Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 46 (2005), No. 2, 423-434.

Fourier-Mukai Transforms and Stable Bundles on Elliptic Curves

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Abstract. We prove Atiyah's classification results about indecomposable vector bundles on an elliptic curve by applying the Fourier-Mukai transform. We extend our considerations to semistable bundles and construct the universal stable sheaves.

MSC 2000: 14H60, 14H52
Keywords: Fourier-Mukai transform, semistability, vector bundle, elliptic curve

1. Introduction

After Grothendieck's proof that every vector bundle on $\mathbb{P}^1$ decomposes as a direct sum of line bundles, Atiyah's groundbreaking paper of 1957 provided an answer to the next case: On elliptic curves there are more vector bundles in the sense that nontrivial extensions appear. However, when turning to special bundles, for example stable or indecomposable, it turns out that, in many cases, there is a unique one once rank and determinant are fixed.

In perspective, this means that moduli spaces of stable bundles with prescribed numerical values (including the determinant) are empty or contain a single point. This is in contrast to $\mathbb{P}^1$ where those moduli spaces are empty when bundles of rank 2 or higher are considered.

We will show another way to obtain Atiyah's results. The methods we use are standard by now, namely semistability of sheaves and the Fourier-Mukai transform. However, they allow rather short proofs of many important results. Note that facts about vector bundles on elliptic curves have always been a basic pillar for the study of (moduli spaces of) vector bundles on elliptic fibrations, see for example [5], [13], [6], [3]. There, the Fourier-Mukai transform has been put to good use and in some sense we are reversing the historical development here.

After finishing this paper we learnt that Polishchuk has given a proof of Proposition 4 using the Fourier transform with the Poincaré bundle in Chapter 14 of his book [12].

2. Notation

Let E be an elliptic curve with fixed basepoint $p_0 \in E$ over an algebraically closed field of characteristic 0 (except where noted, the characteristic is actually arbitrary). Then we can identify E and its Jacobian of degree 0 line bundles $\hat{E} := \mathrm{Pic}^0(E)$ via $E \to \hat{E}, x \mapsto \mathcal{O}_E(x-p_0)$. We will write $t_a : E \to E,\ p \mapsto p+a$ for the group law on E in order to distinguish between addition of points and of divisors. The choice of p_0 also allows defining the normalized Poincaré line bundle on $E \times E$ by $\mathcal{P} := \mathcal{O}_{E\times E}(\Delta - E \times p_0 - p_0 \times E)$, i.e. $\mathcal{P}|_{E\times\{x\}} \cong \mathcal{O}_E(x-p_0)$ and $\mathcal{P}|_{\{p_0\}\times E} \cong \mathcal{O}_E$.

Fourier-Mukai transforms

We denote by $D(E)$ the derived category $D^b(\mathrm{Coh} E)$ of complexes of quasi-coherent sheaves on E with bounded coherent cohomology. See [9] or [8] for details. A complex $K^\bullet$ will be enumerated as $\cdots \to K^{-1} \to K^0 \to K^1 \to \cdots$, and, as usual, we denote by $K^\bullet[n]$ the complex $K^\bullet$ shifted n places to the left. $D(E)$ will always be considered as a triangulated category. To avoid confusion with sheaf cohomology, we will denote the n-th homology of $K^\bullet$ by $h^n(K^\bullet)$. If all homology vanishes except $h^n(K^\bullet)$, we will say that $K^\bullet$ is concentrated in degree n.

The Poincaré bundle now defines a functor as follows:

$$\mathrm{FM}_{\mathcal{P}} : D(E) \to D(E), \quad F \mapsto p_{2*}(\mathcal{P} \otimes p_1^* F).$$

Consider this functor as a correspondence on the derived level. (Here, all functors are derived without further notice. However, in the formula above only p_{2*} is a non-exact functor and we will write $R^0 p_{2*}$ for the usual direct image functor.) The facts known from the algebra of correspondences are valid (see Chapter 16 of [7]). As is customary by now, a functor like the above (with an arbitrary object, a so-called kernel, of $D(E \times E)$ instead of $\mathcal{P}$) is called a Fourier-Mukai transform if it gives rise to an equivalence $\mathrm{FM}_{\mathcal{P}}$ of triangulated categories. Mukai showed in [11] that Poincaré bundles actually give equivalences on all Abelian varieties. He also proved an involution property valid for principal polarized Abelian varieties which in our case reads as

$$\mathrm{FM}_{\mathcal{P}} \circ \mathrm{FM}_{\mathcal{P}} = (-1)^*[-1].$$

All results concerning $\mathrm{FM}_{\mathcal{P}}$ that we use in this article are rather easy to obtain because of the simple form of $\mathcal{P}$. For example, the involution property can be shown like this: the composition $\mathrm{FM}_{\mathcal{P}} \circ \mathrm{FM}_{\mathcal{P}}$ has as kernel $p_{13*}(p_{12}^*\mathcal{P} \otimes p_{23}^*\mathcal{P}) =: p_{13*}K$. Using cohomology and base change together with the definition of $\mathcal{P}$, we see $R^1 p_{13*}K \otimes k(a,b) = H^1(E, \mathcal{O}_E(a+b-2p_0))$. This already shows that $R^1 p_{13*}K$ is a line bundle supported on $\Delta' := \{(x,-x) : x \in E\}$ because $H^1(\mathcal{O}_E(a+b-2p_0) \neq 0 \iff \mathcal{O}_E(a+b-2p_0)$ is trivial $\iff a = -b$ in the group law of E. Furthermore, $R^0 p_{13*}K = 0$ (and hence $p_{13*}K = R^1 p_{13*}K[-1]$ is concentrated in degree 1) which follows, for instance, from computing $\mathrm{ch}(p_{13*}K)$ using Grothendieck-Riemann-Roch. Finally, we note $\mathrm{FM}_{\mathcal{P}}(\mathcal{O}_E) = k(p_0)[-1]$ and $\mathrm{FM}_{\mathcal{P}}(k(p_0)) = \mathcal{O}_E$ and so the line bundle on Δ'

mentioned above is trivial. Altogether we obtain $\mathrm{FM}_{\mathcal{P}}^2 = \mathrm{FM}_{\mathcal{O}_{\Delta'}[-1]} = (-1)^*[-1]$. This also proves that $\mathrm{FM}_{\mathcal{P}}$ is an equivalence.

The classical version of the transform defined above is the ring endomorphism of the even cohomology ring

$$\mathrm{FM}_{\mathrm{ch}(\mathcal{P})} : H^0(E) \oplus H^2(E) \to H^0(E) \oplus H^2(E), \quad \alpha \mapsto p_{2*}(\mathrm{ch}(\mathcal{P}).p_1^*(\alpha)),$$

which is usually called a correspondence on E.

Any choice of kernel in $D(E \times E)$ which we continue to call $\mathcal{P}$ then gives a commutative diagram:

$$\begin{tikzcd} D(E) \arrow[r] \arrow[d, "\mathrm{FM}_{\mathcal{P}}"] & H^{2*}(E) \arrow[d, "\mathrm{FM}_{\mathrm{ch}(\mathcal{P})}"] \\ D(E) \arrow[r] & H^{2*}(E) \end{tikzcd}$$

Here[1] the map $D(E) \to H^{2*}(E)$ sends a complex $F^\bullet$ to $\sum_i (-1)^i \mathrm{ch}(F^i)$. Similar and compatible transforms exist on the K-group $K(E)$ and on the Chow ring $CH(E)$. In the sequel will denote the fundamental classes of the curve and a point by $[E]$ and by $[pt]$, respectively. All calculations could just as well take place in the Chow ring.

The Chern character of the Poincaré bundle in $H^*(E \times E)$ is readily read off from the definition as

$$\mathrm{ch}(\mathcal{P}) = 1 + [\Delta] - [E \times pt] - [pt \times E] - [pt \times pt]$$

(using $N_{\Delta/E\times E} = \mathcal{O}_E$ for $[\Delta]^2 = \deg(c_1(N_{\Delta/E\times E})) = 0$) and hence $\mathrm{FM}_{\mathrm{ch}(\mathcal{P})}(r[E] + d[pt]) = p_{2*}(r[E \times E] + r[\Delta] - r[E \times p_0] - r[p_0 \times E] - [p_0 \times p_0] + d[pt \times E]) = d[E] - r[pt]$.

We reiterate that $\mathrm{FM}_{\mathrm{ch}(\mathcal{P})}$ is the automorphism

$$\mathrm{FM}_{\mathrm{ch}(\mathcal{P})} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} : H^{2*}(E) \to H^{2*}(E), \quad r[E] + d[pt] \mapsto d[E] - r[pt].$$

Semistable sheaves

The facts we need concerning semistable sheaves are the following. See e.g. [14] or [10] for details. Note that semistable sheaves are automatically torsion free, hence vector bundles in our setting.

- The slope of a coherent sheaf F is $\mu(F) := \deg(F)/\mathrm{rk}(F)$. The sheaf F is called semistable if no subsheaf has a slope greater than $\mu(F)$. Equivalently, F is semistable if there is no quotient of F whose slope is smaller than $\mu(F)$. F is called stable if there is no proper subsheaf whose slope is greater or equal than $\mu(F)$.
- A sheaf F, which is not semistable, contains a unique semistable sheaf F' of maximal slope, the so-called maximal destabilizing subsheaf. It is determined by $\mu(U) \le \mu(F')$ for all $U \subseteq F$ and $\mu(U) = \mu(F') \implies U \subseteq F'$.

[1] However, note that for varieties X with nontrivial tangent bundle the correct definition is $\sum_i (-1)^i \mathrm{ch}(F^i)\sqrt{\mathrm{td}_X}$.

- There are no nontrivial morphisms $F \to G$ if F and G are semistable with $\mu(F) > \mu(G)$. Similarly, any nonzero morphism $F \to G$ between stable sheaves with $\mu(F) = \mu(G)$ is an isomorphism.
- If, in a short exact sequence of coherent sheaves, two sheaves are semistable of the same slope μ, then the third is also semistable with slope μ. This means that the category of semistable sheaves with fixed slope is closed under kernels, cokernels and extensions. In particular, it is Abelian.

3. The stable case: rank and degree coprime

Lemma 1. *Let F be a locally free sheaf of rank r and degree d. Then we have the implications* (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) *with*

(i) *F is stable,*
(ii) *F is simple,*
(iii) *F is indecomposable,*
(iv) *F is semistable.*

If moreover r and d are coprime, then we also have (iv) $\Longrightarrow$ (i)*, so that all four properties are equivalent.*

Proof. The implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) are valid for arbitrary varieties and sheaves, as is (iv) $\Longrightarrow$ (i) if $(r,d) = 1$. So assume now that F is indecomposable. Take the maximal destabilizing subsheaf $F' \subset F$. This would lead to an exact sequence $0 \to F' \to F \to F'' \to 0$ with $\mathrm{Hom}(F', F'') = 0$ because the quotient F'' can be filtered by semistable bundles, all of which have slope smaller than $\mu(F')$ (by the uniqueness of F'). But from Serre duality we infer $\mathrm{Ext}^1(F'', F') = \mathrm{Hom}(F', F'')^\vee = 0$. Since F is indecomposable we finally have $F'' = 0$ and $F = F'$ is indeed semistable. □

Lemma 2. *Let $r > 0$ and d be integers and L a line bundle of degree d.*

a) *A stable vector bundle on E with rank r and degree d exists $\Longleftrightarrow$ $(r,d) = 1$.*
b) *If $(r,d) = 1$, there is a unique stable bundle of rank r and determinant L.*

Proof. For a) fix integers r and d with $(r,d) = 1$. Remember that we have chosen an origin p_0 on E. There is another elliptic curve $\tilde{E}$ together with a morphism $\pi_r : \tilde{E} \to E$ such that $E = \tilde{E}/G$ is a finite quotient of order r, and $G \cong \mathbb{Z}/(r)$ acts without fixed points on $\tilde{E}$. (Either take a line bundle M on E of order r and set $\tilde{E} := \mathbf{Spec}(\mathcal{O}_E \oplus M \oplus \cdots \oplus M^{r-1})$, or else use the unramified covering of E given by a subgroup of $\pi_1(E) = \mathbb{Z}^2$ of index r.) The fiber $\pi_r^{-1}(p_0)$ consists of r points, among which we chose a base point $\tilde{p}_0$ for $\tilde{E}$. After that, we can also chose a generator $\tilde{g}$ of $\pi_r^{-1}(p_0)$ (considered as a subgroup of $\tilde{E}$).

Now take a line bundle $\tilde{L}$ on $\tilde{E}$ of degree d, e.g. $\mathcal{O}_{\tilde{E}}(d\tilde{p}_0)$. The projection $\pi_r : \tilde{E} \to E$ is a finite, unramified morphism, and thus $V := \pi_{r*}\tilde{L}$ is a sheaf concentrated in degree 0, locally free of rank r and degree d. It is simple because

$$\begin{aligned}\mathrm{Hom}_E(V,V) &= \mathrm{Hom}_E(\pi_{r*}\tilde{L}, \pi_{r*}\tilde{L}) = \mathrm{Hom}_{\tilde{E}}(\pi_r^*\pi_{r*}\tilde{L}, \tilde{L}) \\ &= \mathrm{Hom}_{\tilde{E}}(\bigoplus_{g\in G} g^*\tilde{L}, \tilde{L}) = \bigoplus_{g\in G} H^0(\tilde{L} \otimes g^*\tilde{L}^\vee) = k\end{aligned}$$

using that only $\mathcal{O}_{\tilde{E}}$ has nontrivial sections among line bundles of degree 0. By the lemma, V is also stable. The other direction of a) will be a consequence of Proposition 4.

For b) we note that by Grothendieck-Riemann-Roch $\mathrm{ch}(V) = \mathrm{ch}(\pi_{r*}\tilde{L}) = \pi_{r*}\mathrm{ch}(\tilde{L}) = \pi_{r*}(1_{\tilde{E}} + \tilde{D}) = r \cdot 1_E + \pi_r(\tilde{D})$. Thus, $\det(V) = \mathcal{O}_E(\pi_r(c_1(\tilde{L}))$. To get a stable bundle with prescribed determinant $L \in \mathrm{Pic}^d(E)$, we simply take $\tilde{L}$ to be an r-th root of $\pi_r^* L$.

Now if V_1 and V_2 are two stable bundles of same rank r and determinant, then the homomorphism bundle $F := V_1 \otimes V_2^\vee$ has rank r^2 and trivial determinant. By stability, we have either $H^0(F) = H^1(F) = k$ or $H^0(F) = H^1(F) = 0$, depending on whether $V_1 \cong V_2$ or not. The claim follows from $\mathrm{FM}_{\mathcal{P}}(F) = T[-1]$ where T is a torsion sheaf containing the origin p_0 because then $H^1(F) = k$. The homological consideration yields $\mathrm{ch}(\mathrm{FM}_{\mathcal{P}}(F)) = -r^2[pt]$. From this and cohomology and basechange, we see that $h^1(\mathrm{FM}_{\mathcal{P}}(F))$ is nonzero torsion. Hence, there exists an $L_1 \in \mathrm{Pic}^0$ such that $V_1 \otimes L_1 \cong V_2$. On the other hand, $h^0(\mathrm{FM}_{\mathcal{P}}(F))$ is the usual push-forward of a bundle, hence torsion free and thus zero. This shows that $\mathrm{FM}_{\mathcal{P}}(F) = T[-1]$ is torsion of length r^2 sitting in degree 1. A local computation, given below, will show that T is actually reduced so that T consists of all r^2 torsion points of order r. Then we have in particular $p_0 \in \mathrm{supp}(T)$ and thus $V_1 \cong V_2$.

Let $[L] \in \mathrm{Pic}^0(E)$ be a point in the support of the torsion sheaf T and choose a parameter t in $[L]$. We want to show that T is annihilated by t. Let $D = k[\varepsilon]/\varepsilon^2$ be the ring of dual numbers over k and $\mathrm{Spec}(D) \to \mathrm{Pic}^0(E)$ be the map corresponding to the ring morphism which sends t to ε. We consider the restriction $\tilde{L}$ of the Poincare sheaf $\mathcal{P}$ to $E \times \mathrm{Spec}(D)$. Then there is a nonsplitting short exact sequence $0 \to L \to \tilde{L} \to L \to 0$. If V_1 is stable, then the short exact sequence $0 \to L \otimes V_1 \to \tilde{L} \otimes V_1 \to L \otimes V_1 \to 0$ does not split either. To see this, we consider the exact sequence $0 \to \mathcal{O}_E \to \mathcal{E}nd(V_1 \otimes L) \to \mathcal{E}nd_0(V_1 \otimes L) \to 0$ (this works in characteristic 0 or if $\mathrm{char}(k)$ does not divide r). Since $V_1 \otimes L$ is stable, we conclude that $H^0(\mathcal{E}nd_0(V_1 \otimes L)) = 0$, and eventually that the map $H^1(\mathcal{O}_E) \to H^1(\mathcal{E}nd(V_1 \otimes L))$ is injective. Thus, in other words, $\mathrm{Ext}^1(L, L) \to \mathrm{Ext}^1(L \otimes V_1, L \otimes V_1)$ is injective. Suppose now that T is not annhilated by t. Then the map $\mathrm{Hom}(V_2, \tilde{L} \otimes V_1) \to \mathrm{Hom}(V_2, L \otimes V_1)$ is surjective. Let $\psi : V_2 \xrightarrow{\sim} L \otimes V_1$ be an isomorphism and $\tilde{\psi} : V_2 \to \tilde{L} \otimes V_1$ be its lift. However, then the image of $\tilde{\psi}$ splits the short exact sequence $0 \to L \otimes V_1 \to \tilde{L} \otimes V_1 \to L \otimes V_1 \to 0$ which is a contradiction. $\square$

Remark. The assertions of the lemma can be rephrased using the moduli space $\mathcal{M}(r, d)$ of stable vector bundles of rank r and degree d:

a) $\mathcal{M}(r, d) \neq \emptyset \iff (r, d) = 1$,

b) $\det : \mathcal{M}(r, d) \xrightarrow{\sim} \mathrm{Pic}^d(E)$ is an isomorphism if $(r, d) = 1$.

Universal bundles

Proposition 3. *Given coprime r and d, there is a universal bundle $\mathcal{G}$ on $E \times E$ parametrizing stable bundles of rank r and degree d, i.e. $\mathrm{FM}_{\mathcal{G}} : D(E) \to D(E)$ is an equivalence such that all $\mathrm{FM}_{\mathcal{G}}(k(p))$ are stable of rank r and degree d.*

Proof. The above construction of stable bundles can also be described in terms of Fourier-Mukai transforms. Consider the graph $\Gamma \subset \tilde{E} \times E$ of π_r and its structure sheaf $\mathcal{O}_\Gamma \in D(\tilde{E} \times E)$

as a kernel. Then we have $\pi_{r*} = \mathrm{FM}_{\mathcal{O}_\Gamma}$. Furthermore, consider next the Poincaré bundle $\tilde{\mathcal{P}}^d$ of degree d line bundles on $\tilde{E}$. We will assume that $\tilde{\mathcal{P}}^d$ is normalized by requiring it to be symmetric. Then the composition $\mathrm{FM}_{\mathcal{O}_\Gamma} \circ \mathrm{FM}_{\tilde{\mathcal{P}}^d} : D(\tilde{E}) \to D(E)$ takes points (i.e. skyscraper sheaves $k(\tilde{x})$) to stable bundles on E with correct rank and degree. However, this map is overparametrized (and hence the composite kernel is not a universal bundle): two points $\tilde{x}$ and $\tilde{y}$ lead to the same bundle if they are in the same π_r-fiber. (Equivalently, two divisors D and $D' := t_{\tilde{g}}^* D$ of degree d give isomorphic bundles $\pi_{r*}\mathcal{O}_{\tilde{E}}(D) \cong \pi_{r*}\mathcal{O}_{\tilde{E}}(D')$.) Thus, it is necessary to divide out the G-action. This is possible if and only if the composite kernel $\mathcal{K} \in D(\tilde{E} \times E)$ (explicitly, $\mathcal{K} := p_{13*}(p_{12}^* \tilde{\mathcal{P}}^d \otimes p_{23}^* \mathcal{O}_\Gamma)$) descends. This in turn means that there is a $\mathcal{G} \in D(E \times E)$ such that $\mathcal{K} = (\pi_r \times \mathrm{id}_E)^* \mathcal{G}$. A necessary and sufficient condition for this is the existence of a G-linearization on $\mathcal{K}$.

Note that (the generator $\tilde{g}$ of) $G \cong \mathbb{Z}/(r)$ acts on $\tilde{E} \times E$ by translation with $\tilde{g}$ on the first factor and trivially on the second. We write $t := t_{(\tilde{g},p_0)}$ for this translation. A G-linearization is a set of isomorphisms $\lambda_g : g^*\mathcal{K} \xrightarrow{\sim} \mathcal{K}$ satisfying the obvious compatibility. Because G is cyclic, it is sufficient and convenient to consider only for the generator. Now

$$\begin{aligned} & t^*\mathcal{K} = \mathcal{K} \\ \iff\ & \mathrm{FM}_{t^*\mathcal{K}} = \mathrm{FM}_{\mathcal{K}} \\ \iff\ & \pi_{r*} \circ \mathrm{FM}_{t^*\tilde{\mathcal{P}}^d} = \pi_{r*} \circ \mathrm{FM}_{\tilde{\mathcal{P}}^d} \\ \iff\ & \pi_{r*} \circ t_{\tilde{g}}^* \circ \mathrm{FM}_{\mathcal{P}^d} \circ \mathrm{FM}_{\mathcal{P}^d}^{-1} = \pi_{r*} \\ \iff\ & \pi_{r*} \circ (t_{\tilde{g}}^{-1})_* = \pi_{r*} \\ \iff\ & \pi_r \circ t_{\tilde{g}}^{-1} = \pi_r \end{aligned}$$

and thus $\mathcal{K}$ is G-linearizable if and only if $\pi_r(\tilde{g}) = \pi_r(\tilde{p}_0)$ – which is the case by definition.

So, we see that $\mathcal{K} = (\pi_r \times \mathrm{id}_E)^*\mathcal{G}$ descends and it remains to show that $\mathcal{G}$ is a universal bundle. This follows at once from

$$\begin{aligned} \mathrm{FM}_{\mathcal{K}}(k(\tilde{x})) &= p_{2*}((\pi_r \times \mathrm{id}_E)^*\mathcal{G} \otimes p_1^*(k(\tilde{x}))) \\ &= p_{2*}(\iota_{\tilde{x}}^*(\pi_r \times \mathrm{id}_E)^*\mathcal{G}) \\ &= p_{2*}(\mathcal{G}|_{\{\pi_r(\tilde{x})\} \times E}) \\ &= \mathrm{FM}_{\mathcal{G}}(k(\pi_r(\tilde{x}))) \end{aligned}$$

with $\iota_{\tilde{x}} : \{\tilde{x}\} \times E \hookrightarrow \tilde{E} \times E$.

Thus $\mathrm{FM}_{\mathcal{G}}$ parametrizes all stable bundles of rank r and degree d like $\mathrm{FM}_{\mathcal{K}}$, too. The difference is that $\mathrm{FM}_{\mathcal{G}}$ is a universal bundle (that it is a locally free sheaf is clear from the construction) because $\mathrm{FM}_{\mathcal{G}}(k(x))$ and $\mathrm{FM}_{\mathcal{G}}(k(y))$ are stable with the same slope but different determinants $\mathcal{P}_x^d$ and $\mathcal{P}_y^d$. A criterion[2] of Bridgeland (see [4]) now states that $\mathrm{FM}_{\mathcal{G}} : D(E) \to D(E)$ is actually an equivalence. □

Remark. The above construction has a connection with the derived McKay correspondence (see [2] for details). The statement is that for the variety $\tilde{E}$ with its G-action, there is an

[2]The criterion states that a functor $F : D(E) \to D(E)$ is an equivalence if and only if it is fully faithful on points, i.e. $\mathrm{Ext}_E^i(k(x), k(y)) = \mathrm{Hom}_{D(E)}^i(F(k(x)), F(k(y)))$ for all $x, y \in E$. This also holds for general varieties if the canonical sheaf is trivial.

equivalence $D^G(\tilde{E}) = D(E)$ (where $D^G(\tilde{E})$ is the derived category of the Abelian category of G-linearized sheaves on $\tilde{E}$). The construction of $\mathcal{G}$ implies that $\mathrm{FM}_{\mathcal{K}} : D^G(\tilde{E}) \xrightarrow{\sim} D(E)$ establishes such an equivalence.

4. The general case: arbitrary rank and degree

Here we consider vector bundles of arbitrary rank r and degree d. We denote $\tilde{r} := r/(r,d)$ and $\tilde{d} := d/(r,d)$. From the results of the previous section we dispose of a universal bundle $\mathcal{G}$ for stable bundles of rank $\tilde{r}$ and degree $\tilde{d}$. Our aim is the following description of semistable sheaves on E.

Proposition 4. *Let $\mathcal{S}(r,d)$ be the set of all isomorphism classes of semistable bundles of rank r and degree d. There is an isomorphism between $\mathcal{S}(r,d)$ and the set* $\mathrm{Torsion}_{\mathrm{length}=(r,d)}$ *of torsion sheaves of length (r,d)*

$$\mathrm{FM}_{\mathcal{G}} : \mathrm{Torsion}_{\mathrm{length}=(r,d)} \xrightarrow{\sim} \mathcal{S}(r,d).$$

Proof. Remember that $\mathcal{G}$ was the universal bundle on $E \times E$ constructed in Proposition 3. The Fourier-Mukai transform $\mathrm{FM}_{\mathcal{G}}$ here is meant in the same direction as there, i.e. taking points to stable bundles.

First, take an arbitrary torsion sheaf T on E of length (r,d). Then, it is obvious that $\mathrm{FM}_{\mathcal{G}}(T)$ is a locally free sheaf of rank $\tilde{r}(r,d) = r$ concentrated in $[0]$. It has degree d because of $\mathrm{ch}(\mathrm{FM}_{\mathcal{G}}(T)) = \mathrm{FM}_{\mathrm{ch}(\mathcal{G})}(\mathrm{ch}(T)) = (r,d)(\tilde{r}[E] + \tilde{d}[pt])$. Finally, it is semistable because all T can be filtered in a composition series, and hence F is a successive extension of stable bundles of rank $\tilde{r}$ and degree $\tilde{d}$.

On the other hand, let F be semistable with rank r and degree d. We are looking for a T with $\mathrm{FM}_{\mathcal{G}}(T) = F$. In order to do this, we will utilize the transform $\mathrm{FM}_{\mathcal{G}^\vee}$ with the dual of the universal bundle as kernel. We need two facts about this: first, $\mathrm{FM}_{\mathcal{G}^\vee} = \mathrm{FM}_{\mathcal{G}}^{-1}[-1]$ (see the original paper of Mukai, [11] for this) and second, that $\mathcal{G}^\vee$ is the universal bundle parametrizing stable bundles on E with rank $\tilde{r}$ and degree $-\tilde{d}$. As a last preliminary, we need some homological information concerning $\mathrm{FM}_{\mathcal{G}^\vee}$. The relations $\mathrm{FM}_{\mathrm{ch}(\mathcal{G}^\vee)}(\tilde{r}[E]+\tilde{d}[pt]) = -[pt]$ and $\mathrm{FM}_{\mathrm{ch}(\mathcal{G}^\vee)}([pt]) = \tilde{r}[E] + e[pt]$ follow from $\mathcal{G}$ being universal and locally free of rank $\tilde{r}$. This allows us to write $\mathrm{FM}_{\mathrm{ch}(\mathcal{G}^\vee)}$ as a matrix (i.e. an automorphism of $H^0(E,\mathbb{Z}) \oplus H^2(E,\mathbb{Z})$), and analogously for $\mathrm{FM}_{\mathrm{ch}(\mathcal{G})}$

$$\mathrm{FM}_{\mathrm{ch}(\mathcal{G}^\vee)} = \begin{pmatrix} -\tilde{d} & \tilde{r} \\ -\frac{1+\tilde{d}e}{\tilde{r}} & e \end{pmatrix}, \quad \mathrm{FM}_{\mathrm{ch}(\mathcal{G})} = -\mathrm{FM}^{-1}_{\mathrm{ch}(\mathcal{G}^\vee)} = \begin{pmatrix} -e & \tilde{r} \\ \frac{1+\tilde{d}e}{-\tilde{r}} & \tilde{d} \end{pmatrix}.$$

Now F has a two-term resolution $0 \to A^{-1} \to A^0 \to F \to 0$ such that $R^0 p_{2*}(\mathcal{G}^\vee \otimes p_1^* A^i) = 0$ for both $i = 0, -1$ using sufficiently antiample twists. Applying $\mathrm{FM}_{\mathcal{G}^\vee}$ to this yields a triangle which can also be written as $\mathrm{FM}_{\mathcal{G}^\vee}(F) = [B^{-1} \xrightarrow{\beta} B^0]$ with $B^i = \mathrm{FM}_{\mathcal{G}^\vee}(A^i)$ or as an exact sequence $0 \to h^0\mathrm{FM}_{\mathcal{G}^\vee}(F) \to B^{-1} \to B^0 \to h^1\mathrm{FM}_{\mathcal{G}^\vee}(F) \to 0$. Note that B^{-1} and B^0 are locally free sheaves concentrated in degrees 0 and 1, respectively. We prove first that β is injective: Assuming the opposite, there is an injection $\mathcal{O}_E(-M) \hookrightarrow \ker(\beta)$ for some $M \gg 0$.

$$\begin{tikzcd} \mathcal{O}_E(-M) \arrow[r] \arrow[d, hook] & 0 \arrow[d] \\ B^{-1} \arrow[r, "\alpha"] & B^0 \end{tikzcd}$$

Application of $\mathrm{FM}_\mathcal{G}$ to the complex morphism $\mathcal{O}_E(-M) \to B^\bullet$ yields a map $\gamma : \mathrm{FM}_\mathcal{G}(\mathcal{O}_E(-M)) \to \mathrm{FM}_\mathcal{G}(\mathrm{FM}_{\mathcal{G}^\vee}(F)) = F[-1]$.

By increasing M some more, if necessary, we can assume that $\mathrm{FM}_\mathcal{G}(\mathcal{O}_E(-M))$ is concentrated in degree 1, i.e. that $R^0p_{2*}(\mathcal{G} \otimes p_1^*\mathcal{O}_E(-M)) = 0$. Then, γ is a morphism between bundles sitting in degree 1 and the homological consideration above shows

$$\mathrm{FM}_{\mathrm{ch}(\mathcal{G})}(\mathrm{ch}(\mathcal{O}_E(-M))) = \begin{pmatrix} -e & \tilde{r} \\ \frac{1+\tilde{d}e}{-\tilde{r}} & d \end{pmatrix} \begin{pmatrix} 1 \\ -M \end{pmatrix} = \begin{pmatrix} -e - \tilde{r}M \\ \frac{1+\tilde{d}e}{-\tilde{r}} - \tilde{d}M \end{pmatrix}.$$

Note that $\mathrm{FM}_\mathcal{G}(\mathcal{O}_E(-M))$ is simple and hence stable by Lemma 1 because $\mathcal{O}_E(-M)$ is simple. Thus, the morphism γ is one between stable bundles with slopes

$$\mu(\mathrm{FM}_\mathcal{G}(\mathcal{O}_E(-M))) = \frac{(1+\tilde{d}e)/\tilde{r} + \tilde{d}M}{e + \tilde{r}M} = \frac{1}{\tilde{r}(e+\tilde{r}M)} + \frac{\tilde{d}}{\tilde{r}} > \frac{\tilde{d}}{\tilde{r}} = \frac{d}{r} = \mu(F)$$

which is a contradiction for $M \gg 0$.

By now we know that β is injective, or, rephrasing the same fact, $\mathrm{coker}(\beta)[-1] = \mathrm{FM}_{\mathcal{G}^\vee}(F)$ is concentrated in degree 1. The numerical invariants of the cokernel are

$$\mathrm{ch}(\mathrm{FM}_{\mathcal{G}^\vee}(F)) = \mathrm{FM}_{\mathrm{ch}(\mathcal{G}^\vee)}(\mathrm{ch}(F)) = \begin{pmatrix} -\tilde{d} & \tilde{r} \\ -\frac{1+\tilde{d}e}{\tilde{r}} & e \end{pmatrix} \begin{pmatrix} r \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ -(r,d) \end{pmatrix}$$

and this proves that $T := \mathrm{coker}(\beta)$ is a torsion sheaf of length (r,d) with $\mathrm{FM}_\mathcal{G}(T) = F$, as claimed. $\square$

Remark. The bijection between torsion sheaves and semistable bundles given by the proposition also allows the identification of indecomposable objects on both sides. Explicitly, torsion sheaves of the form $k[\varepsilon]/\varepsilon^l$ give rise to indecomposable bundles and vice versa. Especially, we obtain an equivalence

$$\mathrm{FM}_\mathcal{G} : E \xrightarrow{\sim} \{F \in \mathcal{S}(r,d) \text{ indecomposable}\}, \quad p \mapsto \mathrm{FM}_\mathcal{G}(k(p)[\varepsilon]/\varepsilon^{(r,d)}).$$

In this way, we have reproven Atiyah's main theorem ([1], II.7).

Note that the equivalence also allows us to describe the endomorphism groups of semistable bundles. Thus, if the torsion sheaf T corresponding to $F \in \mathcal{S}(r,d)$ has indecomposable summands $T = T_1 \oplus \cdots \oplus T_s$, then we have

$$\mathrm{End}_E(F) = \mathrm{End}_E(T) = \bigoplus_{i=1}^{s} \mathrm{End}_E(T_i) = \bigoplus_{i=1}^{s} T_i = T.$$

Remark. There is a natural equivalence relation on the set $\mathcal{S}(r,d)$, the so-called S-equivalence. Two semistable bundles V_1 and V_2 are S-equivalent if the graded objects of their Jordan-Hölder filtrations are isomorphic: $\mathrm{gr}_{\mathrm{JH}}(V_1) \cong \mathrm{gr}_{\mathrm{JH}}(V_2)$.

Stable bundles form one-point equivalence classes. But the presence of properly semistable bundles (which in our setting is equivalent with $(r,d) \neq 1$) implies that $\mathcal{S}(r,d)$ is then neither reduced nor separated.

The quotient $\mathcal{M}(r,d) := \mathcal{S}(r,d)/S-$equivalence is the moduli space of (semistable) bundles of rank r and degree d. Using our description of $\mathcal{S}(r,d)$, we can include the moduli space in our picture:

$$\begin{array}{ccc} \mathrm{Torsion}_{\mathrm{length}=\tilde{r}} & \xrightarrow{\mathrm{FM}_{\mathcal{G}}} & \mathcal{S}(r,d) \\ \big\downarrow{\scriptstyle S} & & \big\downarrow{\scriptstyle S} \\ \mathrm{Div}^r_{\mathrm{eff}} & \xrightarrow{\mathrm{FM}_{\mathcal{G}}} & \mathcal{M}(r,d)^{ss} \end{array}$$

So we see that $\mathcal{M}(r,d)^{ss}$ has the structure of a $\mathbb{P}^{r-1}$-bundle over E in view of the map $\mathrm{Div}^r_{\mathrm{eff}} = \mathrm{Sym}^r(E) \to \mathrm{Pic}^r(E)$ whose fibers are complete linear systems. Especially, it is reduced and separated.

Remark. A particular instance of a universal bundle is the Poincaré bundle $\mathcal{P}$ itself. It corresponds to $r=1$, $d=0$. We get an equivalence between torsion sheaves of length r and locally free semistable sheaves of rank r and degree 0:

$$\mathrm{FM}_{\mathcal{P}} : \mathrm{Torsion}_{\mathrm{length}=r} \xrightarrow{\sim} \mathcal{M}(r,0).$$

Remark. Another description for stable bundles of degree 1 is the bijection

$$\mathrm{FM}_{\mathcal{P}} : \mathrm{Pic}^{-r}(E) \xrightarrow{\sim} \mathcal{M}(r,1).$$

Taking a line bundle L of degree $-r$, we see that $\mathrm{FM}_{\mathcal{P}}(L)$ is concentrated in [1] (there is no R^0 because of the negative degree) and locally free of rank $\dim H^1(L) = r$. Furthermore, writing $\mathrm{FM}_{\mathcal{P}}(L) = F[-1]$ we see that F is a simple sheaf because $\mathrm{FM}_{\mathcal{P}}$ is fully faithful as an equivalence. By Lemma 1 it is also stable.

For the other direction, take an $F \in \mathcal{M}(r,1)$. To see that $\mathrm{FM}_{\mathcal{P}}(F)$ is a line bundle of degree r concentrated in [0], note that with F also $F^\vee$ and $F \otimes M$ are stable, for all $M \in \mathrm{Pic}^0(E)$. Now by cohomology and base change it is enough to show $H^1(F) = 0$ as this implies $h^1\mathrm{FM}_{\mathcal{P}}(F) = 0$ and then we get rank and degree of $\mathrm{FM}_{\mathcal{P}}(F)$ by the homological computation. But if we had $H^1(F) \neq 0$, then by Serre duality there is a nontrivial morphism $\mathcal{O}_E \to F^\vee$ between stable sheaves of slopes 0 and $-1/r$ which is impossible.

Finally, we mention the following characterizations of semistable bundles on elliptic curves.

Lemma 5. *Let F be a vector bundle of rank r and degree d on E. Further, let V be a fixed semistable bundle of rank r^2+r and degree $rd+d+1$. Then the following conditions are equivalent:*

(i) *F is semistable.*

(ii) *There exists a nontrivial sheaf* G *such that* $H^*(F \otimes G) = 0$.
(iii) *The sheaf* G *in* (ii) *can be chosen of rank* $r/(r,d)$.
(iv) $\mathrm{Hom}_E(V, F) = 0$.

Proof. Trivial are iii) $\Longrightarrow$ ii) and i) $\Longrightarrow$ iv) because $\mu(F) < \mu(V)$.

For ii) $\Longrightarrow$ i), assume that F is not semistable and take the maximal destabilizing subsheaf $F' \subset F$. Then F/F' is torsion free, hence locally free, and thus $F' \otimes G \subset F \otimes G$. On the other hand, we have $\chi(F \otimes G) = 0$ by assumption and $\chi(F' \otimes G) > 0$ – which constitutes a contradiction because $\chi = \deg$ on elliptic curves.[3]

i) $\Longrightarrow$ iii): Take G' to be a stable bundle with $\mu(G') = \mu(F)$. Then, $F^\vee \otimes G'$ is semistable of degree 0, and we get $\mathrm{FM}_{\mathcal{P}}(F^\vee \otimes G') = T[-1]$ with a torsion sheaf T of length $r^2/(r,d)$. Now, for a line bundle L corresponding to a point outside of T, we have $H^1(F^\vee \otimes G' \otimes L) = 0$. Thus, $G := G' \otimes L$ suffices.

iv) $\Longrightarrow$ i): Again, assume that F is not semistable and take a maximal destabilizing subsheaf $D \subset F$. Now rank and degree of V are chosen in such a way that $\mu(D) > \mu(V) > \mu(F)$ always holds. The assertion now follows from $\mathrm{Hom}_E(D, F) \neq 0$ due to $D \hookrightarrow F$ and $\mathrm{Hom}_E(V, D) = H^0(D \otimes V^\vee) \neq 0$ due to $\deg(D \otimes V^\vee) > 0 \Longleftrightarrow \mu(D) > \mu(V)$. □

5. Multiplicative structure in degree 0

Atiyah considered the ring generated by (isomorphism classes of) indecomposable vector bundles with degree zero, the multiplication being given by the tensor product. Note that this is a subring of $K^0(E)$.

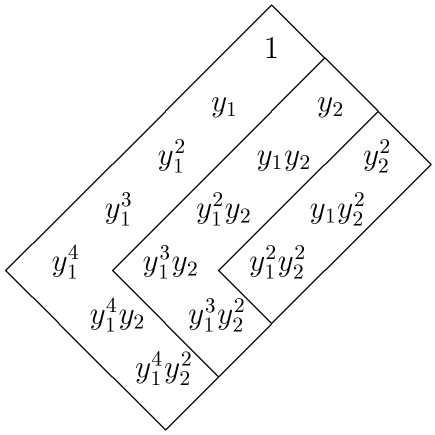
We can approach the products using the following formulae of Mukai:

$$\mathrm{FM}_{\mathcal{P}}(A \otimes B) = \mathrm{FM}_{\mathcal{P}}(A) * \mathrm{FM}_{\mathcal{P}}(B)[1], \quad \mathrm{FM}_{\mathcal{P}}(A * B) = \mathrm{FM}_{\mathcal{P}}(A) \otimes \mathrm{FM}_{\mathcal{P}}(B)$$

where $A * B := m_*(pr_1^* A \otimes pr_2^* B)$ and $m : E \times E \to E$ is the addition.

Denoting by F_r the unique semistable sheaf of rank r and determinant $\mathcal{O}_E$, we get $F_r \otimes F_s = \mathrm{FM}_{\mathcal{P}}(T_r * T_s)$ (T_r is the vector space $k[\varepsilon]/\varepsilon^r$ sitting only in p_0). Thus we have to compute $T_r * T_s$. But since everything is concentrated in a point (in an Artinian situation, actually), we can work in the following setting: Let $m : k[x] \to k[y_1, y_2], x \mapsto y_1 + y_2$ be the map which on spectra is the addition map $\mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$. Then, the $k[y_1, y_2]$-module $k[y_1, y_2]/(y_1^r, y_2^s)$ of finite length corresponds to $T_r \boxtimes T_s$. Now, $m_*(k[y_1, y_2]/(y_1^r, y_2^s))$ is just the same k-vector space considered as a $k[x]$-module via m. Multiplication with x gives (assume $r \leq s$) $x \cdot 1 = y_1 + y_2$, $x \cdot y_1 = y_1^2 + y_2, \ldots, x \cdot y_1^{r-1} = y_1^{r-1} y_2$. We now change the basis of $k[y_1, y_2]/(y_1^r, y_2^s)$ from $y_1^i y_2^j$ ($i = 0, \ldots, r-1$, $j = 0, \ldots, s-1$) to $(y_1 + y_2)^a y_2^b$ (with $b = 0, \ldots, r-1$ assuming that $r \leq s$ and $a = 0, \ldots, r+s-1-2b$ because $(y_1 + y_2)^a y_2^b \neq 0$ if and only if there is a $k \in \{0, \ldots, a\}$ with $a + b - s < k < r$).

[3]Condition (ii) of the proposition is a criterion for μ-stability on general varieties, whereas conditions (iii) and (iv) are peculiar to elliptic curves.



An example with $r = 5$, $s = 3$.

Hence we arrive at the following formula:

$$E_r \otimes E_s = \bigoplus_{k=1}^{\min(r,s)} E_{r+s+1-2k}.$$

This corresponds to Atiyah's theorem III.8.

Note that things are different in characteristic p if $p < r+s$. For example, if $\mathrm{char}(k) = 2$, we have $T_2 * T_2 = T_2 \oplus T_2$.

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Received October 15, 2004