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A Generalization of a Construction due to Van Nypelseer

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Abstract. We give a construction leading to new geometries from Steiner systems or arbitrary rank two geometries. Starting with an arbitrary rank two residually connected geometry Γ, we obtain firm, residually connected, $(IP)_2$ and flag-transitive geometries only if Γ is a thick linear space, the dual of a thick linear space or a $(4,3,4)$-gon. This construction is also used to produce a new firm and residually connected rank six geometry on which the Mathieu group M_{24} acts flag-transitively.

1. Introduction

Let $\mathcal{S} := S(t,k,v)$ be a Steiner system. We associate to $\mathcal{S}$ an incidence structure $\Gamma(\mathcal{S})$ of rank $t+1$ (see Construction 2 in Section 3).

When $t = 2$, we define $\Gamma(\mathcal{S})$ not only for Steiner systems but more generally for an arbitrary rank two geometry $\mathcal{S}$. In the latter case, we show that if $\Gamma(\mathcal{S})$ is a firm, residually connected, $(IP)_2$ and flag-transitive geometry then $\mathcal{S}$ is either a linear space or a $(4,3,4)$-gon.

For $t > 2$, we classify all Steiner systems $\mathcal{S} = S(t,k,v)$ such that $\Gamma(\mathcal{S})$ is a firm, residually connected, $(IP)_2$ and flag-transitive geometry.

When applied to the well known series of Steiner systems associated to the Mathieu groups M_i with $i = 21, 22, 23$ and 24, our construction produces a family of geometries for the Mathieu groups. The last element of this family is a rank six geometry on which the Mathieu group M_{24} acts flag-transitively. It is the first rank six firm, residually connected, flag-transitive and $(IP)_2$ geometry known for the Mathieu group M_{24}.

The paper is organised as follows. In Section 2, we give some definitions and fix notation. In Section 3, we describe our construction. In Section 4, we apply it to Steiner systems $S(2,k,v)$ and more generally to arbitrary rank two geometries. In Section 5, we look at Steiner systems $S(t,k,v)$ with $3 \leq t < k$.

Acknowledgements. We would like to thank Francis Buekenhout for many interesting discussions we had together during the writing of this paper. We also thank Antonio Pasini who suggested several improvements for this paper.

2. Some definitions and notation

Most of the following ideas arise from [18] (see also [5], Chapter 3 or [16]).

An *incidence structure* over a finite set I is a triple $\Gamma = (X, t, *)$ where X is a set of objects, $t : X \to I$ is a type function and $*$ is a symmetric incidence relation on X such that two objects of the same type are incident if and only if they are equal. A *flag* is a set of pairwise incident elements of Γ and a *chamber* is a flag of type I. An incidence structure Γ is a *geometry* provided that every flag is contained in a chamber. Moreover, we say that Γ is *firm* (resp. *thick*) provided that every flag of corank 1 is contained in at least two (resp. at least three) chambers.

The *residue* of a flag F of Γ is the incidence structure $(X_F, *_F, t_F)$ over the set of types $I \backslash t(F)$ where X_F is the set of elements of Γ not in F and incident to F. Moreover $*_F$ and t_F are the restrictions of $*$ and t to X_F and $I \backslash t(F)$. If Γ is a geometry, then obviously Γ_F is also a geometry. A geometry Γ is *residually connected* provided that every residue of rank at least two of Γ has a connected incidence graph. Observe that we regard a geometry as the residue of its empty flag and therefore residual connectedness implies connectedness. Moreover, when dealing with geometries of rank two, 'connected' and 'residually connected' mean the same.

Let $G \leq Aut(\Gamma)$ be a group of automorphisms of Γ. We say that G acts *flag-transitively* on Γ (also that Γ is *flag-transitive*) provided that G acts transitively on the set of all chambers of Γ, hence also on the set of all flags of any given type J where J is a subset of I. Moreover, as in [7], we say that G acts *locally 2-transitively* on Γ and we write $(2T)_1$ for this provided that for each flag F of corank 1 of Γ, the stabilizer G_F of F in G acts two-transitively on the residue Γ_F. If we do not precise what G is, we assume $G = Aut(\Gamma)$.

We refer to [5], Chapter 3, for the definition of diagram of a geometry.

We say that a geometry Γ satisfies the intersection property of rank two $(IP)_2$ provided that every rank two residue of Γ is either a partial linear space or a generalized digon.

If an incidence structure Γ is a firm, residually connected, $(IP)_2$ and flag-transitive geometry we say that Γ is an $\mathcal{A}$-geometry. The motivation for this notation is that if Γ is an $\mathcal{A}$-geometry, then it satisfies a series of axioms, namely, it is a geometry, it is firm, residually connected, $(IP)_2$ and flag-transitive. Therefore $\mathcal{A}$ denotes this set of axioms.

Let Γ be a rank two geometry with points and lines and such that every line is incident to $s+1$ points and every point is incident to $t+1$ lines. An *antiflag* of Γ is a set consisting of a point p and a line L of Γ such that p and L are not incident in Γ. Given an antiflag (p, L) of Γ, we denote by $\alpha(p, L)$ the number of points on L collinear with p in the collinearity graph of Γ. If for all antiflags (p, L) of Γ, the number $\alpha(p, L)$ is a constant $\alpha(\neq 0)$, we say that Γ is a *partial geometry.*

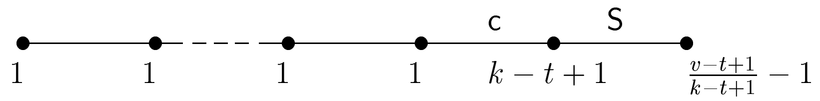
3. The construction

Let $S(t, k, v)$ be a Steiner system, i.e. a set Ω of v points, with subsets of k points of Ω called *blocks* and such that given any t points of Ω, there is exactly one block containing them. Obviously, we have $t \leq k \leq v$. Observe that when $t = 2$, the $S(2, k, v)$ is nothing else but a linear space where all lines have k points.

We recall the following construction due to Francis Buekenhout to produce a geometry Γ of rank t.

Construction 1. [2] *Take a Steiner system $S(t, k, v)$. Let Γ be an incidence structure of rank t defined as follows. Elements of type i are the i-sets of points for $i = 1, \ldots, t-1$. Elements of type t are the blocks of the $S(t, k, v)$. Incidence is symmetrized inclusion.*

It is obvious that for any Steiner system $S(t, k, v)$, the incidence structure obtained using Construction 1 is a firm and residually connected geometry provided that $t < k$ and $\frac{v-t+1}{k-t+1} \geq 2$. The diagram of Γ is easy to compute. We give it below. The symbol S denotes the Steiner system $S(2, k-t+1, v-t+1)$.



Taking the Steiner systems $S(5, 6, 12)$ and $S(5, 8, 24)$, Buekenhout produced, for instance, geometries for the Mathieu sporadic groups M_{11}, M_{12}, M_{22}, M_{23} and M_{24}. Their diagrams are given in Figure 1.

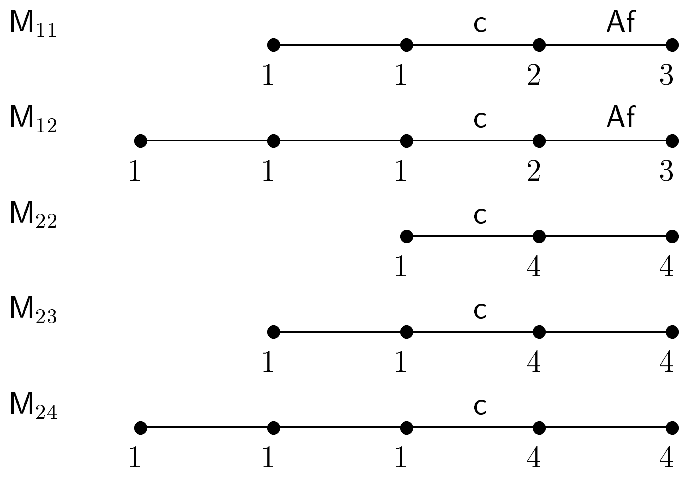


Figure 1. Buekenhout geometries associated to the Mathieu groups

We now give a construction to produce an incidence structure of rank $t+1$ from a Steiner system $S(t,k,v)$.

Construction 2. *Take a Steiner system $\mathcal{S} = S(t,k,v)$ defined on a set Ω of v points. Let $\Gamma(\mathcal{S})$ be an incidence structure of rank $t+1$ defined as follows.*
Elements of type i are the i-sets of points for $i = 1,\dots,t-2$. Elements of type t are the blocks of $\mathcal{S}$. Elements of type $t-1$ are ordered pairs consisting of a $(t-1)$-set x_{t-1} of points together with a $(k-t+1)$-set x_{k-t+1} of points such that $x_{t-1} \cup x_{k-t+1}$ is a block of $\mathcal{S}$. Elements of type $t+1$ are the elements of a copy Ω_1 of Ω. Incidence is symmetrized inclusion except for the following elements.

- *An element of type $t+1$ and an element of type i (with $i = 1,\dots,t-2,t$) are incident if and only if they are disjoint as subsets of the set Ω.*
- *An element p of type $t+1$ is incident with an element (x_{t-1}, x_{k-t+1}) of type $t-1$ if and only if $p \in x_{k-t+1}$.*
- *An element B of type t is incident with an element (x_{t-1}, x_{k-t+1}) of type $t-1$ if and only if $x_{t-1} \subset B$ and $x_{k-t+1} \cap B = \emptyset$.*

As pointed out by Antonio Pasini, this construction is an enrichment of a construction due to Thomas Meixner [14].

Lemma 1. *$\Gamma(\mathcal{S})$ is a firm geometry if and only if $k \geq t+1$ and $\frac{v-t+1}{k-t+1} \geq 3$.*

Proof. Suppose that $\Gamma(\mathcal{S})$ is a firm geometry. Then there are at least three blocks containing a set of $t-1$ points. Hence $\frac{v-t+1}{k-t+1} \geq 3$ and $k-t+1 \geq 2$.

Suppose now that $k \geq t+1$ and $\frac{v-t+1}{k-t+1} \geq 3$. We have to check that every flag of $\Gamma(\mathcal{S})$ is contained in at least two chambers. Let us first show that every flag is contained in at least one chamber. Suppose that $\mathcal{F}$ is a maximal flag of $\Gamma(\mathcal{S})$ not contained in a chamber of $\Gamma(\mathcal{S})$. Then, $\mathcal{F}$ must contain an element p of type $t+1$. Therefore the residue of p in $\Gamma(\mathcal{S})$ is an incidence structure but not a geometry. Obviously, $\mathcal{F}\backslash\{p\}$ must contain an element of type 1, say x_1. Therefore the residue of $\{p, x_1\}$ in $\Gamma(\mathcal{S})$ is an incidence structure but not a geometry. We use the same argument repeatedly to arrive at the conclusion that $\mathcal{F}$ must contain an element of type i for each $i = 1,\dots,t-1,t+1$. Given such a flag, it determines a block $\mathcal{B}$ of the $S(t,k,v)$. Given a block $\mathcal{B}$ in a $S(t,k,v)$ and a set T of $t-1$ points of $\mathcal{B}$, there are exactly $\frac{v-t+1}{k-t+1} - 1$ blocks $\mathcal{B}'$ such that $\mathcal{B} \cap \mathcal{B}' = T$. Hence there are $\frac{v-t+1}{k-t+1} - 1$ elements of type t incident to $\mathcal{F}$. By hypothesis, $\frac{v-t+1}{k-t+1} \geq 3$ and thus $\mathcal{F}$ is not maximal, a contradiction. It is now easy to see that every flag of corank 1 is contained in at least two chambers and thus that $\Gamma(\mathcal{S})$ is a firm geometry. □

We will see in the next section that residual connectedness does not hold every time.

4. The case where $t = 2$

In this case, the Steiner system $\mathcal{S} = S(2,k,v)$ is a linear space and $\Gamma(\mathcal{S})$ is an incidence structure of rank three. Let us deal with the case of an arbitrary rank two geometry and define our construction for such a geometry.

4.1. Construction 2 generalized in rank two

Let $\mathcal{S}$ be a rank two geometry. We may assume without loss of generality that $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \sim)$ where $\mathcal{P}$ and $\mathcal{L}$ denote the points and lines of $\mathcal{S}$, i.e. the elements of one type and the other. The elements of $\mathcal{L}$ are subsets of the point-set $\mathcal{P}$. The incidence relation $\sim$ is inclusion. Observe that $\mathcal{L}$ is a multiset in the sense that the same subset of the point-set may appear several times in $\mathcal{L}$.

Construction 3. *Let $\Gamma(\mathcal{S})$ be an incidence structure of rank 3 defined as follows. Elements of type 1 (resp. 2, 3) are the points (resp. lines, flags of size 2) of $\mathcal{S}$. Incidence is defined as follows.*

- *A flag $C = (p, L)$ is incident with a line L' if and only if $L \cap L' = p$.*
- *A flag $C = (p, L)$ is incident with a point p' if and only if $p' \in L$ and $p' \neq p$.*
- *A point p and a line L are incident if and only if $p \notin L$.*

As Francis Buekenhout mentioned to us, this construction was first used by Van Nypelseer on projective planes (see [15]). It gave rise to a family of geometries of type $\mathsf{Af}.\mathsf{Af}^*$ (see [13]) that is also available in this paper in Section 4.2.

In order to produce a firm geometry with Construction 3, we readily see that $\mathcal{S}$ must be a thick geometry. Hence we now assume that $\mathcal{S}$ is thick.

It is natural to ask what are the rank two geometries for which Construction 3 gives an $\mathcal{A}$-geometry in the sense of Section 2. It is obvious that if we take as $\mathcal{S}$ a generalized digon, the set of elements of type 2 of $\Gamma(\mathcal{S})_p$ is empty for any point $p \in \mathcal{P}$. So the incidence structure we get is not an $\mathcal{A}$-geometry.

The following lemma is obvious.

Lemma 2. *If $\mathcal{S}$ is a partial linear space and $\mathcal{S}^*$ is its dual, then $\Gamma(\mathcal{S}) \cong \Gamma(\mathcal{S}^*)$.*

Observe that the condition for $\mathcal{S}$ to be a partial linear space is needed in Lemma 2. Indeed, take the Fano plane $PG(2,2)$. Construct a geometry $\mathcal{S}$ of rank two in the following way: the points of $\mathcal{S}$ are the points of $PG(2,2)$. The lines of $\mathcal{S}$ are the lines of $PG(2,2)$ taken twice. Incidence is symmetrized inclusion. Obviously, $\mathcal{S}$ is a geometry which is not a partial linear space. In that case, if we take a point p' and a flag $C = (p, L)$ of $\mathcal{S}$ such that $p' \in L$ and $p' \neq p$, we have $p' \sim C$ but $\mid p'^* \cap p^* \mid = 2$ and thus $p'^* \not\sim C^*$.

We recall that the *point-diameter* (resp. *line-diameter*) of $\mathcal{S}$ is the length of the longest path from a point (resp. line) of $\mathcal{S}$ to any other element of $\mathcal{S}$ in the incidence graph of $\mathcal{S}$. The *gonality* is equal to half the girth of the incidence graph of $\mathcal{S}$.

Following roughly [3], we call $\mathcal{S}$ a (d_p, g, d_l)-gon provided that the point-diameter (resp. gonality, line-diameter) of $\mathcal{S}$ is equal to d_p (resp. g, d_l).

Let us give an example which shows that, given a rank two geometry $\mathcal{S}$, the geometry $\Gamma(\mathcal{S})$ is not necessarily residually connected. Take the incidence geometry $\mathcal{S}$ consisting of points and lines of the Desargues configuration. Take a point p and a line L in $\mathcal{S}$ such that p and L are at distance five in the incidence graph of $\mathcal{S}$. In the residue $\Gamma(\mathcal{S})_p$, the line L is not incident to any flag. Hence $\Gamma(\mathcal{S})$ is not a residually connected geometry.

If $\mathcal{S}$ is a rank two geometry with points and lines, we define a *proper triangle* of $\mathcal{S}$ to be a set of three pairwise collinear points of $\mathcal{S}$ that is not contained in any line of $\mathcal{S}$. We say that an edge $\{x, y\}$ of the collinearity graph is *near* a point z if $\{x, y, z\}$ is a proper triangle of $\mathcal{S}$. We define property (Δ) as follows.

(Δ) for every point p of $\mathcal{S}$, the graph $\mathcal{S}(p)$ formed by the points collinear with p and the edges near p is connected.

Lemma 3. *Let $\mathcal{S}$ be a thick connected rank two geometry and let $G \leq Aut(\mathcal{S})$.*

1. (Δ) *holds on $\mathcal{S}$ and $\mathcal{S}^* \Leftrightarrow \Gamma(\mathcal{S})$ is residually connected $\Rightarrow$ the gonality (resp. the point-diameter) of $\mathcal{S}$ is at most 3 (resp. 4).*
2. *If G is flag-transitive on $\Gamma(\mathcal{S})$ and $\mathcal{S}$ is a partial linear space then G is $(2T)_1$ on $\mathcal{S}$.*
3. *Suppose G is flag-transitive and $(2T)_1$ on $\mathcal{S}$. If $\Gamma(\mathcal{S})$ is a geometry then $\mathcal{S}$ is a partial linear space.*
4. *Suppose $\mathcal{S}$ is a partial linear space. Then G is flag-transitive on $\Gamma(\mathcal{S})$ if and only if G is transitive on the paths of length four of $\mathcal{S}$.*

Proof. Left to the reader. □

Corollary 1. *Let $\mathcal{S}$ be a thick connected partial linear space and $G \leq Aut(\mathcal{S})$. Let v be the number of points of $\mathcal{S}$ and s (resp. t) the number of points (resp. lines) incident to a line (resp. point). If $vt(s-1)(t-1)$ does not divide the order of G then G is not flag-transitive on $\Gamma(\mathcal{S})$.*

Proof. This is an immediate consequence of part 4 of Lemma 3. □

Theorem 1. *Let $\mathcal{S}$ be a thick connected partial linear space and let $G \leq Aut(\mathcal{S})$ be flag-transitive on $\mathcal{S}$. If $\Gamma(\mathcal{S})$ is a firm, residually connected and flag-transitive geometry, then G acts transitively on the paths of length four of $\mathcal{S}$. Moreover, $\mathcal{S}$ is either a linear space, the dual of a linear space or a $(4,3,4)$-gon.*

Proof. Part 1 of Lemma 3 restricts the partial linear spaces $\mathcal{S}$ to the following ones: $(4,3,4)$-gons, $(3,3,4)$-gons, $(4,3,3)$-gons and $(3,3,3)$-gons. Part 4 of Lemma 3 finishes the proof. □

If G is transitive on the paths of length four of $\mathcal{S}$, then obviously, G is $(2T)_1$ on $\mathcal{S}$. The pairs $(\mathcal{S}, G)$ where $\mathcal{S}$ is a thick linear space and G is a group that acts flag-transitively and locally two-transitively on $\mathcal{S}$ are given in the following theorem.

Theorem 2. [6] *Let Γ be a thick finite linear space of v points. Let G be a group acting flag-transitively and faithfully on Γ. If Γ is $(2T)_1$ then one of the following occurs:*

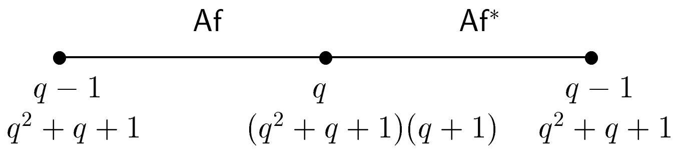
1. $\Gamma = AG(2,4)$, *with* $G = A\Gamma L(1,16)$.
2. $\Gamma = PG(n,q)$, $v = \frac{q^{n+1}-1}{q-1}$, $PSL(n+1,q) \trianglelefteq G \leq P\Gamma L(n+1,q)$ *with* $n \geq 2$.
3. $\Gamma = PG(3,2)$, $v = 15$ *with* $G \cong A_7$.
4. $\Gamma = AG(n,q)$, $v = p^d = q^n$, $G = p^d : G_0$ *with* $SL(n,q) \trianglelefteq G_0$, $q \geq 3$ *and* $n \geq 2$.
5. Γ *is a hermitian unital* $U_H(q)$, $v = q^3+1$, $PSU(3,q) \trianglelefteq G \leq P\Gamma U(3,q)$.

For the $(4,3,4)$-gons, no classification exists. Examples are known. There is one for $P\Gamma L(2,8)$ (see geometry 2 of rank two for $P\Gamma L(2,8)$ in [7]). There exists another one for $PSL(3,4)$ due to Gottschalk (see [9]). The group M_{22} also has such a geometry (see the supplement of [12]). Unfortunately, none of these geometries satisfies the necessary condition given by Corollary 1. There is also a $(4,3,4)$-gon $\mathcal{S}$ for the group $Alt(9)$ (see geometry (36) in [4]). It satisfies the hypotheses of Corollary 1 but $\Gamma(\mathcal{S})$ is not residually connected. Some partial geometries are also $(4,3,4)$-gons. In Subsection 4.3, we produce two infinite families of $(4,3,4)$-gons arising as residues of $\Gamma(AG(2,q))$ (where q is a finite field). One of them is a well known family of partial geometries. We also show that only one $(4,3,4)$-gon out of these two infinite families gives an $\mathcal{A}$-geometry using Construction 3.

Let us look at the thick linear spaces appearing in Theorem 2.

4.2. Projective planes

Theorem 3. *Let $\mathcal{S} = PG(2,q)$, $q \neq 2$. Then $\Gamma(\mathcal{S})$ is a firm and residually connected geometry having the following diagram.*



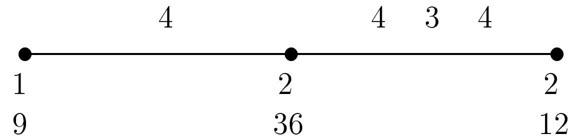
Moreover, the groups $PSL(3,q) \leq G \leq P\Gamma L(3,q)$ act flag-transitively on $\Gamma(\mathcal{S})$.

Proof. We refer to [13] for the proof of this theorem. □

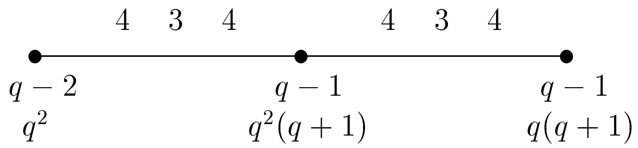
4.3. Affine planes

We denote by E_n an elementary abelian group of order n.

Theorem 4. *Let $\mathcal{S} = AG(2,q)$, $q \neq 2$. Then $\Gamma(\mathcal{S})$ is a firm and residually connected geometry.*
If $q = 3$, the diagram of $\Gamma(\mathcal{S})$ is the following.



If $q > 3$, the diagram of $\Gamma(\mathcal{S})$ is the following.



Moreover, the groups $G = E_{q^2} : G_0$*, with* $G_0 \trianglerighteq SL(2,q)$ *and* $q > 2$ *act flag-transitively on* $\Gamma(\mathcal{S})$*. Finally, if* $q = 4$*, the group* $A\Gamma L(1,16)$ *also acts flag-transitively on* $\Gamma(\mathcal{S})$*.*

The proof of this theorem uses only elementary combinatorial or group-theoretic arguments. Hence we leave it to the interested reader.

Theorem 4 produces two infinite families of $(4,3,4)$-gons. We call them $\Gamma(AG(2,q))_p$ and $\Gamma(AG(2,q))_L$ since they arise as residues of points and lines of $\Gamma(AG(2,q))$. In fact, $\Gamma(AG(2,q))_L$ is the complement of L minus all lines parallel to L in $AG(2,q)$, whereas $\Gamma(AG(2,q))_p$ is the complement of p minus all lines through p in $AG(2,q)$. Their diagrams are given below.

$\boxed{\Gamma(AG(2,q))_L}$

	4 3 4	
$q-2$		$q-1$
q^2-q		q^2

$\boxed{\Gamma(AG(2,q))_p}$

	4 3 4	
$q-1$		$q-1$
q^2-1		q^2-1

One may easily check that the geometries $\Gamma(AG(2,q))_p$ are not partial geometries. Indeed, in $\Gamma(AG(2,q))_p$, given a flag $C = (p,L)$ and a line L' such that $C \not\sim L$, we have $\alpha(C,L') = q$ if $L'//L$ in $AG(2,q)$ and $\alpha(C,L') = q-1$ otherwise.

The geometries $\Gamma(AG(2,q))_L$ are partial geometries such that $\alpha = q-2$. They are the so called duals of Bruck nets of order $q-1$ and degree $q-2$ (see [8], Section 1.2).

We have $Aut(\Gamma(AG(2,q))_p) \cong Aut(AG(2,q))_p$ and $Aut(\Gamma(AG(2,q))_L) \cong Aut(AG(2,q))_L$.

It is natural to ask whether Construction 3 gives an $\mathcal{A}$-geometry when applied to these $(4,3,4)$-gons.

Theorem 5. *Let* $\mathcal{S} = \Gamma(AG(2,q))_p$*. Then* $\Gamma(\mathcal{S})$ *is not flag-transitive.*

Proof. In $AG(2,q)$, take two lines, say L_1 and L_2 having a point p_2 in common and such that $p \in L_1$ but $p \notin L_2$. Take a third point p_3 on L_1 but not on L_2. Choose a point p_4 on L_2 distinct from p_2. Denote by L_3 the line through p and p_4. We have $(p_2,L_1) \sim L_2 \sim (p_4,L_3)$ in $\mathcal{S}$. Now, there is exactly one line through p_4 which has no point in common with L_1. The other lines through p_4 have at least one point in common with L_1. Hence, the automorphism group of $\mathcal{S}$ cannot act transitively on the paths of length four of $\mathcal{S}$ and Lemma 3 yields that $\Gamma(\mathcal{S})$ is therefore not flag-transitive. □

Theorem 6. *Let* $\mathcal{S} = \Gamma(AG(2,q))_L$ *with* $q > 3$*. Then* $\Gamma(\mathcal{S})$ *is firm, residually connected and flag-transitive if and only if* $q = 4$*. In that case,* $\Gamma(\mathcal{S})$ *has the following diagram.*

		6 4 6	
1	1		2
12	48		16

Proof. Left to the reader. □

4.4. Projective spaces $PG(n, q)$ with $n \geq 3$

Theorem 7. *Let $\mathcal{S}$ be the linear space obtained by taking the points and lines of a projective space $PG(n, q)$ ($n \geq 3$) over a finite field of characteristic $q \neq 2$. Incidence is symmetrised inclusion. Then $\Gamma(\mathcal{S})$ is not residually connected.*

Proof. The residue of a line L in $\Gamma(\mathcal{S})$ contains all points not on L and all flags consisting of a point of L and a line through this point. So given a point p not on L, the only points that are at finite distance of p in $\Gamma(\mathcal{S})_L$ are those in the plane containing p and L. Therefore $\Gamma(\mathcal{S})$ is not residually connected. □

4.5. Affine spaces $AG(n, q)$ with $n \geq 3$

Theorem 8. *Let $\mathcal{S}$ be the linear space obtained by taking the points and lines of an affine space $AG(n, q)$ ($n \geq 3$) over a finite field of characteristic $q \neq 2$. Incidence is symmetrised inclusion. Then $\Gamma(\mathcal{S})$ is not residually connected.*

Proof. Similar to the proof of Theorem 7. □

4.6. Hermitian unitals $U_H(q)$

This is the last case to consider in Theorem 2. Some experiments with Magma [1], permit to apply Construction 2 to the smallest examples. Figure 2 gives the diagram of $\Gamma(U_H(q))$ for $q \leq 16$. All of them are firm, residually connected, $(IP)_2$ and flag-transitive geometries.

Conjecture 1. *$\Gamma(U_H(q))$ is a firm, residually connected and flag-transitive geometry for all $q > 2$.*

5. The case where $t > 2$

In this section, we classify Steiner systems $\mathcal{S} := S(t, k, v)$ with $t > 2$ such that $\Gamma(\mathcal{S})$ (obtained by Construction 2) is an $\mathcal{A}$-geometry.

The Property (∗). *Let $\mathcal{S} = S(t, k, v)$ be a Steiner system. Let $\Gamma_1(\mathcal{S})$ be the rank t geometry obtained from $\mathcal{S}$ using Construction 1. Suppose $G := Aut(\mathcal{S})$ acts flag-transitively on Γ_1 and let $C = \{x_1, x_2, \ldots, x_t\}$ (with $t(x_i) = i$) be a chamber of Γ_1. Suppose moreover that G_C acts transitively on the blocks containing x_{t-1} and distinct from x_t. Finally, suppose that the stabilizer of such a block B in G_C acts transitively on the points of $B \backslash x_{t-1}$. Then we say that $\mathcal{S}$ satisfies property (∗).*

Theorem 9. *G acts flag-transitively on $\Gamma(\mathcal{S}) \Leftrightarrow \mathcal{S}$ satisfies Property (∗) above $\Rightarrow G$ is t-transitive on the points of $\mathcal{S}$.*

Proof. Straightforward. □

The Steiner systems $\mathcal{S} := S(t, k, v)$ with $t \geq 3$ such that $G \leq Aut(\mathcal{S})$ acts t-transitively on the points of $\mathcal{S}$ have been classified by William M. Kantor. We give the classification below.

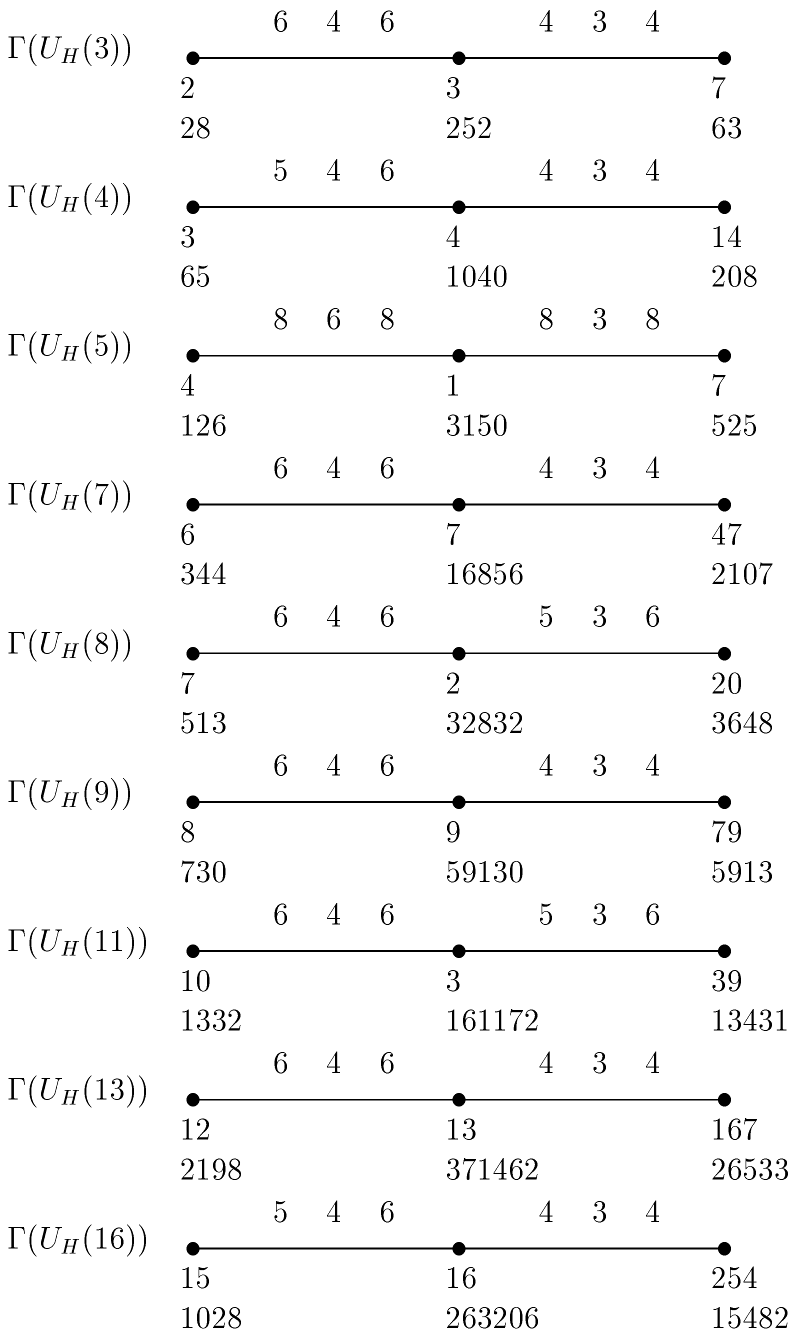


Figure 2. The geometries $\Gamma(U_H(q))$ for $q \leq 16$

Theorem 10. [10] *Let $\mathcal{S} = S(t,k,v)$ with $k \geq t+1 \geq 4$, and let $G \leq Aut(\mathcal{S})$ be t-transitive on the points of $\mathcal{S}$. Then either*

1. *$\mathcal{S}$ consists of the points and planes of $AG(d,2)$ for some d, and G is $2^d : GL(d,2)$ or $2^4 : A_7$ (and $d = 4$);*
2. *the blocks of $\mathcal{S}$ are all the images of $\{\infty\} \cup GF(q)$ under $PGL(2,q^e)$, $e \geq 2$, and $G \trianglerighteq PSL(2,q^e)$; or*
3. *$\mathcal{S}$ is an $S(4,5,11)$, $S(5,6,12)$, $S(3,6,22)$, $S(4,7,23)$, or $S(5,8,24)$ and $G \trianglerighteq \mathsf{M}_v$.*

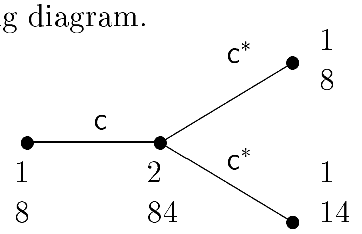
Theorem 10 shows that there are not a lot of $S(t,k,v)$ with $3 \leq t < k$ which may satisfy Property $(*)$. We look at each of them and determine whether Construction 2 applied to them gives an $\mathcal{A}$-geometry or not.

Lemma 4. *If $\mathcal{S}$ is as in* 1. *of Theorem* 10 *then $\Gamma(\mathcal{S})$ is residually connected if and only if $d = 3$.*

Proof. Take a point p and a plane π containing p in $AG(d,2)$. The elements in $\Gamma(\mathcal{S})_{p,\pi}$ are the points of $AG(2,q)$ not on π and the flags consisting of a line l of π containing p and a plane π' containing l and distinct from π. Take a point p' in $\Gamma(\mathcal{S})_{p,\pi}$. Then p' and π are in a unique subspace $\mathcal{H}$ of $AG(d,2)$.
If $d \neq 3$, then $\mathcal{H}$ is a proper subspace of dimension 3 of $AG(d,2)$. The only flags of $\Gamma(\mathcal{S})_{p,\pi}$ are contained in $\mathcal{H}$. Hence p' is not connected to any point p'' in $AG(d,2)\backslash\mathcal{H}$ and $\Gamma(\mathcal{S})$ is not residually connected.
If $d = 3$, one may easily check that $\Gamma(\mathcal{S})$ is indeed residually connected. □

Take the affine space $AG(3,2)$. Obviously, it may be seen as a $S(3,4,8)$. Using Construction 1, we get a c-extended projective plane, actually a copy of $AG(3,2)$. It satisfies Property $(*)$. Therefore, we apply Construction 2 to $S(3,4,8)$ and $\Gamma(S(3,4,8))$ is a rank four flag-transitive geometry having the following diagram.

$\boxed{\Gamma(S(3,4,8))}$

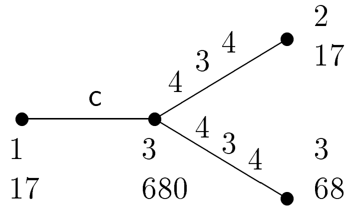


Lemma 5. *If $\mathcal{S}$ is as in* 2. *of Theorem* 10 *then $\Gamma(\mathcal{S})$ is residually connected if and only if $q = 4$ and $e = 2$.*

Proof. We have $\mathcal{S} = S(3, q+1, q^e+1)$. For an element p of type 1 of $\Gamma(\mathcal{S})$, the residue $\Gamma(\mathcal{S})_p$ is isomorphic to $\Gamma(S(2,q,q^e))$ and $Aut(\Gamma(\mathcal{S})_p) \cong G_p$ which is a group of affine type. Therefore, we can apply the results of Section 3, in particular Theorems 6 and 8 to conclude that if $q \neq 4$ or $e \neq 2$, then $\Gamma(\mathcal{S})$ is not residually connected. Now, if $q = 4$ and $e = 2$, the residue $\Gamma(\mathcal{S})_p$ is $\Gamma(AG(2,4))$ with $A\Gamma L(1,16)$. It is residually connected by Theorem 6. □

One may easily check that the $S(3,5,17)$ of Lemma 5 satisfies Property $(*)$ and therefore, $\Gamma(S(3,5,17))$ is a firm, residually connected and flag-transitive geometry. We give its diagram below.

$\boxed{\Gamma(S(3,5,17))}$



Let us now look at the family of Buekenhout geometries we mentioned in Figure 1. Obviously, those associated to the Steiner systems $S(3,6,22)$, $S(4,7,23)$ and $S(5,8,24)$ satisfy Property $(*)$. In Figure 3, we give the geometries $\Gamma(\mathcal{S})$ obtained for $\mathcal{S}$ a Steiner system $S(2,5,21)$, $S(3,6,22)$, $S(4,7,23)$ and $S(5,8,24)$.

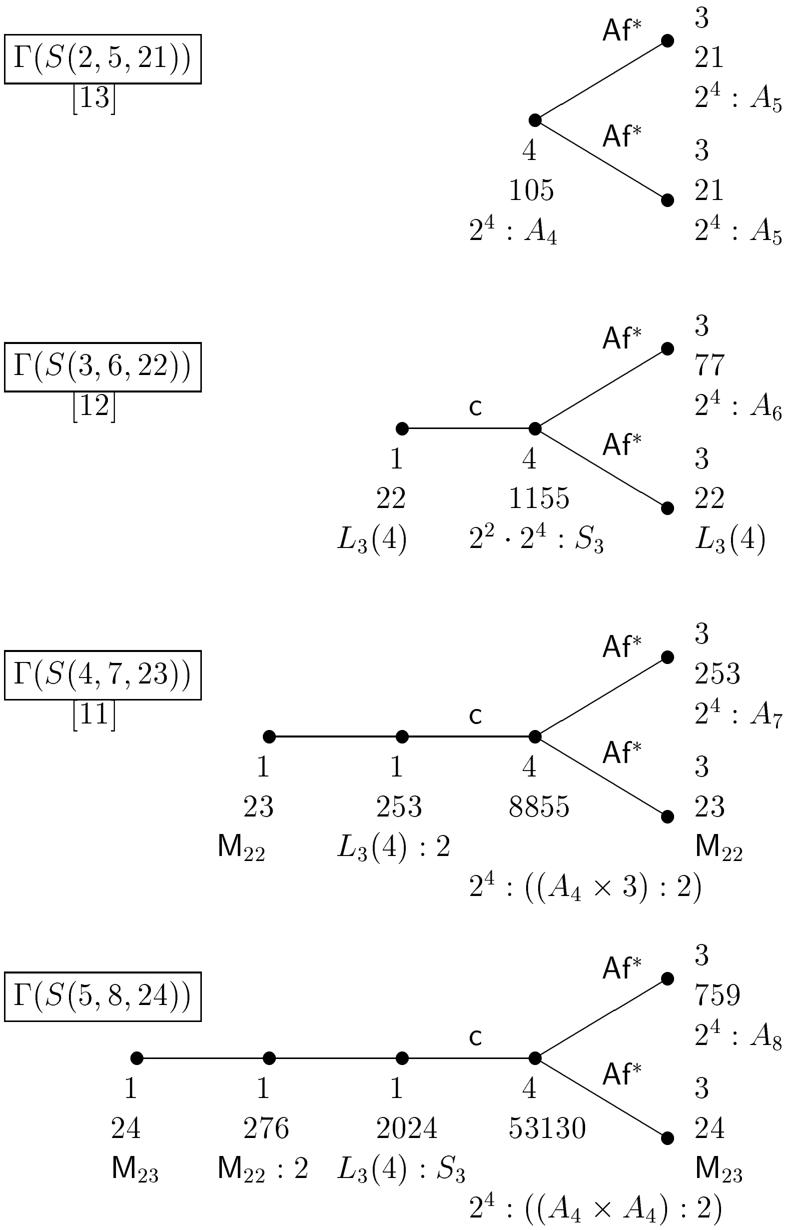


Figure 3. Geometries for the Mathieu groups M_i with $i = 21$, 22, 23 and 24

All geometries in Figure 3 except the last one are known. The last one is new. It is the first rank six geometry on which the Mathieu group M_{24} acts flag-transitively. It has two non-isomorphic residues of rank five which are extensions of dual affine planes for the groups M_{23} and $2^4 : Alt(8)$ (see geometries $\Gamma_{23}(4)$ and $\Gamma_8(4)$ in [17]). Moreover, Construction 2 unifies all geometries appearing in Figure 3.

Observe that the Steiner systems $S(5,6,12)$ and $S(4,5,11)$ do not satisfy Property $(*)$. Hence there is no hope of producing a flag-transitive geometry of rank 5 (resp. 6) for M_{11} (resp. M_{12}) using Construction 2.

To conclude, we summarize the analysis of the Steiner systems appearing in Theorem 10 in the following theorem.

Theorem 11. *Let $\mathcal{S} = S(t,k,v)$ with $3 \leq t < k$ and let $G = Aut(\mathcal{S})$. The geometry $\Gamma(\mathcal{S})$ is a firm, residually connected geometry on which G acts flag-transitively if and only if $\mathcal{S}$ is one of the following.*

1. *$\mathcal{S}$ consists of the points and planes of $AG(3,2)$ and G is $AGL(3,2)$;*
2. *the blocks of $\mathcal{S}$ are all the images of $\{\infty\} \cup GF(4)$ under $PGL(2,16)$ and $G \trianglerighteq P\Gamma L(2,16)$; or*
3. *$\mathcal{S}$ is an $S(3,6,22)$, $S(4,7,23)$ or $S(5,8,24)$ and $G \trianglerighteq \mathsf{M}_v$.*

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