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Spin Groups over a Commutative Ring and the Associated Root Data (Odd Rank Case)

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Abstract. Spin and Clifford groups as group schemes of semi-regular quadratic spaces of odd rank over a commutative ring are shown to be smooth and reductive. Analogously to the hyperbolic case smooth open neighborhoods of unit sections, called big cells, are constructed and examined. Jordan pairs again play a role through an imbedding into hyperbolic space whose rank is higher by one. The property reductive is now proved by constructing maximal tori and their associated root data explicitly.

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Introduction

Spin groups, attached to quadratic forms via Clifford algebras, are essentially objects as classical as Clifford algebras themselves and go back to Lipschitz [9]. His natural construction was refined over time [3], [1], and it is now rather a part of the mathematical folklore that spin groups as group schemes [4], [5], are smooth and reductive. Missing verification is performed partly for the case where the basic quadratic forms are hyperbolic [7]; the proof has already involved lengthy calculation, but combined with nice relations to Jordan pairs [10], [11]. In fact, by étale descent, [7] has actually covered the case for regular quadratic spaces of even rank [8]. The aim of the present article is to give an odd rank counterpart for the semi-regular case. This will complete an expected form of verification.

It turns out that the job is a natural continuation of [7], with nearly the same format of arguments. Always we start with a finitely generated projective module M over an arbitrary commutative base ring k, and follow mostly [7] for notation and conventions. (In fact, here and in [7] as well, M is preferably supposed faithfully projective, i.e., with rank everywhere positive since the trivial case $M = 0$ involves apparent exceptions.) Instead of the hyperbolic $\mathbf{H}(M)$ as the basic quadratic space, we now consider the orthogonal direct sum $\mathbf{H}(M)\perp\langle 1\rangle$ with the trivial rank one space. The wanted case is then covered by fppf descents [8, IV, (3.2)]. It is easy to continue taking the exterior power $\bigwedge(M)$ as the space of spinors (1.1–1.3), and we again enlarge the spin group to the special Clifford group equipped with projection to the special orthogonal group (1.4). Our first goal is to establish their smoothness (1.5), for which we construct smooth open neighborhoods of unit sections called *big cells* (1.7–1.9, 2.4). Contrary to [7] we have no longer direct relations to Jordan pairs, but pursuing analogy to examine the induced group germ structures on big cells as their own interests (2.4–2.10) retrieves a role of Jordan pairs; an imbedding transports the matters to the case of rank one higher which is hyperbolic (2.1–2.2). The property reductive is proved simultaneously with constructing maximal tori in the case where M is free with a base $e_1, e_2, \ldots, e_m$ (3.2–3.3). We see also the expected type B_m of the associated root data (3.1).

1. Constructions, smoothness

1.1. An element $\mathfrak{e}$. We shall use a specific identification of the Clifford algebra $\mathrm{C}(\mathbf{H}(M)\perp\langle 1\rangle)$, analogous to the natural $\mathrm{C}(\mathbf{H}(M)) \cong \mathrm{End}(\bigwedge(M))$ for the hyperbolic case but not the graded tensor product $\mathrm{C}(\mathbf{H}(M))\hat{\otimes}_k\mathrm{C}(\langle 1\rangle)$ itself. An important role is played by the element

$$\mathfrak{e} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{End}(\textstyle\bigwedge(M)), \tag{1.1.1}$$

where the matrix is relative to the decomposition $\bigwedge(M) = \bigwedge^+(M)\bigoplus\bigwedge^-(M)$ and acting from the left. We begin by observing some properties. Recall that the identification $\mathrm{C}(\mathbf{H}(M)) \cong \mathrm{End}(\bigwedge(M))$ describes the universal map as $L : M\bigoplus M^* \to \mathrm{End}(\bigwedge(M))$ sending $(x, f) \in M\bigoplus M^*$ to $l_x + d_f$, the sum of the left wedge-product by x and the left interior product by f (L being denoted V in [7, 3.1]), and that the adjectives even, odd refer to the 'checker-board' grading [8] of $\mathrm{End}(\bigwedge(M)a)$. Moreover, we follow [8, p. 195] to call the unique anti-automorphism of any Clifford algebra extending $-\mathrm{Id}$ (resp. Id) of the basic module the *standard* (resp. *canonical*) involution (the 'standard' being called 'main' in [7]). In our case $\mathrm{C}(\mathbf{H}(M)) \cong \mathrm{End}(\bigwedge(M))$, the element $\mathfrak{e}$ has square unit and commutes (resp. anti-commutes) with even (resp. odd) elements; so the conjugation $s \mapsto \mathfrak{e}s\mathfrak{e}$ by $\mathfrak{e}$ is just the automorphism extending $-\mathrm{Id}_{M\bigoplus M^*}$, whence interchanges the standard and canonical involutions. Furthermore one has $(l_x\mathfrak{e} + \mathfrak{e}d_f)^2 = \langle x, f\rangle$, from which by universality follows a unique algebra homomorphism

$$\varphi : \mathrm{End}(\textstyle\bigwedge(M)) \longrightarrow \mathrm{End}(\textstyle\bigwedge(M)), \tag{1.1.2}$$

such that $\varphi(l_x + d_f) = l_x\mathfrak{e} + \mathfrak{e}d_f$; clearly φ preserves the grading also. We claim that φ is an isomorphism with inverse itself. Indeed, since $\varphi^2(l_x + d_f) = \varphi((l_x - d_f)\mathfrak{e}) = (l_x + d_f)\mathfrak{e}\varphi(\mathfrak{e})$ it suffices to prove $\varphi(\mathfrak{e}) = \mathfrak{e}$, and by localizing there is no harm in assuming M free with a base $e_1, e_2, \ldots, e_m$. Let (e_i^*) denote the base of M^* dual to (e_i). Decomposing $\mathbf{H}(M) = \perp_{i=1}^m \mathbf{H}(k \cdot e_i)$ and endowing each $\mathbf{H}(k \cdot e_i)$ with a base $((e_i, 0), (0, e_i^*))$, we consider the element $z \in \mathrm{End}(\bigwedge(M))$ defined by the formula (2.3.1) in [8, p. 204]; the ingredients z_i (resp. b_i) being our $l(e_i)d(e_i^*)$ (resp. 1) and constituting z as a polynomial with integral coefficients. An easy verification proves

$$z = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{End}(\bigwedge(M)), \tag{1.1.3}$$

while we have $\varphi(z_i) = \varphi(l(e_i))\varphi(d(e_i^*)) = l(e_i)d(e_i^*) = z_i$, showing that z is fixed by φ. Hence so is $\mathfrak{e} = 1 - 2z$, by (1.1.1), (1.1.3). In fact, it can be checked simultaneously that both standard and canonical involutions send z to z or $1 - z$ according as m is even or odd. We shall record here an immediate consequence:

Both involutions applied to $\mathfrak{e}$ multiply it with the factor $(-1)^m$, where the rank m is understood as a locally constant integer-valued function on $\mathrm{Spec}(k)$.

1.2. Identification of $\mathbf{C(H(}M\mathbf{)}\perp\langle \mathbf{1}\rangle\mathbf{)}$. Let $k[w] = k[T]/(T^2 - 1)$ denote the quadratic extension defined by $w^2 - 1 = 0$. We consider $k[w]$ as $\mathbf{Z}/2\mathbf{Z}$-graded by $k[w]^+ := k \cdot 1$, $k[w]^- := k \cdot w$. Whereas $k[w]$ might be regarded as the Clifford algebra $\mathrm{C}(\langle 1 \rangle)$, we forget this at present and call the unique automorphism of $k[w]$ with $w \mapsto -w$ the *conjugation.* Moreover, we treat $\mathrm{End}(\bigwedge(M)) \bigotimes_k k[w]$ as a graded k-algebra with the grading induced from $k[w]$ by tensoring with $\mathrm{End}(\bigwedge(M))$. Let us consider the linear map

$$\begin{gathered} \tilde{L} : M \bigoplus M^* \bigoplus k \longrightarrow \mathrm{End}(\bigwedge(M)) \bigotimes_k k[w] \\ \tilde{L}(x, f, t) := (l_x + d_f + t\mathfrak{e}) \bigotimes w. \end{gathered}$$

Just as in the construction (1.1.2) of φ we see that $\tilde{L}$ composed with the squaring recovers now the quadratic form of $\mathbf{H}(M)\perp\langle 1 \rangle$, whence a unique extension

$$\Phi_M : \mathrm{C}(\mathbf{H}(M)\perp\langle 1 \rangle) \longrightarrow \mathrm{End}(\bigwedge(M)) \bigotimes_k k[w]$$

as an algebra homomorphism. Clearly Φ_M preserves the gradings also. In fact,

1.3. Lemma. *Φ_M is an isomorphism and identifies the standard involution of* $\mathrm{C}(\mathbf{H}(M)\perp\langle 1 \rangle)$ *with* (the canonical involution) $\otimes$ (the conjugation) *if the rank m of M is even, and with* (the standard involution) $\otimes\, 1$ *if m is odd.*

Proof. The latter statement is clear from the observation at the end of (1.1). We prove the former. Since both members have the same rank (as modules), it suffices to check the surjectivity. Moreover, the identification made in [8, p. 210] of $\mathrm{C}(\mathbf{H}(M) \cong \mathrm{End}(\bigwedge(M)))$ with the even part $\mathrm{C}^+ \subset \mathrm{C}$ of $\mathrm{C} := \mathrm{C}(\mathbf{H}(M)\perp\langle 1 \rangle)$ yields readily that $\Phi_M|\mathrm{C}^+ : \mathrm{C}^+ \to \mathrm{End}(\bigwedge(M))$ ($\subset \mathrm{End}(\bigwedge(M)) \bigotimes_k k[w]$ obviously) equals the isomorphism φ constructed in (1.1.2). Hence we are reduced to proving $1 \otimes w \in \mathrm{im}(\Phi_M)$. Now let $e_0 \in \mathrm{C}$ denote the imbedded element $(0, 0, 1) \in M \bigoplus M^* \bigoplus k \subset \mathrm{C}$, so that $\Phi_M(e_0) = \mathfrak{e} \otimes w$. Localizing without loss of generality, we again employ the argument in (1.1) to find that the elements $\Phi_M(x, 0, 0)\Phi_M(0, f, 0) = (l_x \otimes w)(d_f \otimes w) = (l_x d_f) \otimes 1$, for $x \in M$, $f \in M^*$, generate a subalgebra containing $z \otimes 1$, cf. (1.1.3), whence an element $\tilde{z} \in \mathrm{C}$ with $\Phi_M(\tilde{z}) = z \otimes 1$. It follows that $1 \otimes w = (\mathfrak{e} \otimes 1)(\mathfrak{e} \otimes w) = \Phi_M((1 - 2\tilde{z})e_0)$.

1.4. The group schemes. In the following, we treat Φ_M as an identification and take $\bigwedge(M)$ as the space of spinors. Namely, working in $\mathbf{GL}(\bigwedge(M))$ we define the *special Clifford group* $\mathbf{CL}^+(\mathbf{H}(M)\perp\langle 1\rangle)$ to be the normalizer of imbedded $M \bigoplus M^* \bigoplus k$:

$$\begin{gathered} L : M \bigoplus M^* \bigoplus k \longrightarrow \operatorname{End}(\bigwedge(M)) \\ L(x, f, t) := l_x + d_f + t\mathfrak{e}, \end{gathered} \tag{1.4.1}$$

and the *spin group* $\mathbf{Spin}(\mathbf{H}(M)\perp\langle 1\rangle) \subset \mathbf{CL}^+(\mathbf{H}(M)\perp\langle 1\rangle)$ to be the kernel of the character ν given by $\nu(s) := \overline{s}s$. Here, $\overline{?} : s \mapsto \overline{s}$ denotes the involution $\operatorname{End}(\bigwedge(M)) \tilde{\to} \operatorname{End}(\bigwedge(M))^{\mathrm{op}}$ locally canonical or standard according as M has rank even or odd, and we call ν again the *spinor character*, cf. [7, 3.1]. Moreover, in view of [8, IV, (5.1.1)], we define the *special orthogonal group* $\mathbf{SO}(\mathbf{H}(M)\perp\langle 1\rangle) \subset \mathbf{GL}(M \bigoplus M^* \bigoplus k)$ to be the kernel of det restricted to the full orthogonal group; an obvious adaptation of [8, IV, (6.3.1)] being read that the vector representation π, given by $sL(\xi)s^{-1} =: L(\pi(s)\cdot\xi)$, is an fppf epimorphism $\pi : \mathbf{CL}^+(\mathbf{H}(M)\perp\langle 1\rangle) \to \mathbf{SO}(\mathbf{H}(M)\perp\langle 1\rangle)$ with kernel $\mathbf{G}_{\mathbf{m}k}$. The k-groups constructed in this way are the main objects studied here. From the constructions follows easily (same argument as in [7, 3.2]) that they are all affine finitely presented k-group schemes. Our ultimate interest is to prove in addition the properties smooth with connected and reductive fibers. Here, we shall settle the following result as the first goal:

1.5. Theorem.

(a) *The k-groups $\mathbf{CL}^+(\mathbf{H}(M)\perp\langle 1\rangle)$, $\mathbf{Spin}(\mathbf{H}(M)\perp\langle 1\rangle)$, and $\mathbf{SO}(\mathbf{H}(M)\perp\langle 1\rangle)$ are all smooth with connected fibers.*

(b) *The homomorphisms $\mathbf{CL}^+(\mathbf{H}(M)\perp\langle 1\rangle) \to \mathbf{SO}(\mathbf{H}(M)\perp\langle 1\rangle)$ and $\mathbf{Spin}(\mathbf{H}(M)\perp\langle 1\rangle) \to \mathbf{SO}(\mathbf{H}(M)\perp\langle 1\rangle)$, both induced by the vector representation, are faithfully flat and finitely presented.*

Note that the part (a) implies (b) similarly to [7, 3.8]. In order to prove (a), we shall proceed analogously to the hyperbolic case [7]. Whereas the proof itself will be completed in (1.9) below, arguments in the course fit well to further discussions succeeding next. We begin with the connectedness. The problem being actually same as in [7, 3.3] on the fibers where $\mathbf{H}(M)\perp\langle 1\rangle$ remains non-degenerate, the doubtful part is characteristic two. Over an algebraically closed field k of characteristic two, the spin group $\mathrm{Spin}_{2n+1}(k)$ projects isomorphically onto $\mathrm{SO}_{2n+1}(k)$ and the Clifford group projects onto the latter. Therefore, only the connectedness of the Clifford group remains to be proved. The same argument as in [7, 3.3] applies after all, since we have the following

1.6. Lemma. *Let k be a perfect field of characteristic two, and (V, q) a regular quadratic space over k (necessarily of even-dimension). Construct the orthogonal direct sum $(V, q)\perp\langle 1\rangle$ and denote the generator $(0, 1) \in V \bigoplus k$ of the summand $\langle 1\rangle$ by e_0. Then the products ξe_0 for all non-singular $\xi \in V \bigoplus k$ form a set of generators for the special Clifford group $\mathrm{CL}^+((V, q)\perp\langle 1\rangle)$.*

Proof. Since k is a field, the vector representation π maps $\mathrm{CL}^+((V, q)\perp\langle 1\rangle)$ onto $\mathrm{SO}((V, q)\perp\langle 1\rangle)$ with kernel k^* and the announced set X clearly contains k^*. Therefore it suffices to see that the image $\pi(X)$ generates $\mathrm{SO}((V,q)\perp\langle 1\rangle)$. Now since we are in characteristic two, a classical argument (cf. §23 (p. 52 ff) of [6]) shows that $\mathrm{SO}((V,q)\perp\langle 1\rangle)$ stabilizes the line $k\cdot e_o \subset V\bigoplus k$ and goes to, via extracting (V,V)-entries $\mathrm{End}(V\bigoplus k)\to\mathrm{End}(V)$, the symplectic group $\mathrm{Sp}(V,b_q)$ of the associated bilinear form b_q. In fact, this yields an identification $\mathrm{SO}((V,q)\perp\langle 1\rangle)\cong\mathrm{Sp}(V,b_q)$ since the field k is perfect as well. Moreover, an easy computation shows that, for non-singular $\xi=a+t\cdot e_0\in V\bigoplus k$ with $\lambda:=(q(a)+t^2)^{-1}$, the transported element $\pi(\xi e_0)\in\mathrm{Sp}(V,b_q)$ is just the transvection $\tau_{\lambda,a}: x\mapsto x+\lambda b_q(x,a)a$, and again by the perfectness of k such $\tau_{\lambda,a}$'s exhaust all symplectic transvections. The assertion now follows from Prop. 4 (p. 10) of [6].

1.7. Open subschemes $\boldsymbol{\Omega}$, $\boldsymbol{\Omega_1}$. The situation is now same as in [7, 3.5], and we proceed analogously to prove the smoothness; thus what we want are smooth open neighborhoods of unit sections. Again the same function χ on $\mathbf{W}(\mathrm{End}(\bigwedge(M)))$ is to be considered, which extracts the $\mathrm{End}(\bigwedge^m(M))$ ($\cong k$)-entries of matrices relative to the decomposition of $\bigwedge(M)$ distinguishing the top-term $\bigwedge^m(M)$ [7, 3.5]. Moreover, let χ_1 denote the function on $\mathbf{W}(\mathrm{End}(M\bigoplus M^*\bigoplus k))$ extracting the determinants of $\mathrm{End}(M)$-entries. Since both χ and χ_1 have value one at unit sections, their obvious restrictions to our k-groups define principal open subschemes containing each unit section; among them are, say $\Omega\subset\mathbf{CL}^+(\mathbf{H}(M)\perp\langle 1\rangle)$ defined by χ, and $\Omega_1\subset\mathbf{SO}(\mathbf{H}(M)\perp\langle 1\rangle)$ defined by χ_1. Our next aim is to prove that they answer the question. This will be done through cell-decompositions below, which enlarge the previous ones [7, 3.6, 3.7] for the hyperbolic case.

1.8. Imbedded subgroups. We write $M^+:=M$, $M^-:=M^*$ and consider for each $\sigma=\pm$ a multiplication $(u,y)\bullet(u',y'):=(u+u'-y\wedge y', y+y')$ in the k-scheme underlying $\mathbf{W}(\bigwedge^2(M^\sigma)\bigoplus M^\sigma)$. It is immediate that $\bullet$ defines a group structure with unit $(0,0)$ and inversion $(u,y)\mapsto(-u,-y)$. The so obtained k-group is denoted $\mathbf{W}(\bigwedge^2(M^\sigma)\bigoplus M^\sigma]$, which is smooth with unipotent fibers since it is an extension of $\mathbf{W}(\bigwedge^2(M^\sigma))$ by $\mathbf{W}(M^\sigma)$. The multiplication $\bullet$ anticipates that the earlier defined homomorphism $Y_\sigma:\mathbf{W}(\bigwedge^2(M^\sigma))\to\mathbf{GL}(\bigwedge(M))$ [7, 3.6.1] is now extended to $\mathbf{W}(\bigwedge^2(M^\sigma)\bigoplus M^\sigma]$ as

$$\begin{array}{rcccl}\multicolumn{5}{c}{\mathbf{W}(\bigwedge^2(M)\bigoplus M]\xrightarrow{Y_+}\mathbf{GL}(\bigwedge(M))\xleftarrow{Y_-}\mathbf{W}(\bigwedge^2(M^*)\bigoplus M^*]}\\ Y_+(u,y) &:=& Y_+(u)(1+l_y\mathfrak{e}) &=& (1+l_y\mathfrak{e})Y_+(u),\\ Y_-(v,g) &:=& Y_-(v)(1+d_g\mathfrak{e}) &=& (1+d_g\mathfrak{e})Y_-(v).\end{array}\tag{1.8.1}$$

Moreover, a straightforward verification proves these $Y_\pm$ to normalize $M\bigoplus M^*\bigoplus k\subset\mathrm{End}(\bigwedge(M))$ (1.4.1) with

$$\mathbf{W}(\bigwedge^2(M)\bigoplus M]\xrightarrow{X_+}\mathbf{GL}(M\bigoplus M^*\bigoplus k)\xleftarrow{X_-}\mathbf{W}(\bigwedge^2(M^*)\bigoplus M^*]$$

$$X_+(u,y):=\begin{pmatrix}1 & u-y\otimes y & 2y\\ 0 & 1 & 0\\ 0 & -y & 1\end{pmatrix},\quad X_-(v,g):=\begin{pmatrix}1 & 0 & 0\\ v-g\otimes g & 1 & 2g\\ -g & 0 & 1\end{pmatrix}\tag{1.8.2}$$

the induced actions, namely to factor through $\mathbf{CL}^+(\mathbf{H}(M)\perp\langle 1\rangle)$ with $\pi Y_\pm=X_\pm$. Needless to say, matrices in (1.8.2) act from the left and identifications $M^\sigma\bigotimes_k M^\sigma\cong\mathrm{Hom}(M^{-\sigma},M^\sigma)$, $\sigma=\pm$, are made so that $(x\otimes y)(f)=\langle y,f\rangle x$. In fact, $Y_\pm$ takes values in $\mathbf{Spin}(\mathbf{H}(M)\perp\langle 1\rangle)=$

$\ker(\nu)$ (1.4), since both standard and canonical involutions of $\mathrm{End}(\bigwedge(M))$ have the same effect on $Y_+(u)$ (resp. $Y_-(v)$) and since the involution denoted $\overline{?} : s \mapsto \overline{s}$ (1.4) converts $1 + l_y\mathfrak{e}$ (resp. $1 + d_g\mathfrak{e}$) to $1 - l_y\mathfrak{e}$ (resp. $1 - d_g\mathfrak{e}$), cf. the last observation in (1.1). On the other hand, the same homomorphism

$$\begin{gathered} Y_0 : \mathbf{G_{m}}_k \times \mathbf{GL}(M) \longrightarrow \mathbf{GL}(\textstyle\bigwedge(M)) \\ Y_0(t,h) := t\det(h)^{-1} \textstyle\bigwedge(h) \end{gathered} \tag{1.8.3}$$

as in [7, 3.6.3] clearly factors through $\mathbf{CL}^+(\mathbf{H}(M)\perp\langle 1\rangle)$ with $\pi Y_0 = X_0\mathrm{pr}_2$, where

$$\begin{gathered} X_0 : \mathbf{GL}(M) \longrightarrow \mathbf{GL}(M \textstyle\bigoplus M^* \bigoplus k) \\ X_0(h) := \begin{pmatrix} h & 0 & 0 \\ 0 & h^{*-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \end{gathered} \tag{1.8.4}$$

and since the standard and canonical involutions actually differ only by the conjugation $s \mapsto \mathfrak{e}s\mathfrak{e}$, the composite νY_0 after all equals the same character $\nu_0 : (t,h) \mapsto t^2\det(h)^{-1}$ as in [7, 3.6.5]; in particular we know that $\ker(\nu Y_0)$ is a smooth k-group (the assumption faithfully projective for M being used here). In order to establish the desired smoothness, namely the part (a) of Theorem 1.5, it suffices therefore to prove the following

1.9. Proposition. *The morphisms*

$$\begin{gathered} \Psi : \mathbf{W}(\textstyle\bigwedge^2(M^*) \bigoplus M^*] \times (\mathbf{G_{m}}_k \times \mathbf{GL}(M)) \times \mathbf{W}(\textstyle\bigwedge^2(M) \bigoplus M] \longrightarrow \mathbf{CL}^+(\mathbf{H}(M)\perp\langle 1\rangle) \\ \Psi((v,g),(t,h),(u,y)) := Y_-(v,g)Y_0(t,h)Y_+(u,y), \end{gathered}$$

$$\begin{gathered} \Psi_1 : \mathbf{W}(\textstyle\bigwedge^2(M^*) \bigoplus M^*] \times \mathbf{GL}(M) \times \mathbf{W}(\textstyle\bigwedge^2(M) \bigoplus M] \longrightarrow \mathbf{SO}(\mathbf{H}(M)\perp\langle 1\rangle) \\ \Psi_1((v,g),h,(u,y)) := X_-(v,g)X_0(h)X_+(u,y) \end{gathered}$$

are open immersions with images Ω, Ω_1.

Proof. Calculating the product $X_-(v,g)X_0(h)X_+(u,y) =: p$ shows readily that Ψ_1 is monomorphic with $\chi_1(p) = \det(h)$. In fact, this yields soon the monomorphicity of Ψ, since $\pi\Psi$ coincides with Ψ_1 modulo $\mathrm{pr}_2 : \mathbf{G_{m}}_k \times \mathbf{GL}(M) \to \mathbf{GL}(M)$ and the scalar t equals $\chi(Y_-(v,g)Y_0(t,h)Y_+(u,y))$, as follows similarly to [7, 3.6.6]. Moreover, such recovery of t makes the following two statements sufficient to complete our proof:

1° any point s in Ω goes to Ω_1 by π;

2° any point s_1 in Ω_1 is an image under Ψ_1.

Without loss of generality, we may assume s, s_1 with value in k, and further the top-wedge $\bigwedge^m(M)$ trivialized by a base ω; using notation introduced in [7, 1.6].

We prove 1°. Put $t := \chi(s) \in k^*$ and $g := -t^{-1} \cdot \omega_-^{m-1}((s\cdot\omega)_{m-1}) \in M^*$, $v := t^{-1} \cdot \omega_-^{m-2}((s\cdot\omega)_{m-2}) \in \bigwedge^2(M^*)$, in other words $s\cdot\omega =: tz$ with $z \in \bigwedge(M)$ expressed so that ω, $(-1)^m g \dashv \omega$, $v \dashv \omega$ are the components of top-three degrees; moreover, let

$$\begin{pmatrix} h & * & * \\ b & * & * \\ f & * & * \end{pmatrix} \tag{1.9.1}$$

denote the matrix of $\pi(s)$, namely $h \in \mathrm{End}(M)$, $b \in \mathrm{Hom}(M, M^*)$, $f \in M^*$ with $sl(x) = (l(h \cdot x) + d(b(x)) + \langle x, f\rangle \mathfrak{e})s$ identically in $x \in M$. The last members being operated on ω, it becomes an easy adaptation of the argument in [7, 3.7] to obtain $f = -gh(= -h^* \cdot g)$, $b = vh + g \otimes f$. On account of (1.8.2), this converts (1.9.1) to

$$X_-(v, g) \begin{pmatrix} h & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}, \tag{1.9.2}$$

showing h invertible as claimed.

To prove 2°, we change notation so that (1.9.1) denotes now s_1. So h is invertible by assumption, and from the fact that (1.9.1) transforms an element $(h^{-1} \cdot x, 0, 0) \in M \bigoplus M^* \bigoplus k$ to $(x, b(h^{-1} \cdot x), \langle x, fh^{-1}\rangle)$ and keeps the quadratic form of $\mathbf{H}(M)\perp\langle 1\rangle$ invariant follows $\langle x, b(h^{-1} \cdot x)\rangle + \langle x, fh^{-1}\rangle^2 = 0$ identically in $x \in M$. This says that, if $g := -fh^{-1} \in M^*$, the map $v := bh^{-1} + g \otimes g \in \mathrm{Hom}(M, M^*)$ is alternating. Hence we again arrive at (1.9.2), expressing s_1. It remains to convert the latter matrix in (1.9.2) to the form $X_0(h)X_+(u, y)$. In fact, we have a more general result as follows (note that it implies $\det(s') = \alpha$), the verification of which is straightforward and similar to the argument above to be omitted: If a matrix s' of the latter form in (1.9.2) belongs to the full orthogonal group of $\mathbf{H}(M)\perp\langle 1\rangle$, it can be written as $\mathrm{diag}(1, 1, \alpha)X_0(h)X_+(u, y)$ with some $\alpha \in k$, $u \in \bigwedge^2(M)$, $y \in M$.

1.10. Calculation of $\chi(\mathbf{Y_+(u, y)Y_-(v, g)})$. Theorem (1.5) being thus established, we close this section with some incidental observations which anticipate partly the next section. Again trivializing the top-wedge $\bigwedge^m(M) = k \cdot \omega$ we call attention to ω acted upon by $Y_+(u, y)Y_-(v, g)$. From (1.8.1) follows easily that the degree-m term of $Y_+(u, y)Y_-(v, g) \cdot \omega$ equals that of $Y_+(u)Y_-(v) \cdot \omega - l_y Y_+(u)Y_-(v)d_g \cdot \omega$. Extracting the coefficient of ω, which is $\chi(Y_+(u, y)Y_-(v, g))$ by definition, is similar to [7, 3.6.7] and yields

$$\chi(Y_+(u, y)Y_-(v, g)) = \delta(u,\ v) - \langle y \wedge \exp(u),\ g \wedge \exp(v)\rangle, \tag{1.10.1}$$

where $\delta(u, v) := \langle \exp(u), \exp(v)\rangle$ as in [7, 2.3]. Needless to say, (1.10.1) itself is valid generally, regardless of whether $\bigwedge^m(M) \cong k$ or not. Further the last pairing in (1.10.1) equals $\langle Y_-(v)Y_+(u) \cdot y,\ g\rangle$, and supposing (u, v) quasi-invertible with [7, 2.6.1] soon converts $Y_-(v)Y_+(u) \cdot y$ to $\delta(u, v)\exp(u^v) \wedge y'$, where we set

$$y' := (1 + uv)^{-1} \cdot y \in M, \qquad g' := (1 + vu)^{-1} \cdot g \in M^* \tag{1.10.2}$$

in general. Since $\langle y', g\rangle = \langle y, g'\rangle$, it follows that $\langle y \wedge \exp(u), g \wedge \exp(v)\rangle = \delta(u, v)\langle y', g\rangle = \delta(u, v)\langle y, g'\rangle$. This shows

$$\chi(Y_+(u, y)Y_-(v, g)) = \delta(u,\ v)(1 - \langle y', g\rangle) = \delta(u, v)(1 - \langle y, g'\rangle), \tag{1.10.3}$$

in view of (1.10.1); the assumption (u, v) quasi-invertible being in practice harmless and convenient for later calculation. On the other hand, let us imbed M into the direct sum $M \bigoplus k$ and denote the element $(0, 1) \in M \bigoplus k$ by e; similar conventions apply to $M^* \subset M^* \bigoplus k = M^* \bigoplus k \cdot e^*$ with an identification $M^* \bigoplus k \cong (M \bigoplus k)^*$ such that $e^* = \mathrm{pr}_2$:

$M \bigoplus k \to k$. Then one has $\exp(u + y \wedge e) = \exp(u) \wedge (1 + y \wedge e)$, $\exp(v + g \wedge e^*) = \exp(v) \wedge (1 + g \wedge e^*)$, and it is easy to see that their pairing is just the right-hand side of (1.10.1). Again using the notation δ we arrive at

$$\chi(Y_+(u, y)Y_-(v, g)) = \delta(u + y \wedge e,\ v + g \wedge e^*). \tag{1.10.4}$$

Now (1.10.4) suggests a role played by the Jordan pair $(\bigwedge^2(M \bigoplus k),\ \bigwedge^2(M^* \bigoplus k))$. This will be exposed in the next section.

2. Imbedding, group germ structure

2.1. An imbedding ι and its Clifford transforms. There exists a morphism $\mathbf{H}(M) \perp\langle 1\rangle \to \mathbf{H}(M \bigoplus k)$ of quadratic modules given by the linear map

$$\begin{gathered} \iota : M \bigoplus M^* \bigoplus k \longrightarrow (M \bigoplus k) \bigoplus (M^* \bigoplus k) \\ \iota(x, f, t) := (x + te,\ f + te^*). \end{gathered}$$

On account of natural identifications (1.1.2), one may describe the induced map $\mathrm{C}(\iota)$ between Clifford algebras as the unique homomorphism

$$\mathrm{C}(\iota) : \mathrm{End}(\textstyle\bigwedge(M)) \bigotimes_k k[w] \longrightarrow \mathrm{End}(\textstyle\bigwedge(M \bigoplus k)) \tag{2.1.1}$$

such that $\mathrm{C}(\iota)\cdot((l_x + d_f + t\mathfrak{e}) \otimes w) = l_{x+te} + d_{f+te^*}$. Furthermore, modulo the obvious inclusion $\mathrm{End}(\bigwedge^+(M \bigoplus k)) \times \mathrm{End}(\bigwedge^-(M \bigoplus k)) \to \mathrm{End}(\bigwedge^+(M \bigoplus k))$, the induced map $\mathrm{C}^+(\iota)$ between even parts is

$$\begin{gathered} \mathrm{C}^+(\iota) : \mathrm{End}(\textstyle\bigwedge(M)) \longrightarrow \mathrm{End}(\textstyle\bigwedge^+(M \bigoplus k)) \times \mathrm{End}(\textstyle\bigwedge^-(M \bigoplus k)) \\ \mathrm{C}^+(\iota) \cdot s := \mathrm{C}(\iota) \cdot (s \otimes 1). \end{gathered} \tag{2.1.2}$$

Since everything is compatible with scalar extensions, one may consider the matters scheme-theoretically. We are interested in how $\mathrm{C}^+(\iota)$ transforms the special Clifford group $\mathbf{CL}^+(\mathbf{H}(M) \perp\langle 1\rangle)$. Let us introduce homomorphisms

$$\begin{gathered} T_+ : \mathbf{W}(M) \longrightarrow \mathbf{GL}(M \bigoplus k) \qquad T_- : \mathbf{W}(M^*) \longrightarrow \mathbf{GL}(M \bigoplus k) \\ T_+(y) := \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}, \qquad T_-(g) := \begin{pmatrix} 1 & 0 \\ -g & 1 \end{pmatrix}, \end{gathered} \tag{2.1.3}$$

$$\begin{gathered} T_0 : \mathbf{GL}(M) \longrightarrow \mathbf{GL}(M \bigoplus k) \\ T_0(h) := \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}, \end{gathered} \tag{2.1.4}$$

where the matrices are relative to the decomposition $M \bigoplus k$ and acting from the left.

2.2. Proposition.

(a) $\mathrm{C}^+(\iota)$ *induces a homomorphism*

$$\mathbf{CL}^+(\mathbf{H}(M)\perp\langle 1\rangle) \longrightarrow \mathbf{CL}^+(\mathbf{H}(M\bigoplus k)) \tag{2.2.1}$$

between special Clifford groups such that

$$\mathrm{C}^+(\iota)\cdot Y_+(u,y) = Y_+(u+y\wedge e)Y_0(1,T_+(y)), \tag{2.2.2}$$

$$\mathrm{C}^+(\iota)\cdot Y_-(v,g) = Y_-(v+g\wedge e^*)Y_0(1,T_-(g)), \tag{2.2.3}$$

$$\mathrm{C}^+(\iota)\cdot Y_0(t,h) = Y_0(t,T_0(h)). \tag{2.2.4}$$

(b) *The induced homomorphism* (2.1.1) *commutes with the spinor characters; moreover, it commutes with the functions both denoted* χ *in* (1.7) *and in* [7, 3.5].

We note that (2.2.1) induces a homomorphism between spin groups also, on account of (b); in fact between special orthogonal groups as well, since (2.2.1) keeps the imbedded subgroups $\mathbf{G}_{\mathbf{m}k}$ invariant and since (1.5.b), [7, 3.13.b] describe the special orthogonal group as a faithfully flat and finitely presented quotient of $\mathbf{CL}^+$. It will be needless to say that Y_+, Y_-, Y_0 in the right-hand sides of (2.2.2, 3, 4) should be understood following [7] with its M being replaced by our $M\bigoplus k$. Moreover, by Remark in [11, p. 31], $\mathbf{CL}^+(\mathbf{H}(M\bigoplus k))$ is generated as group sheaf by the images of Y_+, Y_-, Y_0; therefore the formulas (2.2.2, 3, 4) are actually sufficient for establishing part (a), and further for the first part of (b) as well. Now we see from construction (2.1) that so far as $z\in\bigwedge^+(M)$, $z^*\in\bigwedge^+(M^*)$ are even, the expressions l_z, d_{z^*} are invariant under $\mathrm{C}^+(\iota)$, in the sense that one has $\mathrm{C}^+(\iota)\cdot l_z = l(\bigwedge(\mathrm{in}_1)\cdot z)\in\mathrm{End}(\bigwedge(M\bigoplus k))$, etc.; similarly we have $\mathrm{C}^+(\iota)\cdot(l_y\mathfrak{e}) = l_y(l_e+d_{e^*})$, $\mathrm{C}^+(\iota)\cdot(d_g\mathfrak{e}) = d_g(l_e+d_{e^*})$. Therefore, applying $\mathrm{C}^+(\iota)$ to the definitions (1.8.1) soon yields

$$\begin{aligned}\mathrm{C}^+(\iota)\cdot Y_+(u,y) &= Y_+(u+y\wedge e)(1+l_yd_{e^*}),\\ \mathrm{C}^+(\iota)\cdot Y_-(v,g) &= Y_-(v+g\wedge e^*)(1+d_gl_e),\end{aligned}$$

whereas on account of [7, 1.3.2, 3.6.3] with the definition (2.1.3) we have $1+l_yd_{e^*} = \bigwedge(1+y\otimes e^*) = Y_0(1,T_+(y))$, $1+d_gl_e = \bigwedge(1-e\otimes g) = Y_0(1,T_-(g))$. This proves (2.2.2, 3). As for the remaining (2.2.4) and the last part of (b), they become clear through another description of $\mathrm{C}^+(\iota)$ below.

2.3. Lemma. $\mathrm{C}^+(\iota)$, *constructed in* (2.1.2), *coincides with the map with components* $\mathrm{End}(\bigwedge(M)) \to \mathrm{End}(\bigwedge^+(M\bigoplus k))$, $\mathrm{End}(\bigwedge(M)) \to \mathrm{End}(\bigwedge^-(M\bigoplus k))$ *being the isomorphisms of transporting structures through*

$$\begin{matrix}\Theta_+ : \bigwedge(M) \xrightarrow{\sim} \bigwedge^+(M\bigoplus k) & \quad \Theta_- : \bigwedge(M) \xrightarrow{\sim} \bigwedge^-(M\bigoplus k)\\ \Theta_+(z) := z_+ + z_-\wedge e, & \quad \Theta_-(z) := z_- + z_+\wedge e\end{matrix} \tag{2.3.1}$$

(suffixes $\pm$ indicating the components relative to the $\pm$-decomposition of the exterior algebra)

Proof. The coincidence of images of $l_x+d_f\in\mathrm{End}(\bigwedge(M))$, for any $(x,f)\in M\bigoplus M^*$, is to be proved. We work in the whole $\mathrm{End}(\bigwedge(M\bigoplus k))$ and relative to the decomposition

$$\begin{matrix}\bigwedge(M)\bigoplus\bigwedge(M) \xrightarrow{\sim} \bigwedge(M\bigoplus k)\\ (a,b)\mapsto a+b\wedge e\end{matrix}$$

describe any element of $\mathrm{End}(\bigwedge(M\bigoplus k))$ as two-by-two matrices with entries in $\mathrm{End}(\bigwedge(M))$ acting from the left. Since $(a+b\wedge e)_\pm = a_\pm + b_\mp \wedge e = \Theta_\pm(a_\pm + b_\pm)$, and since $\Theta_\pm((l_x + d_f)\cdot(a_\pm + b_\pm)) = (l_x+d_f)\cdot b_\pm + ((l_x+d_f)\cdot a_\pm)\wedge e$, the transported action of $l_x + d_f$ on $\bigwedge(M\bigoplus k)$ is $a + b\wedge e \mapsto (l_x+d_f)\cdot b + ((x_x + d_f)\cdot a)\wedge e$. Therefore what we must check takes of the form

$$\mathrm{C}^+(\iota)\cdot((l_x+d_f)\otimes 1) = \begin{pmatrix} 0 & l_x+d_f \\ l_x+d_f & 0 \end{pmatrix}. \tag{2.3.2}$$

Moreover, straightforward calculation shows

$$l_{x+te} = \begin{pmatrix} l_x & 0 \\ t\mathfrak{e} & l_x \end{pmatrix}, \qquad d_{f+te^*} = \begin{pmatrix} d_f & t\mathfrak{e} \\ 0 & d_f \end{pmatrix},$$

so that by the characterization of the map $\mathrm{C}(\iota)$ (2.1.1), we have

$$\mathrm{C}(\iota)\cdot((l_x+d_f+t\mathfrak{e})\otimes w) = \begin{pmatrix} l_x+d_f & t\mathfrak{e} \\ t\mathfrak{e} & l_x+d_f \end{pmatrix}. \tag{2.3.3}$$

We shall prove (2.3.2) by factoring $(l_x+d_f)\otimes 1$ into $((l_x+d_f)\otimes w)(\mathfrak{e}\otimes w)(\mathfrak{e}\otimes 1)$; on account of (2.3.3), it remains to check

$$\mathrm{C}(\iota)\cdot(\mathfrak{e}\otimes 1) = \begin{pmatrix} \mathfrak{e} & 0 \\ 0 & \mathfrak{e} \end{pmatrix}. \tag{2.3.4}$$

To see (2.3.4), we may localize to make $M = \bigoplus_{i=1}^m k\cdot e_i$ free with a base. Let us consider the previous elements z, $z_i \in \mathrm{End}(\bigwedge(M))$ (1.1). Since $z_i = l(e_i)d(e_i^*)$ by definition, $\mathrm{C}(\iota)\cdot(z_i\otimes 1)$ has the same expression understood within $\mathrm{End}((M\bigoplus k))$. Moreover, since l_e $(= l(e))$ anti-commutes with both $l(e_i)$ and $d(e_i^*)$, it commutes with $\mathrm{C}(\iota)\cdot(z_i\otimes 1)$, a fortiori with $\mathrm{C}(\iota)\cdot(z\otimes 1)$. Hence $\mathrm{C}(\iota)\cdot(z\otimes 1)$ transforms $b\wedge e = l_e\mathfrak{e}\cdot b$ to $l_e z\mathfrak{e}\cdot b = (\mathfrak{e}z\mathfrak{e}\cdot b)\wedge e$, as well as $a \in \bigwedge(M) \subset \bigwedge(M\bigoplus k)$ to $z\cdot a$. It follows that

$$\mathrm{C}(\iota)\cdot(z\otimes 1) = \begin{pmatrix} z & 0 \\ 0 & \mathfrak{e}z\mathfrak{e} \end{pmatrix}.$$

On account of $1-2z = \mathfrak{e}$, $1 - 2\mathfrak{e}z\mathfrak{e} = \mathfrak{e}(1-2z)\mathfrak{e} = \mathfrak{e}$, we get the desired (2.3.4).

2.4. Big cells. By the *big cell* of $\mathbf{CL}^+(\mathbf{H}(M)\perp\langle 1\rangle)$ (resp. $\mathbf{CL}^+(\mathbf{H}(M\bigoplus k))$), we shall understand the open subscheme denoted by Ω in (1.7) (resp. in [7, 3.5] with its M being replaced by our $M\bigoplus k$). A direct consequence of the last statement in (2.2.b) is that the notion big cell is preserved under the morphism (2.2.1) viewed as a base change. Moreover, since the right-hand sides of (2.2.2, 3) are commuting products and since the big cell of $\mathbf{CL}^+(\mathbf{H}(M\bigoplus k))$ is invariant under the multiplications by $Y_0(1, T_+(y))$, $Y_0(1, T_-(g))$, it follows ([7, 3.11]) that a product $Y_+(u,y)Y_-(v,g)$ lies in the big cell of $\mathbf{CL}^+(\mathbf{H}(M)\perp\langle 1\rangle)$ if and only if $(u+y\wedge e,\ v+g\wedge e^*)$ is quasi-invertible in the Jordan pair $(\bigwedge^2(M\bigoplus k), \bigwedge^2(M^*\bigoplus k))$. This fact itself is visible from (1.10.4), however, one would like to proceed further to know how such $Y_+(u,y)Y_-(v,g)$ decomposes according to the cell-decomposition (1.9). On account of [7, 2.6], the job is likely to involve quasi-inverses for the pair $(\bigwedge^2(M\bigoplus k), \bigwedge^2(M^*\bigoplus k))$

decomposed as $\bigwedge^2(M \bigoplus k) \cong \bigwedge^2(M) \bigoplus M$, $\bigwedge^2(M^* \bigoplus k) \cong \bigwedge^2(M^*) \bigoplus M^*$. We begin by introducing various polynomials, in terms of which the desired components will be described.

2.5. The polynomial map $((\mathbf{u}, \mathbf{y}), (\mathbf{v}, \mathbf{g})) \mapsto \mathbf{h}$. For a moment, some notations h, t, etc. will be specified so that they depend on $((u, y), (v, g))$ to represent certain polynomial or rational maps on $(\bigwedge^2(M) \bigoplus M) \times (\bigwedge^2(M^*) \bigoplus M^*)$. Possible scalar extensions $k \to R$ should be understood as well, however, apparent and harmless restrictions to k-valued points will be made tacitly. This being said, we define

$$h := 1 + uv - u(g) \otimes g + y \otimes v(y) + (\langle y, g\rangle - 2)y \otimes g \in \mathrm{End}(M) \tag{2.5.1}$$

for all $(u, y) \in \bigwedge^2(M) \bigoplus M$, $(v, g) \in \bigwedge^2(M^*) \bigoplus M^*$. In the case where (u, v) is quasi-invertible, i.e., $1 + uv$ is invertible (see [7, 1.5]), straightforward calculation shows that h factors as

$$\begin{aligned} h &= (1 + uv)(1 + y' \otimes v(y))(1 + y'' \otimes g) \\ &= (1 + y \otimes g'')(1 - u(g) \otimes g')(1 + uv), \end{aligned} \tag{2.5.2}$$

where $y' \in M$, $g' \in M^*$ are as in (1.10.2) and now $y'' := -u(g') + (\langle y', g\rangle - 2)y' \in M$, $g'' := v(y') + (\langle y, g'\rangle - 2)g' \in M^*$. Besides $\langle y', g\rangle = \langle y, g'\rangle$, we have $\langle y', v(y)\rangle = \langle y, v^u(y)\rangle = 0$, $\langle u(g), g'\rangle = \langle u^v(g), g\rangle = 0$ by (1.10.2) with [7, 1.5.5], and using these shows $1 + \langle y'', g\rangle = (1 - \langle y', g\rangle)^2$, $1 + \langle y, g''\rangle = (1 - \langle y, g'\rangle)^2$; since $\det(1 + x \otimes f) = 1 + \langle x, f\rangle$ in general [7, 1.3.2], it follows that

$$\det(h) = \det(1 + uv)(1 - \mu)^2 \qquad \text{with} \qquad \mu := \langle y',\ g\rangle = \langle y,\ g'\rangle. \tag{2.5.3}$$

On account of [7, 2.6.2] and of (1.10.3, 4), this amounts to

$$\det(h) = t^2 \quad \text{with} \quad t := \chi(Y_+(u, y)Y_-(v, g)) = \delta(u + y \wedge e,\ v + g \wedge e^*), \tag{2.5.4}$$

which now holds for all (u, v) by density. Therefore, that h is invertible is another equivalent condition to the previous: $Y_+(u, y)Y_-(v, g)$ is in the big cell $\Leftrightarrow$ $(u + y \wedge e,\ v + g \wedge e^*)$ is quasi-invertible.

2.6. Rational maps $((\boldsymbol{u}, \boldsymbol{y}), (\boldsymbol{v}, \boldsymbol{g})) \mapsto \boldsymbol{x}, \boldsymbol{f}$. Supposing h invertible we set

$$\begin{aligned} x &:= h^{-1} \cdot (u(g) + (1 - \langle y, g\rangle)y) && \in M, \\ f &:= h^{*-1} \cdot (- v(y) + (1 - \langle y, g\rangle)g) && \in M^*. \end{aligned} \tag{2.6.1}$$

If, in addition, (u, v) is quasi-invertible then the scalar $1 - \mu$ is invertible by (2.5.3) and again straightforward verifications using (2.5.2) prove

$$x = (1 - \mu)^{-1} \cdot (u(g') + y'), \qquad f = (1 - \mu)^{-1} \cdot (-v(y') + g'). \tag{2.6.2}$$

This amounts to $h \cdot (u(g') + y') = (1 - \mu) \cdot (u(g) + (1 - \langle y, g\rangle)y)$, etc., however, calculating each image $h \cdot u(g')$, $h \cdot y'$, $h^* \cdot v(y')$, $h^* \cdot g'$ with the aid of (2.5.2) in fact precedes (2.6.2) and yields similar relations

$$h^{-1} \cdot y = (1 - \mu)^{-2} \cdot (y' + \mu u(g')), \qquad h^{*-1} \cdot g = (1 - \mu)^{-2} \cdot (g' - \mu v(y')) \tag{2.6.3}$$

as well. Combining (2.6.2) with (2.6.3) shows readily

$$h^{-1}\cdot y = (1-\mu)^{-1}\cdot(x-u(g')), \qquad h^{*-1}\cdot g = (1-\mu)^{-1}\cdot(f+v(y')), \tag{2.6.4}$$

$$h^{-1}\cdot y - \langle y,f\rangle x = (1-\mu)^{-1}\cdot y', \qquad h^{*-1}\cdot g - \langle x,g\rangle f = (1-\mu)^{-1}\cdot g'. \tag{2.6.5}$$

Moreover, since $\langle u(g')+y', -v(y)+(1+\langle y,g\rangle)g\rangle$ is easily seen to be $(1-\mu)\langle y,g\rangle$, we get

$$\langle x,\ h^*\cdot f\rangle = \langle h\cdot x,\ f\rangle = \langle y,\ g\rangle \tag{2.6.6}$$

from (2.6.1) and (2.6.2). Note that (2.6.6) holds for all (u,v) by density.

2.7. Rational maps $((u,y),(v,g))\mapsto U, V$. Always supposing h invertible we set

$$\begin{array}{lcll} U & := & \bigwedge^2(h)^{-1}\cdot(u+uvu+uv(y)\wedge y-u(g)\wedge y) & \in \bigwedge^2(M), \\ V & := & \bigwedge^2(h^*)^{-1}\cdot(v+vuv+g\wedge vu(g)+g\wedge v(y)) & \in \bigwedge^2(M^*). \end{array} \tag{2.7.1}$$

Recall [7, 1.5] that composites like uvu are taken under identifications $\bigwedge^2(M)\subset \mathrm{Hom}(M^*,M)$, etc. Moreover, since $\bigwedge^2(h)\cdot U = hUh^*$, $\bigwedge^2(h^*)\cdot V = h^*Vh$ by [7, 1.4.6] and since $h(x\otimes x)h^* = (h\cdot x)\otimes(h\cdot x)$, $h^*(f\otimes f)h = (fh)\otimes(fh)$ obviously, it becomes straightforward after sandwiching members between h and h^* to verify

$$\begin{array}{l} U = (u+y\otimes y)h^{*-1} - x\otimes x = h^{-1}(u-y\otimes y)+x\otimes x, \\ V = (v-g\otimes g)h^{-1} + f\otimes f = h^{*-1}(v+g\otimes g) - f\otimes f. \end{array} \tag{2.7.2}$$

In the case where (u,v) is quasi-invertible, we have

$$\begin{array}{lcl} h^{-1}(y\otimes y) & = & (1-\mu)^{-1}\cdot(x\otimes y - u(g')\otimes y), \\ (g\otimes g)h^{-1} & = & (1-\mu)^{-1}\cdot(g\otimes f + g\otimes v(y')) \end{array}$$

by (2.6.4), while (2.6.2) shows $u(f) = x-(1-\mu)^{-1}\cdot y$, $v(x) = -f+(1-\mu)^{-1}\cdot g$ and $x\otimes u(f)$ (resp. $v(x)\otimes f$) is equal to the composite $-(x\otimes f)u$ (resp. $v(x\otimes f)$), whence

$$\begin{array}{lcl} x\otimes x & = & -(x\otimes f)u + (1-\mu)^{-1}\cdot x\otimes y, \\ f\otimes f & = & -v(x\otimes f) + (1-\mu)^{-1}\cdot g\otimes f. \end{array}$$

Therefore (2.7.2) yields

$$\begin{array}{lcl} U & = & (h^{-1}-x\otimes f)u + (1-\mu)^{-1}\cdot u(g')\otimes y, \\ V & = & v(h^{-1}-x\otimes f) - (1-\mu)^{-1}\cdot g\otimes v(y'). \end{array} \tag{2.7.3}$$

Moreover, we use (2.7.2) to calculate $U(g)$ and $V(y)$; on account of (2.6.3), (2.6.2) the result is $U(g) = (1-\mu)^{-1}\cdot u(g')$, $V(y) = (1-\mu)^{-1}\cdot v(y')$, however, we proceed further to

$$h^{-1}\cdot y - \langle y,f\rangle x = -U(g)+x, \qquad h^{*-1}\cdot g - \langle x,g\rangle f = V(y)+f, \tag{2.7.4}$$

with aid of (2.6.2), (2.6.5). Note that (2.7.4) holds for all (u,v) by density.

2.8. Proposition. *Let* $(u, y) \in \bigwedge^2(M) \bigoplus M$, $(v, g) \in \bigwedge^2(M^*) \bigoplus M^*$, *and put*

$$\mathfrak{a} := u + y \wedge e \in \bigwedge^2(M \bigoplus k), \qquad \mathfrak{b} := v + g \wedge e^* \in \bigwedge^2(M^* \bigoplus k). \tag{2.8.1}$$

For $(\mathfrak{a}, \mathfrak{b})$ *to be quasi-invertible in the Jordan pair* $(\bigwedge^2(M \bigoplus k), \bigwedge^2(M^* \bigoplus k))$, *it is necessary and sufficient that the endomorphism* $h \in \mathrm{End}(M)$ *defined in* (2.5) *is invertible. In that case, the quasi-inverses are given by*

$$\begin{array}{rclcl} \mathfrak{a}^{\mathfrak{b}} & = & U + (h^{-1} \cdot y - \langle y, f\rangle x) \wedge e & = & \bigwedge^2 (T_-(g)) \cdot (U + x \wedge e), \\ \mathfrak{b}^{\mathfrak{a}} & = & V + (h^{*-1} \cdot g - \langle x, g\rangle f) \wedge e^* & = & \bigwedge^2 (T_+(y)^*) \cdot (V + f \wedge e^*), \end{array} \tag{2.8.2}$$

with $x \in M$, $f \in M^*$, $U \in \bigwedge^2(M)$, $V \in \bigwedge^2(M^*)$ *defined in* (2.6), (2.7), *and* $T_\pm$ *in* (2.1); *moreover, the endomorphism* $1 + \mathfrak{a}\mathfrak{b} \in \mathrm{End}(M \bigoplus k)$ *takes the form*

$$1 + \mathfrak{a}\mathfrak{b} = T_+(y)^{-1} T_-(f) T_0(h) T_+(x) T_-(g)^{-1}. \tag{2.8.3}$$

Proof. The first statement has been observed in (2.5). The remaining involve with linear maps intertwining $M \bigoplus k$ and $M^* \bigoplus k$, e.g. $\bigwedge^2(M \bigoplus k) \subset \mathrm{Hom}(M^* \bigoplus k, M \bigoplus k)$, and similarly to (2.1.3, 4) we shall represent them as two-by-two matrices acting from the left. An immediate consequence of the rule [7, 1.4.1] is that

$$\mathfrak{a} = u + y \wedge e = \begin{pmatrix} u & y \\ -y & 0 \end{pmatrix}, \qquad \mathfrak{b} = v + g \wedge e^* = \begin{pmatrix} v & g \\ -g & 0 \end{pmatrix}. \tag{2.8.4}$$

Matrices for $U + x \wedge e$, $V + f \wedge e^*$ are similar, and from [7, 1.4.6] with the rules of composition like $gU = -U(g)$, etc., follow the second equalities in (2.8.2) as a consequence of (2.7.4). Moreover, $1 + \mathfrak{a}\mathfrak{b}$ is at present supposed invertible and describing $\mathfrak{a}^{\mathfrak{b}} = (1 + \mathfrak{a}\mathfrak{b})^{-1}\mathfrak{a}$, $\mathfrak{b}^{\mathfrak{a}} = \mathfrak{b}(1 + \mathfrak{a}\mathfrak{b})^{-1}$ [7, 1.5]; so the first equalities in (2.8.2) amount to

$$(1 + \mathfrak{a}\mathfrak{b})^{-1}\mathfrak{a} = \begin{pmatrix} U & x' \\ -x' & 0 \end{pmatrix}, \qquad \mathfrak{b}(1 + \mathfrak{a}\mathfrak{b})^{-1} = \begin{pmatrix} V & f' \\ -f' & 0 \end{pmatrix}, \tag{2.8.5}$$

where $x' := h^{-1} \cdot y - \langle y, f\rangle x$, $f' := h^{*-1} \cdot g - \langle x, g\rangle f$. In order to resolve $(1 + \mathfrak{a}\mathfrak{b})^{-1}$, we begin by proving (2.8.3). From (2.8.4) follows

$$1 + \mathfrak{a}\mathfrak{b} = \begin{pmatrix} 1 + uv - y \otimes g & u(g) \\ v(y) & 1 - \langle y, g\rangle \end{pmatrix},$$

and its easy modification shows

$$\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} (1 + \mathfrak{a}\mathfrak{b}) \begin{pmatrix} 1 & 0 \\ -g & 1 \end{pmatrix} = \begin{pmatrix} h & u(g) + (1 - \langle y, g\rangle) y \\ v(y) - (1 - \langle y, g\rangle) g & 1 - \langle y, g\rangle \end{pmatrix}.$$

The last form offers a posteriori motivations for the definitions (2.5.1), (2.6.6), and may be rewritten in the form

$$= \begin{pmatrix} h & h \cdot x \\ -h^{*-1} \cdot f & 1 - \langle h \cdot x, f\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -f & 1 \end{pmatrix} \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

Combining these proves (2.8.3), and it becomes now straightforward to invert $1+\mathfrak{a}\mathfrak{b}$. Note, however, that so far as our proof is concerned, (u,v) may be supposed quasi-invertible by density. In that case (2.6.2) gives $1+\langle x,g\rangle = 1+\langle y,f\rangle = (1-\mu)^{-1}$, so we invert the right-hand side of (2.8.3) by grouping the external two products; through calculation with (2.6.4) and $-\langle h^{-1}\cdot y,g\rangle + (1-\mu)^{-2} = (1-\mu)^{-1}$, cf. (2.6.3), the result is

$$(1+\mathfrak{a}\mathfrak{b})^{-1} = \begin{pmatrix} h^{-1} - x\otimes f & -(1-\mu)^{-1}\cdot u(g') \\ -(1-\mu)^{-1}\cdot v(y') & (1-\mu)^{-1} \end{pmatrix}.$$

Now the desired (2.8.5) follows as a consequence of (2.7.3), (2.6.5).

2.9. More calculation. Returning to (2.4), we consider the product $Y_+(u,y)Y_-(v,g)$, supposed in the big cell of $\mathbf{CL}^+(\mathbf{H}(M)\perp\langle 1\rangle)$, and to be decomposed along the cell-decomposition (1.9). Its image in $\mathbf{CL}^+(\mathbf{H}(M\bigoplus k))$ under $\mathrm{C}^+(\iota)$ equals

$$\begin{aligned} &\mathrm{C}^+(\iota)\cdot(Y_+(u,y)Y_-(v,g)) = \\ &= Y_0(1,T_+(y))Y_+(\mathfrak{a})Y_-(\mathfrak{b})Y_0(1,T_-(g)) \qquad \text{(by (2.2.2, 3), (2.8.1))} \\ &= Y_0(1,T_+(y))Y_-(\mathfrak{b}^{\mathfrak{a}})Y_0(t,1+\mathfrak{a}\mathfrak{b})Y_+(\mathfrak{a}^{\mathfrak{b}})Y_0(1,T_-(g)) \qquad \text{(by [7, 2.6.1])} \end{aligned}$$

with t as in (2.5.4), while on account of commutation relations $Y_0(1,T_+(y))Y_-(\mathfrak{b}^{\mathfrak{a}}) = Y_0(1, T_+(y))Y_-(V+f\wedge e^*)$, $Y_+(\mathfrak{a}^{\mathfrak{b}})Y_0(1,T_-(g)) = Y_0(1,T_-(g))Y_+(U+x\wedge e)$, as follows from [7, 3.11] with (2.8.2), and of a consequence $Y_0(1,T_+(y))Y_0(t,1+\mathfrak{a}\mathfrak{b})Y_0(1,T_-(g)) = Y_0(1,T_-(f))Y_0(t, T_0(h))Y_0(1,T_+(x))$ of (2.8.3), we may proceed further to

$$\begin{aligned} &= Y_-(V+f\wedge e^*)Y_0(1,T_-(f))Y_0(t,T_0(h))Y_0(1,T_+(x))Y_+(U+x\wedge e) \\ &= \mathrm{C}^+(\iota)\cdot(Y_-(V,f)Y_0(t,h)Y_+(U,x)). \end{aligned}$$

Since $\mathrm{C}^+(\iota)$ is a monomorphism, this gives the desired decomposition. We close this section by summarizing the results in

2.10. Theorem. *The following conditions on a pair* $(u,y)\in\bigwedge^2(M)\bigoplus M$, $(v,g)\in\bigwedge^2(M^*)\bigoplus M^*$ *are equivalent:*

(i) $Y_+(u,y)Y_-(v,g)$ *lies in the open subscheme* $\Omega\subset\mathbf{CL}^+(\mathbf{H}(M)\perp\langle 1\rangle)$ *(see* (1.7)*);*

(ii) $(u+y\wedge e,\ v+g\wedge e^*)$ *is quasi-invertible in the Jordan pair* $(\bigwedge^2(M\bigoplus k), \bigwedge^2(M^*\bigoplus k))$*;*

(iii) *the endomorphism* $h\in\mathrm{End}(M)$ *(see* (2.5)*) is invertible.*

In fact, the scalars $\chi(Y_+(u,y)Y_-(v,g))$ *and* $\delta(u+y\wedge e, v+g\wedge e^*)$ *are equal, say to* t*, and one has* $\det(h)=t^2$*. Furthermore under these equivalent conditions, one has*

$$Y_+(u,y)Y_-(v,g) = Y_-(V,f)Y_0(t,h)Y_+(U,x) \tag{2.10.1}$$

with $x\in M$, $f\in M^*$, $U\in\bigwedge^2(M)$, $V\in\bigwedge^2(M^*)$ *being defined in* (2.6), (2.7).

3. Root data, reductivity

3.1. A root datum $\mathcal{R}$ and its variants. The property reductive is still waiting for establishment. We shall subsume it to constructing déploiements [4], and turn attention for a moment to root data. Let X denote the same $\mathbf{Z}$-module as in [7, 4.1], which is free of rank $m+1$ with a base:

$$X = \mathbf{Z}\varepsilon_0 \oplus \mathbf{Z}\varepsilon_1 \oplus \mathbf{Z}\varepsilon_2 \oplus \cdots \oplus \mathbf{Z}\varepsilon_m \tag{3.1.1}$$

and dual to $X^\vee = \bigoplus_{i=0}^m \mathbf{Z}\varepsilon_i^\vee$ with the pairing denoted $\langle\ ,\ \rangle : X \times X^\vee \to \mathbf{Z}$, $\langle \varepsilon_i, \varepsilon_j^\vee \rangle = \delta_{ij}$ (Kronecker's delta). Now let $\Phi \subset X$ denote the subset consisting of $2m^2$-elements

$$\begin{aligned} \alpha_{ij} &:= \varepsilon_i - \varepsilon_j, & \alpha_{ji} &:= -\varepsilon_i + \varepsilon_j, \\ \beta_{ij} &:= \varepsilon_i + \varepsilon_j, & \beta_{ji} &:= -\varepsilon_i - \varepsilon_j, \\ \beta_i &:= \varepsilon_i, & \beta_{-i} &:= -\varepsilon_i, \end{aligned} \tag{3.1.2}$$

where $1 \le i < j \le m$, and $?^\vee : \Phi \xrightarrow{\sim} \Phi^\vee \subset X^\vee$ the bijection such that

$$\begin{aligned} \alpha_{ij}^\vee &:= \varepsilon_i^\vee - \varepsilon_j^\vee, & \alpha_{ji}^\vee &:= -\varepsilon_i^\vee + \varepsilon_j^\vee, \\ \beta_{ij}^\vee &:= \varepsilon_0^\vee + \varepsilon_i^\vee + \varepsilon_j^\vee, & \beta_{ji}^\vee &:= -\varepsilon_0^\vee - \varepsilon_i^\vee - \varepsilon_j^\vee, \\ \beta_i^\vee &:= \varepsilon_0^\vee + 2\varepsilon_i^\vee, & \beta_{-i}^\vee &:= -\varepsilon_0^\vee - 2\varepsilon_i^\vee. \end{aligned} \tag{3.1.3}$$

A straightforward verification proves that the so modified quadruple

$$\mathcal{R} := (X,\ \Phi,\ X^\vee,\ \Phi^\vee) \tag{3.1.4}$$

is a reduced root datum, and that the m roots

$$\varepsilon_i - \varepsilon_{i+1}\ (1 \le i \le m-1) \quad \text{and} \quad \varepsilon_m (= \beta_m) \tag{3.1.5}$$

(understood as the singleton ε_m when $m = 1$) form a system of simple roots. In particular, $\mathcal{R}$ is of type B_m ($:= A_1$ when $m = 1$). Analogously to the D_m-case [7, 4.2] we need the variants $\mathrm{ss}(\mathcal{R}) \to \mathcal{R} \to \mathrm{scon}(\mathcal{R})$ induced by $\mathcal{R}$, and their constructions [4, XXI, 6.5, 6.6] soon show that the previous description [7, 4.2] goes without any changes at the level of underlying modules and linear maps. A change has occurred in the definition of (co)roots, but within mere reinterpretations of notations. Among them are the fundamental weights $(\varpi_i)_{1\le i\le m}$, which we now understand relative to the simple roots (3.1.5); then easy verification shows $\varpi_i = \varepsilon_1 + \cdots + \varepsilon_i$ $(1 \le i \le m-1)$ and $\varpi_m = (\varepsilon_1 + \cdots + \varepsilon_m)/2$ (the same formulas as in §4.5 (VI) (p. 203) of [2, Ch. V]), and modifies the description [7, 4.2.3] of the map $f : X \to \tilde{X}$ $(= \bigoplus_{i=1}^m \mathbf{Z}\varpi_i$, the weight lattice) underlying $\mathcal{R} \to \mathrm{scon}(\mathcal{R})$ to

$$f\left(\sum_{i=0}^m \xi_i\varepsilon_i\right) = \sum_{i=1}^{m-1}(\xi_i - \xi_{i-1})\varpi_i + (\xi_0 + 2\xi_m)\varpi_m. \tag{3.1.6}$$

In fact, $\mathrm{ss}(\mathcal{R})$ is also replaceable by $\mathrm{ad}(\mathcal{R})$, the adjoint datum, since the $\mathbf{Z}$-module denoted $X_1 (= \bigoplus_{i=1}^m \mathbf{Z}\varepsilon_i)$ in [7, 4.2.4] coincides with our root lattice Q; recall that $\mathrm{ad}(\mathcal{R}) = (Q, \Phi, P^\vee, \Phi^\vee)$ with $P^\vee$ the *coweight lattice*, which is by definition the $\mathbf{Z}$-submodule dual to Q of the $\mathbf{Q}$-extension $Q_{\mathbf{Q}}^\vee$ of the coroot lattice $Q^\vee \subset X^\vee$. Since m elements $\varepsilon_i^\star := \frac{1}{2}\varepsilon_0^\vee + \varepsilon_i^\vee \in Q_{\mathbf{Q}}^\vee$

form a base of $P^\vee$ and redescribe the coroots (3.1.3) as $\pm\varepsilon_i^\star \pm \varepsilon_j^\star$, $\pm 2\varepsilon_i^\star$, one may confirm now an expected feature of $\mathrm{ad}(\mathcal{R})$ being dual to the simply connected data of type C_m. So we prefer $\mathrm{ad}(\mathcal{R})$ to $\mathrm{ss}(\mathcal{R})$ in the following, but reserve the notation X_1 and write $\mathrm{ad}(\mathcal{R}) = (X_1, \Phi, P^\vee, \Phi^\vee)$ alternatively, in order to remember certain resemblances with the D_m-case [7].

3.2. Déploiements. Let us return to the group side. We suppose the k-module M free with a base

$$e = (e_1,\ e_2,\ \cdots, e_m) \tag{3.2.1}$$

(the previous notation $e := (0,1) \in M \bigoplus k$ will not be used in sequel). We denote by $\mathbf{D}_k(X)$, etc. the diagonalizable k-tori associated to X, etc., and follow [7, 4.3] to construct inclusions

$$\begin{array}{ll} \eta_e : \mathbf{D}_k(X) \longrightarrow \mathbf{G}_{\mathbf{m}k} \times \mathbf{GL}(M) & \eta_e^1 : \mathbf{D}_k(X_1) \longrightarrow \mathbf{GL}(M) \\ \eta_e(s) := (s(\varepsilon_0),\ \sum_{i=1}^m s(\varepsilon_i) e_i \otimes e_i^*), & \eta_e^1(s_1) := \sum_{i=1}^m s_1(\varepsilon_i) e_i \otimes e_i^*. \end{array} \tag{3.2.2}$$

Moreover, η_e composed with $\mathbf{D}_k(f) : \mathbf{D}_k(\tilde{X}) \to \mathbf{D}_k(X)$, cf. (3.1.6), is denoted by $\tilde{\eta}_e$ and soon described in terms of our fundamental weights as

$$\begin{array}{c} \tilde{\eta}_e : \mathbf{D}_k(\tilde{X}) \longrightarrow \mathbf{G}_{\mathbf{m}k} \times \mathbf{GL}(M) \\ \tilde{\eta}_e(\tilde{s}) := (\tilde{s}(\varpi_m),\ h_e(\tilde{s}))\,, \end{array} \tag{3.2.3}$$

where $h_e(\tilde{s}) := \tilde{s}(2\varpi_m) (= \tilde{s}(\varpi_m)^2)$ if $m = 1$ and

$$h_e(\tilde{s}) := \tilde{s}(\varpi_1) e_1 \otimes e_1^* + \sum_{i=2}^{m-1} \tilde{s}(\varpi_i - \varpi_{i-1}) e_i \otimes e_i^* + \tilde{s}(2\varpi_m - \varpi_{m-1}) e_m \otimes e_m^*$$

if $m \geq 2$. Note that h_e, $\tilde{\eta}_e$, as well as η_e, η_e^1, are actually the same maps as in [7, 4.3] with the appearance of h_e changed by the manner of setting (ϖ_i). Composing η_e, η_e^1, $\tilde{\eta}_e$ with Y_0 or X_0 thus yields inclusions

$$\begin{array}{rcll} D_e & := & Y_0 \circ \eta_e : & \mathbf{D}_k(X) \longrightarrow \mathbf{CL}^+(\mathbf{H}(M) \perp \langle 1 \rangle), \\ D_e^1 & := & X_0 \circ \eta_e^1 : & \mathbf{D}_k(X_1) \longrightarrow \mathbf{SO}(\mathbf{H}(M) \perp \langle 1 \rangle), \\ \tilde{D}_e & := & Y_0 \circ \tilde{\eta}_e : & \mathbf{D}_k(\tilde{X}) \longrightarrow \mathbf{Spin}(\mathbf{H}(M) \perp \langle 1 \rangle). \end{array} \tag{3.2.4}$$

After obvious modifications, the commutative diagram (4.3.6) in [7] yields now a similar one for our group schemes. With these setups, we have

3.3. Theorem.

(a) *The k-group schemes $G := \mathbf{CL}^+(\mathbf{H}(M) \perp \langle 1 \rangle)$, $G_1 := \mathbf{SO}(\mathbf{H}(M) \perp \langle 1 \rangle)$, and $\tilde{G} := \mathbf{Spin}(\mathbf{H}(M) \perp \langle 1 \rangle)$ are all reductive.*

(b) *In the case where M is free with a base $e = (e_1, \ldots, e_m)$, the image $T := \mathrm{im}(D_e)$ (resp. $T_1 := \mathrm{im}(D_e^1)$, $\tilde{T} := \mathrm{im}(\tilde{D}_e)$) is a maximal torus of G (resp. G_1, $\tilde{G}$) and the set $\Phi \subset X$ (resp. $\Phi \subset X_1$, $f(\Phi) \subset \tilde{X}$) is the root system of G (resp. G_1, $\tilde{G}$) relative to T (resp. T_1, $\tilde{T}$), in the sense of* [4, XIX, 3.6].

(c) *The datum consisting of the subtorus T (resp. T_1, $\tilde{T}$) equipped with the isomorphism D_e (resp. D^1_e, $\tilde{D}_e$), and the root system Φ (resp. Φ, $f(\Phi)$) above, is a déploiement of G (resp. G_1, $\tilde{G}$) relative to T (resp. T_1, $\tilde{T}$), in the sense of* [4, XXII, 1.13]. *Further the corresponding root datum is equal to $\mathcal{R}$ (resp.* $\mathrm{ad}(\mathcal{R})$, $\mathrm{scon}(\mathcal{R})$*).*

(d) *The homomorphisms $\tilde{G} \overset{\text{incl.}}{\to} G \overset{\pi}{\to} G_1$ are compatible with these déploiements and correspond to the canonical morphisms* $\mathrm{ad}(\mathcal{R}) \to \mathcal{R} \to \mathrm{scon}(\mathcal{R})$ *of root data.*

In order to prove the part (a), we shall verify the criterion (iii) in Proposition 1.12 of [4, XIX]; this gives the maximalities of tori as well, and in fact has been so used in the hyperbolic case [7]. Therefore in the whole proof of our theorem the same format of reasoning as [7] applies. Let us proceed along [7, 4.4–4.6] mutatis mutandis. The first step has no difficulty, where the Lie algebras $\mathfrak{g} := \mathrm{Lie}(G)$, etc. are decomposed under $\mathbf{D}_k(X)$, etc. and yield an expected appearance of roots; all root spaces being isomorphic to k, and the fixed parts $\mathfrak{g}^0$, etc. equaling $\mathfrak{t} := \mathrm{Lie}(T)$, etc. A non-trivial step is involved with root subgroups. In particular, we need a counterpart of [7, 4.5] for the roots $\beta_{\pm i}$, which serves the Weyl elements required in the criterion [4, XIX, Prop. 1.12 (iii)] on the one side, and relates the coroots to our groups on the other side. This will be done as follows: Introducing an index ρ with $2m$ values $\pm 1, \cdots, \pm m$, we define homomorphisms

$$\begin{aligned} q_\rho : \mathbf{G}_{\mathbf{a}k} &\longrightarrow \mathbf{CL}^+(\mathbf{H}(M)\perp\langle 1\rangle)(= G) \\ q_i(\lambda) &:= Y_+(0, \lambda e_i) = 1 + \lambda l(e_i)\mathfrak{e}, \\ q_{-i}(\lambda) &:= Y_-(0, -\lambda e_i^*) = 1 + \lambda d(e_i^*)\mathfrak{e}, \end{aligned} \tag{3.3.1}$$

where the index i takes values $1, \ldots, m$. By construction q_ρ is monomorphic and normalized by $T \cong \mathbf{D}_k(X)$ with multiplier β_ρ; moreover, it factors through $\mathbf{Spin}(\mathbf{H}(M)\perp\langle 1\rangle) = \tilde{G}$ and, together with the composite $q^1_\rho := \pi \circ q_\rho$ with the vector representation, furnishes the wanted root subgroups. Let us consider a morphism

$$\begin{aligned} B_\rho : \mathbf{G}_{\mathbf{a}k} \times \mathbf{D}_k(X) \times \mathbf{G}_{\mathbf{a}k} &\longrightarrow G \\ B_\rho(\lambda, s, \mu) &:= q_{-\rho}(\lambda) D_e(s) q_\rho(\mu), \end{aligned} \tag{3.3.2}$$

which is monomorphic by construction and by (1.9). In addition we consider its obvious modifications $B^1_\rho : \mathbf{G}_{\mathbf{a}k} \times \mathbf{D}_k(X_1) \times \mathbf{G}_{\mathbf{a}k} \to G_1$ (with $q_{\pm\rho}$, D_e replaced by $q^1_{\pm\rho}$, D^1_e), $\tilde{B}_\rho$: $\mathbf{G}_{\mathbf{a}k} \times \mathbf{D}_k(\tilde{X}) \times \mathbf{G}_{\mathbf{a}k} \to \tilde{G}$ (with D_e replaced by $\tilde{D}_e$). The lemma below is then the desired counterpart of [7, 4.5]. Since the concluding arguments in [7, 4.6] are adapted to our case obviously, this yields an actual finish of our proof.

3.4. Lemma.

(a) *For a product $q_\rho(\lambda)q_{-\rho}(\mu)$ to lie in the image of the morphism B_ρ, it is necessary and sufficient that the scalar $1 + \lambda\mu$ is invertible. In that case, one has*

$$q_\rho(\lambda)q_{-\rho}(\mu) = B_\rho\left(\tfrac{\mu}{1+\lambda\mu},\ \beta^\vee_\rho(1+\lambda\mu),\ \tfrac{\lambda}{1+\lambda\mu}\right). \tag{3.4.1}$$

Furthermore similar statements hold for the q^1_ρ, B^1_ρ's and the q_ρ, $\tilde{B}_\rho$'s.

(b) *The k-valued point $q_\rho(1)q_{-\rho}(-1)q_\rho(1)$ of* $\mathbf{CL}^+(\mathbf{H}(M)\perp\langle 1\rangle)$ *normalizes both* $\mathrm{im}(D_e)$ *and* $\mathrm{im}(\tilde{D}_e)$*, and induces on* $\mathbf{D}_k(X)$ *(resp. on* $\mathbf{D}_k(\tilde{X})$*) the automorphism corresponding to the symmetry $x \mapsto x - \langle x, \beta_\rho^\vee\rangle\beta_\rho$ in X (resp. a similar one corresponding to a symmetry in $\tilde{X}$). Similar statements hold for the k-valued point $q_\rho^1(1)q_{-\rho}^1(-1)q_\rho^1(1)$ of* $\mathbf{SO}(\mathbf{H}(M)\perp\langle 1\rangle)$*, which normalizes* $\mathrm{im}(D_e^1)$.

Proof. As had been mentioned in [7, 4.5], the part (b) is a formal consequence of formulas of the type (3.4.1). Moreover, taking inverses with $B_{-i}(\lambda, s, \mu) = B_i(-\mu, s^{-1}, -\lambda)^{-1}$, etc. reduces the part (a) to the case where $\rho = i$ positive, and it is needless to say that the scalar λ, μ may be supposed in k harmlessly. Now let

$$y := \lambda e_i \in M, \qquad g := -\mu e_i^* \in M^*, \tag{3.4.2}$$

so that $q_i(\lambda) = Y_+(0, y)$, $q_{-i}(\mu) = Y_-(0, g)$ by (3.3.1). A special case of (2.10), applied to the pair $((0, y), (0, g))$, yields the following statement:

For $Y_+(0, y)Y_-(0, g)(= q_i(\lambda)q_{-i}(\mu))$ to lie in Ω, it is necessary and sufficient that the scalar $t := 1 - \langle y, g\rangle$, cf. (1.10.3)*, is invertible; in that case one has*

$$Y_+(0, y)Y_-(0, g) = Y_-(0, t^{-1}g)Y_0(t, h)Y_+(0, t^{-1}y), \tag{3.4.3}$$

where $h := 1 + (\langle y, g\rangle - 2)y \otimes g \in \mathrm{End}(M)$, cf. (2.5.1).

Since $\langle y, g\rangle = -\lambda\mu$ by (3.4.2), we have $t = 1 + \lambda\mu$; moreover, it follows that

$$\begin{aligned} h &= 1 + (\lambda\mu + 2)\lambda\mu e_i \otimes e_i^* \\ &= 1 + (t^2 - 1)e_i \otimes e_i^* = \textstyle\sum_{\kappa\neq i} e_\kappa \otimes e_\kappa^* + t^2 \cdot e_i \otimes e_i^*. \end{aligned}$$

This reads $(t, h) = \eta_e(\beta_i^\vee(t))$ by the definitions (3.1.3) and (3.2.2), whence $Y_0(t, h) = D_e(\beta_i^\vee(t))$ by (3.2.4). Therefore (3.4.3) is actually same as (3.4.1), and the Ω above may be shrunk to $\mathrm{im}(B_i)$. So modified statement remains valid clearly when $(Y_\pm, Y_0, q_{\pm i}, \eta_e, D_e, B_i)$ (resp. $(\eta_e$, $D_e, B_i))$ is replaced by $(X_\pm, X_0, q_{\pm i}^1, \eta_e^1, D_e^1, B_i^1)$ (resp. $(\tilde{\eta}_e, \tilde{D}_e, \tilde{B}_i)$). This completes the proof.

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