}([[N : M] : [L : M]], [[K : M] : [L : M]] \\ &= \mathrm{gcd}([N : L], [K : L]).\end{aligned}$$

(2) Again by Corollary 1.2, $[L : N], [L : K] \in S(R)$, and hence $\mathrm{gcd}([L : N], [L : K])$ exists. By [4, Corollary 3.2(2)] and Lemma 1.3, we have that

$$[L : \mathrm{LCM}(N, K)] = [[L : M] : [\mathrm{LCM}(N, K) : M]] = [[L : M] : [\mathrm{lcm}([N : M], [K : M])]$$

$$= \mathrm{gcd}([[L : M] : [N : M]], [[L : M] : [K : M]]) = \mathrm{gcd}([L : N], [L : K]),$$

as required.
(3)It follows from Lemma 1.3 and [4, Corollary 3.4(1)] that

$$\begin{aligned}[\mathrm{LCM}(\mathrm{GCD}(N,K),L):M] &= \mathrm{lcm}([\mathrm{GCD}(N,K):M],[L:M]) \\ &= \mathrm{lcm}(\mathrm{gcd}([N:M],[K:M]),[L:M]) \\ &= \mathrm{gcd}([\mathrm{lcm}([N:M],[L:M]),\mathrm{lcm}([N:M],[K:M])) \\ &= \mathrm{gcd}(\mathrm{LCM}(N,L):M],[\mathrm{LCM}(N,K):M]) \\ &= [\mathrm{GCD}(\mathrm{LCM}(N,L),\mathrm{LCM}(N,K)):M],\end{aligned}$$

and the result follows.
(4) Again by Lemma 1.3 and [4, Corollary 34.(2)] we infer that

$$\begin{aligned}[\mathrm{GCD}(\mathrm{LCM}(N,K),L):M] &= \mathrm{gcd}([\mathrm{LCM}(N,K):M],[L:M]) \\ &= \mathrm{gcd}(\mathrm{lcm}([N:M],[K:M]),[L:M]) \\ &= \mathrm{lcm}(\mathrm{gcd}([N:M],[L:M]),\mathrm{gcd}([K:M],[L:M])) \\ &= \mathrm{lcm}([\mathrm{GCD}(N,L):M],[\mathrm{GCD}(K,L):M]) \\ &= [\mathrm{LCM}(\mathrm{GCD}(N,L),\mathrm{GCD}(K,L)):M],\end{aligned}$$

and the result is now clear. □

3. Lattice of submodules of multiplication modules

Let R be a GGCD ring and A, B finitely generated faithful multiplication ideals of R. In [4] we defined

$$\Phi_{A,B} = \{I \,:\, I \text{ is an ideal of } R, \quad I|A, \quad \mathrm{gcd}(I,B) = R\},$$

and we investigated this lattice of ideals. We showed for example that $X \in \Phi_{A,B}$ is smallest if and only if the only ideal dividing $[A : X]$ and relatively prime to B is R. In this section we generalize various properties of this lattice of ideals to multiplication modules.

Let M be a GGCD R-module. For $N, K \in S(M)$ we define

$$\Phi^M_{N,K} = \Phi_{N,K} = \{T \,:\, T \text{ is a submodule of } M, \quad T|N, \quad \mathrm{GCD}(T,K) = M\}.$$

$\Phi_{N,K}$ is non-empty since $M \in \Phi_{N,K}$.

The proofs of the following are straightforward.

Lemma 3.1. *Let R be a ring and M a* GGCD *module (equivalently, R is a* GGCD *ring and M is a multiplication module). Let $N, K \in S(M)$ and $I \in S(R)$. Then*

1. *$T \in \Phi_{N,K}$ if and only if $[T : M] \in \Phi_{[N:M],[K:M]}$.*
2. *$I \in \Phi_{[N:M],[K:M]}$ if and only if $IM \in \Phi_{N,K}$.*
3. *T is minimal in $\Phi_{N,K}$ if and only if $[T : M]$ is minimal in $\Phi_{[N:M],[K:M]}$.*
4. *I is minimal in $\Phi_{[N:M],[K:M]}$ if and only if IM is minimal in $\Phi_{N,K}$.*

The following result extends [4, Theorem 3.5].

Proposition 3.2. *Let R be a ring and M a* GGCD *module. Let $N, K \in S(M)$. Then $\Phi_{N,K}$ is a lattice of submodules. Moreover, if $\Phi_{N,K}$ contains a minimal element then it is unique.*

Proof. R is a GCCD ring, and by [4, Theorem 3.5] $\Phi_{[N:M],[K:M]}$ is a lattice of ideals of R. Let $X, Y \in S(M)$, and let $G = \mathrm{GCD}(X, Y)$ and $L = \mathrm{LCM}(X, Y)$. Then $[G : M] = \gcd([X : M], [Y : M])$ and $[L : M] = \mathrm{lcm}([X : M], [Y : M])$. Hence $[G : M], [L : M] \in \Phi_{[N:M],[K:M]}$, and by Lemma 3.1, $G, L \in \Phi_{N,K}$. Suppose that T is a minimal element of $\Phi_{N,K}$. By Lemma 3.1(3), $[T : M]$ is minimal in $\Phi_{[N:M],[K:M]}$, and by [4, Theorem 3.5] $[T : M]$ is the smallest element of $\Phi_{[N:M],[K:M]}$. For any $X \in \Phi_{N,K}$, $[X : M] \in \Phi_{[N:M],[K:M]}$. Hence $[T : M] \subseteq [X : M]$, and this implies that $T \subseteq X$, so that T is smallest in $\Phi_{N,K}$. □

Let M be a multiplication R-module. Let $N, K, T \in S(M)$ and suppose that $\mathrm{GCD}(N, T) = M = \mathrm{GCD}(K, T)$. Then

$$\gcd([N : M], [T : M]) = R = \gcd([K : M], [T : M]),$$

and by [4, Lemma 3.4], $\gcd([N : M][K : M], [T : M]) = R$. Hence

$$\mathrm{GCD}([N : M]K, T) = M = \mathrm{GCD}([K : M]N, T).$$

Lemma 3.3. *Let R be a ring and M a* GGCD *module. Let $N, K \in S(M)$. Then T is the smallest element in $\Phi_{N,K}$ if and only if $[T : M]$ is the smallest element in $\Phi_{[N:M],[K:M]}$.*

Proof. If $[T : M]$ is the smallest element in $\Phi_{[N:M],[K:M]}$ then T is the smallest element in $\Phi_{N,K}$, (see the proof of Proposition 3.2). Conversely, suppose that T is the smallest element in $\Phi_{N,K}$. Let J be an ideal of R such that $J|[[N : M] : [T : M]]$ and $\gcd(J, [K : M]) = R$. By [4, Theorem 3.7] it is enough to show that $J = R$. Now $[[N : M] : [T : M]] = [N : T]$, and hence $JT|N$. $\mathrm{GCD}(JM, K) = M$ since $\gcd(J, [K : M]) = R$. But $\mathrm{GCD}(T, K) = M$. It follows from the remark made above that $\mathrm{GCD}([JM : M]T, K) = M$, and hence $\mathrm{GCD}(JT, K) = M$. Therefore $JT \in \Phi_{N,K}$, and hence $T \subseteq JT \subseteq T$ so that $JT = T$, and this finally gives that $J = R$. □

Theorem 3.4. *Let R be a ring and M a* GGCD *module, and let $N, K \in S(M)$. Then $T \in \Phi_{N,K}$ is smallest if and only if the only submodule dividing $[N : T]M$ and M-coprime to K is M.*

Proof. Lemma 3.3 and [4, Theorem 3.7]. □

Compare the next result with [4, Theorem 3.8].

Proposition 3.5. *Let M be a* GGCD *R-module, let $N, K \in S(M)$, and let $G = \mathrm{GCD}(N, K)$. Then $\Phi_{N,K} = \Phi_{[N:G]M,G}$.*

Proof. Let $T \in \Phi_{N,K}$. By Lemma 3.1, $[T : M] \in \Phi_{[N:M],[K:M]}$, and by [4, Theorem 3.8], $[T : M] \in \Phi_{[[N:M]:[G:M]],[G:M]}$, so that $[T : M] \in \Phi_{[N:G],[G:M]}$. Hence by Lemma 3.1, $T \in \Phi_{[N:G]M,G}$. Conversely, let $T \in \Phi_{[N:G]M,G}$. Then

$$[T : M] \in \Phi_{[[N:G]M:M],[G:M]} = \Phi_{[N:G],[G:M]} = \Phi_{[[N:M]:[G:M]],[G:M]}.$$

By [4, Theorem 3.8], $[T : M] \in \Phi_{[N:M],[K:M]}$, and hence $T \in \Phi_{N,K}$. □

Let M be a GGCD R-module and let $N, K \in S(M)$. Define two sequences of ideals of R and two sequences of submodules of M recursively as follows:

$$I_0 = [N : M], \quad J_0 = [K : M], \quad J_{i+1} = \gcd(I_i, J_i), \quad I_{i+1} = [I_i : J_{i+1}] \quad \text{for all } i \geq 0,$$

$$N_0 = N, \quad K_0 = K, \quad K_{i+1} = \mathrm{GCD}(N_i, K_i), \quad N_{i+1} = [N_i : K_{i+1}]M \quad \text{for all } i \geq 0.$$

Lemma 3.6. *Let M be a* GGCD *R-module and $N, K \in S(M)$ with the sequences I_i, J_i, N_i, K_i as above. Then*

(1) $N_i = I_iM$ *and* $K_i = J_iM$ *for all* $i \geq 0$.

(2) $K_i \subseteq K_{i+1}$, *and* $N_i \subseteq N_{i+1}$ *for all* $i \geq 0$.

(3) $\Phi_{N,K} = \Phi_{N_i,K_i}$ *for all* $i \geq 0$.

Proof. (1) Induction on i. The result is trivial if $i = 0$. If $i \geq 0$ and $N_i = I_iM$ and $K_i = J_iM$, then

$$K_{i+1} = \mathrm{GCD}(N_i, K_i) = \mathrm{GCD}(I_iM, J_iM) = \gcd(I_i, J_i)M = J_{i+1}M,$$

and

$$N_{i+1} = [N_i : K_{i+1}]M = [I_iM : J_{i+1}M]M = [I_i : J_{i+1}]M = I_{i+1}M.$$

(2) $K_i \subseteq K_{i+1}$, since $K_{i+1} = \mathrm{GCD}(N_i, K_i)$. Now $[N_i : K_{i+1}]M = N_{i+1}$. Hence

$$N_i = [N_i : M]M \subseteq [N_i : K_{i+1}]M = N_{i+1}.$$

(3) follows by Proposition 3.5. □

Combining Theorem 3.4, Lemma 3.6 and [4, Theorem 3.9], we have the following.

Corollary 3.7. *Let M be a* GGCD *R-module and $N, K \in S(M)$ with the sequences I_i, J_i, N_i, K_i as above. Then the following are equivalent.*

(1) $\bigcup_{i=1}^{\infty} N_i$ *is the smallest element in* $\Phi_{N,K}$.

(2) $\bigcup_{i=1}^{\infty} N_i \in \Phi_{N,K}$.

(3) $\bigcup_{i=1}^{\infty} N_i \in S(M)$.

(4) $\bigcup_{i=1}^{\infty} N_i = N_n$ *for some positive integer* n.

(5) $N_n = N_{n+1}$ *for some positive integer* n.
(6) $N_n = M$ *for some positive integer* n.

We end this section with a result which may be compared with [4, Theorem 10]. Its proof follows from Lemma 1.3 and [4, Theorem 10].

Theorem 3.8. *Let* M *be a* GGCD R*-module and* $N, K \in S(M)$. *Let* $L = \mathrm{LCM}(N, K)$, *and let* T_N, T_K *be the smallest elements in* $\Phi_{N,[L:N]M}$ *and* $\Phi_{K,[L:K]M}$ *respectively. Then*

(1) $\mathrm{LCM}(T_N, T_K) = \mathrm{LCM}(N, K)$.
(2) $\mathrm{GCD}([N : T_N]M, [K : T_K]\mathrm{GCD}(T_N, T_K)) = M$

$$= \mathrm{GCD}([K : T_K]M, [N : T_N]\mathrm{GCD}(T_N, T_K)).$$

(3) $\mathrm{GCD}(T_N, [\mathrm{LCM}(T_N, T_K) : T_N]M) = M$

$$= \mathrm{GCD}(T_K, [\mathrm{LCM}(T_N, T_K) : T_K]M).$$

4. GCD and LCM of projective modules

In this section we generalize and extend our study in [2] on GCD and LCM of projective ideals. Compare the following result with [2, Theorem 2.1]. We denote by $S^*(M)$ the set of finitely generated projective submodules of M.

Proposition 4.1. *Let* R *be a ring and* M *a finitely generated faithful multiplication* R*-module. Let* $N, K \in S^*(M)$.

(1) *If* $G = \mathrm{GCD}(N, K)$ *exists then* $G \in S^*(M)$ *and* $\mathrm{ann}G = \mathrm{ann}(N + K)$. *In this case* $\mathrm{lcm}(\mathrm{ann}N, \mathrm{ann}K)$ *exists and* $\mathrm{ann}G = \mathrm{lcm}(\mathrm{ann}N, \mathrm{ann}K)$.
(2) *If* $L = \mathrm{LCM}(N, K)$ *exists then* $L \in S^*(M)$ *and*

$$\mathrm{ann}L = \mathrm{ann}([N : M]K) = \mathrm{ann}([K : M]N).$$

In this case $\mathrm{gcd}(\mathrm{ann}N, \mathrm{ann}K)$ *exists and* $\mathrm{ann}L = \mathrm{gcd}(\mathrm{ann}N, \mathrm{ann}K)$.
(3) $L = \mathrm{LCM}(N, K)$ *exists if and only if* $N \cap K \in S^*(M)$, *and in this case* $L = N \cap K$.

Proof. (1) By Lemmas 1.1, 1.3 and [2, Theorem 2.1(1)], we have $[N : M], [K : M] \in S^*(R)$, $\mathrm{gcd}([N : M], [K : M])$ exists and is in $S^*(R)$, and $[G : M] = \mathrm{gcd}([N : M], [K : M])$. Hence $G \in S^*(M)$. Also

$$\begin{aligned}\mathrm{ann}G &= \mathrm{ann}[G : M] = \mathrm{ann}\,\mathrm{gcd}([N : M], [K : M])\\ &= \mathrm{ann}([N : M] + [K : M])\\ &= \mathrm{ann}[N : M] \cap \mathrm{ann}[K : M] = \mathrm{ann}N \cap \mathrm{ann}K\\ &= \mathrm{ann}(N + K).\end{aligned}$$

Since $\mathrm{ann}N \cap \mathrm{ann}K$ is a principal ideal generated by an idempotent and hence is multiplication, $\mathrm{lcm}(\mathrm{ann}N, \mathrm{ann}K)$ exists, and the result follows.

(2) Suppose $L = \mathrm{LCM}(N, K)$ exists. By Lemmas 1.1, 1.3 and [2, Theorem 2.1(2)], $\mathrm{lcm}([N : M], [K : M]) \in S^*(R)$, and hence $L \in S^*(M)$. Moreover,

$$\begin{aligned}\mathrm{ann}L &= \mathrm{ann}[L : M] = \mathrm{ann}\ \mathrm{lcm}([N : M], [K : M])\\ &= \mathrm{ann}([N : M][K : M])\\ &= \mathrm{ann}([[N : M][K : M] : M])\\ &= \mathrm{ann}([N : M]K)\\ &= \mathrm{ann}([K : M]N).\end{aligned}$$

Next, using the fact that finitely generated projective ideals are locally either zero or invertible, it is easy to check that

$$\mathrm{ann}([N : M][K : M]) = \mathrm{ann}[N : M] + \mathrm{ann}[K : M] = \mathrm{ann}N + \mathrm{ann}K.$$

Because $\mathrm{ann}N + \mathrm{ann}K$ is a principal ideal generated by an idempotent and hence is multiplication, we infer that $\gcd(\mathrm{ann}N, \mathrm{ann}K)$ exists, and $\gcd(\mathrm{ann}N, \mathrm{ann}K) = \mathrm{ann}N + \mathrm{ann}K$.
(3) Obvious. □

Lemma 4.2. *Let R be a ring and M a finitely generated faithful multiplication R-module. Let $N, K \in S^*(M)$ and $I, J \in S^*(R)$.*

1. *If $\mathrm{LCM}(N, K)$ exists then so too does $\mathrm{LCM}(IN, IK)$, and in this case $\mathrm{LCM}(IN, IK) = I\mathrm{LCM}(N, K)$.*
2. *If $\mathrm{lcm}(I, J)$ exists then so too does $\mathrm{LCM}(IN, JN)$, and in this case $\mathrm{LCM}(IN, JN) = \mathrm{lcm}(I, J)N$.*

Proof. (1) Since $I \in S^*(R)$, it follows from [1, Theorem 3.1] that

$$[(IN \cap IK) : M] = [IN : M] \cap [IK : M] = I[N : M] \cap I[K : M]$$

$$= I([N : M] \cap [K : M]) = I[(N \cap K) : M],$$

and hence $IN \cap IK = I(N \cap K)$. Suppose that $\mathrm{LCM}(N, K)$ exists. Then

$$I\mathrm{LCM}(N, K) = I(N \cap K) = IN \cap IK \in S^*(M),$$

and hence $\mathrm{LCM}(IN, IK)$ exists and $I\mathrm{LCM}(N, K) = \mathrm{LCM}(IN, IK)$.
(2) Since $N \in S^*(M)$, we use [1, Theorem 3.1] to obtain that $(I \cap J)N = IN \cap JN$. Then

$$\mathrm{lcm}(I, J)N = (I \cap J)N = IN \cap JN \in S^*(M),$$

and hence $\mathrm{LCM}(IN, JN)$ exists, and $\mathrm{LCM}(IN, JN) = \mathrm{lcm}(I, J)N$. □

Compare the next result with [2, Theorem 2.5].

Theorem 4.3. *Let R be a ring and M a finitely generated faithful multiplication R-module. Let $N, K \in S^*(M)$ and $I, J \in S^*(R)$.*

(1) $\mathrm{LCM}(IN, IK)$ *exists if and only if* $\mathrm{LCM}(N + (\mathrm{ann}I)M, K + (\mathrm{ann}I)M)$ *exists, and in this case*
$$\mathrm{LCM}(IN, IK) = I\mathrm{LCM}(N + (\mathrm{ann}I)M, K + (\mathrm{ann}I)M).$$

(2) $\mathrm{LCM}(IN, JN)$ *exists if and only if* $\mathrm{lcm}(I + \mathrm{ann}N, J + \mathrm{ann}N)$ *exists, and in this case*
$$\mathrm{LCM}(IN, JN) = \mathrm{lcm}(I + \mathrm{ann}N, J + \mathrm{ann}N)N.$$

(3) *If* $\mathrm{GCD}(IN, IK)$ *exists then so too does* $\mathrm{GCD}(N + (\mathrm{ann}I)M, K + (\mathrm{ann}I)M)$*, and in this case*
$$\mathrm{GCD}(IN, IK) = I\mathrm{GCD}(N + (\mathrm{ann}I)M, K + (\mathrm{ann}I)M).$$

(4) *If* $\mathrm{GCD}(IN, JN)$ *exists then so too does* $\mathrm{gcd}(I + \mathrm{ann}N, J + \mathrm{ann}N)$*, and in this case*
$$\mathrm{GCD}(IN, JN) = \mathrm{gcd}(I + \mathrm{ann}N, J + \mathrm{ann}N)N.$$

Proof. (1) If $\mathrm{LCM}(IN, IK)$ exists then by Lemma 1.3, $\mathrm{lcm}([IN : M], [IK : M]) = \mathrm{lcm}(I[N : M], I[K : M])$ exists and $\mathrm{LCM}(IN, IK) = \mathrm{lcm}(I[N : M], I[K : M])M$. We obtain from [2, Theorem 2.5] that $\mathrm{lcm}([N : M] + \mathrm{ann}I, [K : M] + \mathrm{ann}I)$ exists, and
$$\mathrm{lcm}(I[N : M], I[K : M]) = I\mathrm{lcm}([N : M] + \mathrm{ann}I, [K : M] + \mathrm{ann}I).$$
As $M \in S^*(M)$, [22, Theorem 11], the result follows by Lemma 4.2(2). The converse follows by Lemma 4.2(1).

(2) Assume that $\mathrm{LCM}(IN, JN)$ exists. Then by Lemma 1.3, $\mathrm{lcm}(I[N : M], J[N : M])$ exists and $\mathrm{LCM}(IN, JN) = \mathrm{lcm}(I[N : M], J[N : M])M$. From Lemma 1.3 and [2, Theorem 2.5(1)] we get that
$$\mathrm{lcm}(I + \mathrm{ann}[N : M], J + \mathrm{ann}[N : M]) = \mathrm{lcm}(I + \mathrm{ann}N, J + \mathrm{ann}N)$$
exists, and
$$[N : M]\mathrm{lcm}(I + \mathrm{ann}N, J + \mathrm{ann}N) = \mathrm{lcm}(I[N : M], J[N : M]).$$
The result is now clear. The converse follows from Lemma 4.2.

(3) Let $G = \mathrm{GCD}(IN, IK)$. Then by Lemma 1.3, $[G : M] = \mathrm{gcd}(I[N : M], I[K : M])$. As $I|I[N : M]$ and $I|I[K : M]$, we have $I|[G : M]$. Hence $G \subseteq IM$ and by Corollary 1.3, $[G : IM] \in S^*(R)$. Now $[N : M] \subseteq [[G : M] : I]$, and $\mathrm{ann}I \subseteq [[G : M] : I]$. Hence
$$[N : M] + \mathrm{ann}I \subseteq [[G : M] : I] \subseteq [G : IM].$$
Similarly $[K : M] + \mathrm{ann}I \subseteq [G : IM]$, and hence $[G : IM]$ is a common divisor of $[N : M] + \mathrm{ann}I$ and $[K : M] + \mathrm{ann}I$, that is $[G : IM]M$ is a common divisor of $N + (\mathrm{ann}I)M$ and $K + (\mathrm{ann}I)M$. If G' is another such common divisor then $IG'|IN$ and $IG'|IK$. It follows that $IG'|G$, so that $G = FIG'$ for some ideal F of R. Next,
$$\begin{aligned}[G : IM] &= [FIG' : IM] = [FG' : M] + \mathrm{ann}I \\ &= F[G' : M] + \mathrm{ann}I = (F + \mathrm{ann}I)([G' : M] + \mathrm{ann}I),\end{aligned}$$

and hence $[G : IM]M = (F+\text{ann}I)(G'+(\text{ann}I)M)$. But $(\text{ann}I)M \subseteq G'$. Thus $[G : IM]M = (F + \text{ann}I)G'$, and hence $G'|[G : IM]M$. This finally gives that

$$[G : IM]M = \text{GCD}(N + (\text{ann}I)M, K + (\text{ann}I)M),$$

and hence

$$G = [G : IM]M = I\text{GCD}(N + (\text{ann}I)M, K + (\text{ann}I)M).$$

(4) The proof is similar to (2) using Lemma 1.3 and [2, Theorem 2.5(2)]. □

Compare the next result with [2, Theorem 2.3].

Theorem 4.4. *Let R be a ring and M a finitely generated faithful multiplication R-module. Let $N, K \in S^*(M)$ and let $L = N \cap K$. Then*

(1) $\text{LCM}(N, K)$ *exists if and only if* $\text{LCM}(N+(\text{ann}L)M, K+(\text{ann}L)M)$ *exists, and in this case*
$$\text{LCM}(N + (\text{ann}L)M, K + (\text{ann}L)M) = \text{LCM}(N, K) + (\text{ann}L)M.$$
(2) *If* $\text{LCM}(N, K)$ *exists then so too does* $\text{GCD}(N+(\text{ann}L)M, K+(\text{ann}L)M,)$ *and in this case*
$$\text{GCD}(N + (\text{ann}L)M, K + (\text{ann}L)M) = \text{GCD}(N, K) + (\text{ann}L)M.$$
(3) *If* $\text{GCD}(N, K)$ *exists then so too does* $\text{GCD}(N+(\text{ann}L)M, K+(\text{ann}L)M,)$ *and in this case*
$$\text{GCD}(N + (\text{ann}L)M, K + (\text{ann}L)M) = \text{GCD}(N, K) + (\text{ann}L)M.$$
(4) *If* $\text{GCD}(N, K)$ *exists for all* $N, K \in S^*(M)$ *then* $\text{LCM}(N, K)$ *exists for all* $N, K \in S^*(M)$.

Proof. (1) Suppose $\text{LCM}(N, K)$ exists. Then by Proposition 4.1 and Corollary 1.2, $\text{LCM}(N, K) = L$, $L \in S^*(M)$, $[L : M] \in S^*(R)$, and it follows that $[L : M](N \cap K) = [L : M]N \cap [L : M]K$. Next,

$$\begin{aligned}[][[L : M](N \cap K) : L] &= [[L : M]N : L] \cap [[L : M]K : L]\\ &= [[L : M]N : [L : M]M] \cap [[L : M]K : [L : M]M]\\ &= ([N : M] + \text{ann}([L : M])) \cap ([K : M] + \text{ann}([L : M]))\\ &= ([N : M] + \text{ann}L) \cap ([K : M] + \text{ann}L) \in S^*(R).\end{aligned}$$

It follows from [1, Theorem 3.1] that

$$(N + (\text{ann}L)M) \cap (K + (\text{ann}L)M) = (([N : M] + \text{ann}L) \cap ([K : M] + \text{ann}L))M \in S^*(M).$$

This shows that $\text{LCM}(N + (\text{ann}L)M, K + (\text{ann}L)M)$ exists. On the other hand,

$$\begin{aligned}[][[L : M](N \cap K) : L]M &= [[L : M](N \cap K) : [L : M]M]M\\ &= ([(N \cap K) : M] + \text{ann}([L : M])M\\ &= ([(N \cap K) : M] + \text{ann}L)M\\ &= (N \cap K) + (\text{ann}L)M\\ &= \text{LCM}(N, K) + (\text{ann}L)M,\end{aligned}$$

so that

$$\mathrm{LCM}(N + (\mathrm{ann}L)M, K + (\mathrm{ann}L)M) = \mathrm{LCM}(N, K) + (\mathrm{ann}L)M.$$

Conversely, suppose that $\mathrm{LCM}(N + (\mathrm{ann}L)M, K + (\mathrm{ann}L)M)$ exists. Then $[N + (\mathrm{ann}L)M : K + (\mathrm{ann}L)M] \in S^*(R)$. But

$$\begin{aligned}[N + (\mathrm{ann}L)M : K + (\mathrm{ann}L)M] &= [N + (\mathrm{ann}L)M : K] \cap [N + (\mathrm{ann}L)M : (\mathrm{ann}L)M] \\ &= [N : K] + [(\mathrm{ann}L)M : K].\end{aligned}$$

Thus $[N : K]K + [(\mathrm{ann}L)M : K]K \in S^*(M)$. Now by [1, Theorem 3.1],

$$\begin{aligned}[N : K]K \cap [(\mathrm{ann}L)M : K]K &= ([N : K] \cap [(\mathrm{ann}L)M : K])K \\ &= ([(N \cap K) : K] \cap [\mathrm{ann}(N \cap K)M : K])K \\ &= [(N \cap K) \cap \mathrm{ann}(N \cap K)M : K]K \\ &= (\mathrm{ann}K)K = 0,\end{aligned}$$

which is a multiplication ideal of R. We obtain from [3, Theorem 3.2] and [22, Theorem 8] that $N \cap K$ is a multiplication module. Finally,

$$\begin{aligned}\mathrm{ann}(N \cap K) &= \mathrm{ann}[(N \cap K) : M] = \mathrm{ann}([N : M] \cap [K : M]) \\ &= \mathrm{ann}([N : M][K : M]) = \mathrm{ann}([N : M]K),\end{aligned}$$

which is generated by an idempotent. By [22, Theorem 11], $N \cap K \in S^*(M)$, and hence $\mathrm{LCM}(N, K)$ exists.

(2) By Corollary 1.3, $[N : M] + \mathrm{ann}L = [[N : M]L : L] \in S^*(R)$. It follows that $N + (\mathrm{ann}L)M = ([N : M] + \mathrm{ann}L)M \in S^*(M)$. Moreover,

$$\begin{aligned}\mathrm{ann}(N + (\mathrm{ann}L)M) &= \mathrm{ann}N \cap \mathrm{ann}((\mathrm{ann}L)M) \\ &= \mathrm{ann}N \cap \mathrm{ann}(\mathrm{ann}L) \subseteq \mathrm{ann}L \cap \mathrm{ann}(\mathrm{ann}L) = 0.\end{aligned}$$

Thus $N + (\mathrm{ann}L)M \in S(M)$. Similarly, $K + (\mathrm{ann}L)M \in S(M)$. The result follows by (1) and Proposition 2.1.

(3) Similar to the proof for ideals, see [2, Theorem 2.3].

(4) $\mathrm{GCD}(N, K)$ exists for all $N, K \in S^*(M)$, hence for all $N, K \in S(M)$. Let $L = N \cap K$. By Proposition 2.5, $\mathrm{LCM}(N + (\mathrm{ann}L)M, K + (\mathrm{ann}L)M)$ exists. The result follows from (1). □

S. Glaz, [14] and [15], defined a ring to be a generalized GCD ring if the following two conditions are satisfied.

(1) R is a p.p. ring (that is, a ring in which every principal ideal is projective).
(2) The intersection of any two finitely generated flat ideals of R is finitely generated and flat.

Since finitely generated flat and finitely generated projective ideals coincide in p.p. rings, one can replace condition (2) by

(2′) The intersection of any two finitely generated projective ideals of R is projective (equivalently, $\mathrm{lcm}(I,J)$ exists for all finitely generated projective ideals I,J of R).

Glaz showed [14, Proposition 3.1] that if $aR\cap bR$ is a finitely generated projective ideal for any non-zero divisors $a,b\in R$, then $aR\cap bR$ is a finitely generated projective ideal for any non-zero $a,b\in R$. Thus R is a GGCD ring if

(1) R is a p.p. ring.

(2″) The intersection of any two invertible ideals of R is invertible.

Suppose that R is a p.p. ring and P is a prime ideal of R. Then R_P is a local p.p. ring, hence an integral domain. Let $I,J\in S^*(R)$ such that $I\cap J\in S^*(R)$, (equivalently $\mathrm{lcm}(I,J)$ exists). Then I_P, J_P and $I_P\cap J_P=(I\cap J)_P$ are invertible ideals of R_P. It follows by [4, Theorem 2.1] (and see the remark made after [2, Corollary 1.5]) that $\gcd(I_P,J_P)$ exists and $\gcd(I_P,J_P)=[IJ:(I\cap J)]_P$. Since $[IJ:(I\cap J)]_P$ is a common divisor of I_P and J_P for all P, $[IJ:(I\cap J)]$ is a common divisor of I and J. Suppose that G is any common divisor of I and J. Then $G_P|I_P$ and $G_P|J_P$, and hence $G_P|[IJ:(I\cap J)]_P$. This finally gives that $G|[IJ:(I\cap J)]$, and hence $\gcd(I,J)=[IJ:(I\cap J)]$. Combining this remark and [2, Corollary 2.4], we get that a ring R is a Glaz GGCD ring if and only if $\gcd(I,J)$ exists for all $I,J\in S^*(R)$.

We generalize this to modules as follows. An R-module M is a *Glaz* GGCD *module* if M is finitely generated faithful multiplication, every cyclic submodule of M is projective, and $N\cap K\in S^*(M)$ (equivalently $\mathrm{LCM}(N,K)$ exists) for all $N,K\in S^*(M)$.

Suppose that R is a Glaz GGCD ring and M is a finitely generated faithful multiplication R-module. Using the fact that a ring R is p.p. if and only if every cyclic submodule of a projective R-module is projective [11], we infer that every cyclic submodule of M is projective since M is projective [22, Theorem 11]. Let $N,K\in S^*(M)$. By Lemma 1.1, $[N:M],[K:M]\in S^*(R)$, and hence

$$[(N\cap K):M]=[N:M]\cap[K:M]\in S^*(R).$$

Hence $N\cap K\in S^*(M)$, and M is a Glaz GGCD module. Conversely, let R be a ring and M a Glaz GGCD module. Then M is a finitely generated faithful multiplication (hence projective) R-module where every cyclic submodule of M is projective. By [11], R is a p.p. ring. Let $I,J\in S^*(R)$. Then $IM,JM\in S^*(M)$, and hence $(I\cap J)M=IM\cap JM\in S^*(M)$. This implies that $I\cap J=[(I\cap J)M:M]\in S^*(R)$ by Lemma 1.1. Hence R is a Glaz GGCD ring. This shows that a ring R is a Glaz GGCD ring if and only if some finitely generated faithful multiplication R-module M is a Glaz GGCD module. On the other hand Glaz GGCD modules are GGCD modules. For let M be a Glaz GGCD module. Then M is finitely generated faithful multiplication. If $N,K\in S(M)$, then $N,K\in S^*(M)$, and hence $N\cap K\in S^*(M)$. By Lemma 1.1(5), $N\cap K$ is a finitely generated multiplication submodule of M. Since K is a faithful module and $[N:M]$ is a faithful ideal of R (Lemma 1.1(7)), we obtain from the proof of Proposition 4.1(2) that $0=\mathrm{ann}([N:M]K)=\mathrm{ann}(N\cap K)$. Hence M is a GGCD module.

Let M be a Glaz GGCD module and $N,K\in S^*(M)$. Let P be a prime ideal of R. Then R is a p.p. ring [11], and hence R_P is an integral domain. It follows that N_P, K_P and $N_P\cap K_P=(N\cap K)_P$ are finitely generated faithful multiplication submodules of M (Lemma 1.1). By

Proposition 2.1, $\mathrm{GCD}(N_P, K_P)$ exists and $\mathrm{GCD}(N_P, K_P) = ([[K : M]N : (N \cap K)]M)_P$. It is easy to see now that $\mathrm{GCD}(N, K)$ exists and $\mathrm{GCD}(N, K) = ([[K : M]N : (N \cap K)])M$. Hence M is a Glaz GGCD module if and only if $\mathrm{GCD}(N, K)$ exists for all $N, K \in S^*(M)$ (Theorem 4.4(4)).

Combining the above remarks, Corollary 1.2, [2, Theorem 1.8 and Corollary 2.4], and the fact that finitely generated flat and finitely generated projective submodules of a finitely generated faithful multiplication module over a p.p. ring coincide, we can state several conditions on a ring R equivalent to GGCD ring as defined by Glaz. We will use $I(R), F(R), F(M)$ to denote the sets of invertible ideals of R, flat ideals of R, and flat submodules of M respectively.

Theorem 4.5. *Let R be a p.p. ring and M a finitely generated faithful multiplication R-module (equivalently, every cyclic submodule of M is projective). Then the following are equivalent.*

(1) *R is a Glaz GGCD ring.*
(2) *M is a Glaz GGCD module.*
(3) *For all $I, J \in I(R)$, $\quad I \cap J \in I(R)$.*
(4) *For all $I, J \in S^*(R)$, $\quad I \cap J \in S^*(R)$.*
(5) *For all $I, J \in I(R)$, $\quad [I : J] \in I(R)$.*
(6) *For all $I, J \in S^*(R)$, $\quad [I : J] \in S^*(R)$.*
(7) *For all $I, J \in F(R)$, $\quad [I : J] \in F(R)$.*
(8) *For all $I, J \in I(R)$, $\quad \mathrm{lcm}(I, J)$ exists and is in $I(R)$.*
(9) *For all $I, J \in S^*(R)$, $\quad \mathrm{lcm}(I, J)$ exists and is in $S^*(R)$.*
(10) *For all $I, J \in F(R)$, $\quad \mathrm{lcm}(I, J)$ exists and is in $F(R)$.*
(11) *For all $I, J \in I(R)$, $\quad \gcd(I, J)$ exists and is in $I(R)$.*
(12) *For all $I, J \in S^*(R)$, $\quad \gcd(I, J)$ exists and is in $S^*(R)$.*
(13) *For all $I, J \in F(R)$, $\quad \gcd(I, J)$ exists and is in $F(R)$.*
(14) *For all $N, K \in S^*(M)$, $\quad \mathrm{LCM}(N, K)$ exists and is in $S^*(M)$.*
(15) *For all $N, K \in F(M)$, $\quad \mathrm{LCM}(N, K)$ exists and is in $F(M)$.*
(16) *For all $N, K \in S^*(M)$, $\quad \mathrm{GCD}(N, K)$ exists and is in $S^*(M)$.*
(17) *For all $N, K \in F(M)$, $\quad \mathrm{GCD}(N, K)$ exists and is in $F(M)$.*
(18) *For all $N, K \in S^*(M)$, $\quad [N : K] \in S^*(R)$.*
(19) *For all $N, K \in F(M)$, $\quad [N : K] \in F(R)$.*
(20) *For all $N, K \in F(M)$, $\quad N \cap K \in F(M)$.*

References

[1] Ali, M. M.; Smith, D. J.: *Some remarks on multiplication and projective modules.* Commun. Algebra **32**(10) (2004), 3897–3909. Zbl pre02130750

[2] Ali, M. M.; Smith, D. J.: *Generalized GCD rings II.* Beitr. Algebra Geom. **44** (2003), 75–98. Zbl 1023.13003

[3] Ali, M. M.; Smith, D. J.: *Finite and infinite collections of multiplication modules.* Beitr. Algebra und Geom. **42** (2001), 557–573. Zbl 0987.13001

[4] Ali, M. M.; Smith, D. J. *Generalized GCD rings.* Beitr. Algebra Geom. **42** (2001), 219–233. Zbl 0971.13002

[5] Anderson, D. D.: *Some remarks on multiplication ideals II.* Commun. Algebra **30** (2000), 3383–3390.cf. ibidem **28**(5) (2000), 2577-2583. Zbl 0965.13003

[6] Anderson, D. D.; *Some remarks on multiplication ideals.* Math. Jap. **25** (1980), 463–469. Zbl 0446.13001

[7] Anderson, D. D.; Anderson, D. F.: *Cancellation modules and related modules.* Lect. Notes Pure Appl. Math. **220**, Dekker, New York (2001), 13–25. Zbl 1037.13005

[8] Anderson, D. D.; Anderson, D. F.: *Generalized GCD domains.* Comment. Math. Univ. St. Pauli **28** (1980), 215–221. Zbl 0434.13001

[9] Anderson, D. D.; Anderson, D. F.: *Some remarks on cancellation ideals.* Math. Jap.**29** (1984), 879–886. Zbl 0552.13001

[10] El-Bast Z. A.; Smith, P. F.: *Multiplication modules.* Commun. Algebra **16** (1988), 755–779. Zbl 0642.13002

[11] Evans, M. W.: *On commutative p.p. rings.* Pac. J. Math. **41** (1972), 687–697. Zbl 0232.16019

[12] Faith, C.: *Algebra: Rings, Modules and Categories. I* Springer-Verlag 1981. Zbl 0508.16001

[13] Gilmer, R.: *Multiplicative Ideal Theory.* Kingston: Queen's 1992. Zbl 0804.13001

[14] Glaz, S.: *Finite conductor rings.* Proc. Am. Math. Soc. **129** (2001), 2833–2843. Zbl 0971.13003

[15] Glaz, S.: *Finite conductor rings with zero divisors.* Non-Noetherian Ring Theory (eds. S. Chapman and S. Glaz), Mathematics and its Applications **520**, Kluwer Academic Publishers, Dordrecht 2000, 251–269. Zbl pre01665644

[16] Kaplansky, I.: *Commutative Rings.* Boston 1974. Zbl 0296.13001

[17] Larsen, M. D.; McCarthy, P. J.: *Multiplicative Theory of Ideals.* New York 1971. Zbl 0237.13002

[18] Low, G. M.; Smith, P. F.: *Multiplication modules and ideals.* Commun. Algebra **18** (1990), 4353–4375. Zbl 0737.13001

[19] Naoum, A. G.; Hasan, M. A. K.: *The residual of finitely generated multiplication modules.* Arch. Math. **46** (1986), 225–230. Zbl 0573.13001

[20] Naoum, A. G.; Hasan, M. A. K.: *On finitely generated projective and flat ideals in commutative rings.* Period. Math. Hung. **16** (1985), 251–260. Zbl 0553.13004

[21] Singh, S.; Al-Shaniafi, Y.: *Multiplication modules.* Commun. Algebra **29** (2001), 2597–2609. Zbl 1036.13003

[22] Smith, P. F.: *Some remarks on multiplication modules.* Arch. Math. **50** (1988), 223–235. Zbl 0615.13003

Received February 17, 2004