Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 46 (2005), No. 2, 311-320.

A Formula for Angles between Subspaces of Inner Product Spaces

Hendra Gunawan **Oki Neswan** **Wono Setya-Budhi**

Department of Mathematics, Bandung Institute of Technology
Bandung 40132, Indonesia
e-mail: hgunawan, oneswan, wono@dns.math.itb.ac.id

Abstract. We present an explicit formula for angles between two subspaces of inner product spaces. Our formula serves as a correction for, as well as an extension of, the formula proposed by Risteski and Trenčevski [13]. As a consequence of our formula, a generalized Cauchy-Schwarz inequality is obtained.

MSC 2000: 15A03, 51N20, 15A45, 15A21, 46B20
Keywords: Angles between subspaces, canonical angles, generalized Cauchy-Schwarz inequality

1. Introduction

The notion of angles between two subspaces of the Euclidean space $\mathbb{R}^d$ has been studied by many researchers since the 1950's or even earlier (see [3]). In statistics, canonical (or principal) angles are studied as measures of dependency of one set of random variables on another (see [1]). Some recent works on angles between subspaces and related topics can be found in, for example, [4, 8, 12, 13, 14]. Particularly, in [13], Risteski and Trenčevski introduced a more geometrical definition of angles between two subspaces of $\mathbb{R}^d$ and explained its connection with canonical angles. Their definition of the angle, however, is based on a generalized Cauchy-Schwarz inequality which we found incorrect. The purpose of this note is to fix their definition and at the same time extend the ambient space to any real inner product space.

Let $(X, \langle\cdot,\cdot\rangle)$ be a real inner product space, which will be our ambient space throughout this note. Given two nonzero, finite-dimensional, subspaces U and V of X with $\dim(U) \leq$

$\dim(V)$, we wish to have a definition of the angle between U and V that can be viewed, in some sense, as a natural generalization of the 'usual' definition of the angle (a) between a 1-dimensional subspace and a q-dimensional subspace of X, and (b) between two p-dimensional subspaces intersecting on a common $(p-1)$-dimensional subspace of X.

To explain precisely what we mean by the word 'usual', let us review how the angle is defined in the above two trivial cases:

(a) If $U = \mathrm{span}\{u\}$ is a 1-dimensional subspace and $V = \mathrm{span}\{v_1, \dots, v_q\}$ is a q-dimensional subspace of X, then the angle θ between U and V is defined by

$$\cos^2\theta = \frac{\langle u, u_V\rangle^2}{\|u\|^2\|u_V\|^2} \tag{1.1}$$

where u_V denotes the (orthogonal) projection of u on V and $\|\cdot\| = \langle\cdot,\cdot\rangle^{\frac{1}{2}}$ denotes the induced norm on X. (Throughout this note, we shall always take θ to be in the interval $[0, \frac{\pi}{2}]$.)

(b) If $U = \mathrm{span}\{u, w_2, \dots, w_p\}$ and $V = \mathrm{span}\{v, w_2, \dots, w_p\}$ are p-dimensional subspaces of X that intersects on $(p-1)$-dimensional subspace $W = \mathrm{span}\{w_2, \dots, w_p\}$ with $p \geq 2$, then the angle θ between U and V may be defined by

$$\cos^2\theta = \frac{\langle u_W^\perp, v_W^\perp\rangle^2}{\|u_W^\perp\|^2\|v_W^\perp\|^2} \tag{1.2}$$

where $u_W^\perp$ and $v_W^\perp$ are the orthogonal complement of u and v, respectively, on W.

One common property among these two cases is the following. In (a), we may write $u = u_V + u_V^\perp$ where $u_V^\perp$ is the orthogonal complement of u on V. Then (1.1) amounts to

$$\cos^2\theta = \frac{\|u_V\|^2}{\|u\|^2},$$

which tells us that the value of $\cos\theta$ is equal to the ratio between the length of the projection of u on V and the length of u. Similarly, in (b), we claim that the value of $\cos\theta$ is equal to the ratio between the volume of the p-dimensional parallelepiped spanned by the projection of $u, w_2, \dots, w_p$ on V and the volume of the p-dimensional parallelepiped spanned by $u, w_2, \dots, w_p$.

Motivated by this fact, we shall define the angle between a p-dimensional subspace $U = \mathrm{span}\{u_1, \dots, u_p\}$ and a q-dimensional subspace $V = \mathrm{span}\{v_1, \dots, v_q\}$ (with $p \leq q$) such that the value of its cosine is equal to the ratio between the volume of the p-dimensional parallelepiped spanned by the projection of $u_1, \dots, u_p$ on V and the p-dimensional parallelepiped spanned by $u_1, \dots, u_p$. As we shall see later, the ratio is a number in $[0,1]$ and is invariant under any change of basis for U and V, so that our definition of the angle makes sense.

In the following sections, an explicit formula for the cosine in terms of $u_1, \dots, u_p$ and $v_1, \dots, v_q$ will be presented. Our formula serves as a correction for Risteski and Trenčevski's. As a consequence of our formula, a generalized Cauchy-Schwarz inequality is obtained. An extension to the case where the subspace V is infinite dimensional, assuming that the ambient space X is infinite dimensional, will also be discussed.

2. Main results

Hereafter we shall employ the standard n-inner product $\langle\cdot,\cdot|\cdot,\dots,\cdot\rangle$ on X, given by

$$\langle x_0,x_1|x_2,\dots,x_n\rangle := \begin{vmatrix} \langle x_0,x_1\rangle & \langle x_0,x_2\rangle & \dots & \langle x_0,x_n\rangle \\ \langle x_2,x_1\rangle & \langle x_2,x_2\rangle & \dots & \langle x_2,x_n\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle x_n,x_1\rangle & \langle x_n,x_2\rangle & \dots & \langle x_n,x_n\rangle \end{vmatrix},$$

and the standard n-norm $\|x_1,x_2,\dots,x_n\| := \langle x_1,x_1|x_2,\dots,x_n\rangle^{\frac{1}{2}}$ (see [6] or [10]). Here we assume that $n \ge 2$. (If $n = 1$, the standard 1-inner product is understood as the given inner product, while the standard 1-norm is the induced norm.) Note particularly that $\langle x_1,x_1|x_2,\dots,x_n\rangle = \det[\langle x_i,x_j\rangle]$ is nothing but the Gram's determinant generated by $x_1,x_2,\dots,x_n$ (see [5] or [11]). Geometrically, being the square root of the Gram's determinant, $\|x_1,\dots,x_n\|$ represents the volume of the n-dimensional parallelepiped spanned by $x_1,\dots,x_n$.

A few noticeable properties of the standard n-inner product are that it is bilinear and commutative in the first two variables. Also, $\langle x_0,x_1|x_2,\dots,x_n\rangle = \langle x_0,x_1|x_{i_2},\dots,x_{i_n}\rangle$ for any permutation $\{i_2,\dots,i_n\}$ of $\{2,\dots,n\}$. Moreover, from properties of Gram's determinants, we have $\|x_1,\dots,x_n\| \ge 0$ and $\|x_1,\dots,x_n\| = 0$ if and only if $x_1,\dots,x_n$ are linearly dependent.

As for inner products, we have the Cauchy-Schwarz inequality for the n-inner product:

$$\langle x_0,x_1|x_2,\dots,x_n\rangle^2 \le \|x_0,x_2,\dots,x_n\|^2\|x_1,x_2,\dots,x_n\|^2$$

for every $x_0,x_1,\dots,x_n$. There is also Hadamard's inequality which states that

$$\|x_1,\dots,x_n\| \le \|x_1\|\cdots\|x_n\|$$

for every $x_1,\dots,x_n$.

Next observe that $\langle x_0,x_1+x_1'|x_2,\dots,x_n\rangle = \langle x_0,x_1|x_2,\dots,x_n\rangle$ for any linear combination x_1' of $x_2,\dots,x_n$. Thus, for instance, for $i = 0$ and 1, one may write $x_i = x_i^* + x_i^\perp$, where x_i^* is the projection of x_i on $\mathrm{span}\{x_2,\dots,x_n\}$ and $x_i^\perp$ is its orthogonal complement, to get

$$\langle x_0,x_1|x_2,\dots,x_n\rangle = \langle x_0^\perp,x_1^\perp|x_2,\dots,x_n\rangle = \langle x_0^\perp,x_1^\perp\rangle\|x_2,\dots,x_n\|^2.$$

(Here $\|x_2,\dots,x_n\|$ represents the volume of the $(n-1)$-parallelepiped spanned by $x_2,\dots,x_n$.)

Using the standard n-inner product and n-norm, we can, for instance, derive an explicit formula for the projection of a vector x on the subspace spanned by $x_1,\dots,x_n$. Let $x^* = \sum_{k=1}^n \alpha_k x_k$ be the projection of x on $\mathrm{span}\{x_1,\dots,x_n\}$. Taking the inner products of x^* and x_l, we get the following system of linear equations:

$$\sum_{k=1}^n \alpha_k\langle x_k,x_l\rangle = \langle x^*,x_l\rangle = \langle x,x_l\rangle, \quad l = 1,\dots,n.$$

By Cramer's rule together with properties of inner products and determinants, we obtain

$$\alpha_k = \frac{\langle x,x_k|x_{i_2(k)},\dots,x_{i_n(k)}\rangle}{\|x_1,x_2,\dots,x_n\|^2},$$

where $\{i_2(k),\dots,i_n(k)\} = \{1,2,\dots,n\}\setminus\{k\}$, $k = 1,2,\dots,n$.

2.1. The claim and its proof

We claim in the introduction that the cosine of the angle θ between the two p-dimensional subspaces $U = \mathrm{span}\{u, w_2, \dots, w_p\}$ and $V = \mathrm{span}\{v, w_2, \dots, w_p\}$ defined by (1.2) is equal to the ratio between the volume of the p-dimensional parallelepiped spanned by the projection of $u, w_2, \dots, w_p$ on V and the volume of the p-dimensional parallelepiped spanned by $u, w_2, \dots, w_p$. That is,

$$\cos^2\theta = \frac{\|u_V, w_2, \dots, w_p\|^2}{\|u, w_2, \dots, w_p\|^2},$$

where u_V denotes the projection of u on V.

To verify this, we first observe that θ satisfies

$$\cos^2\theta = \frac{\langle u, v|w_2, \dots, w_p\rangle^2}{\|u, w_2, \dots, w_p\|^2\|v, w_2, \dots, w_p\|^2}.$$

Indeed, writing $u = u_W + u_W^\perp$ and $v = v_W + v_W^\perp$ (where u_W and v_W are the projection of u and v, respectively, on $W = \mathrm{span}\{w_2, \dots, w_p\}$), we obtain

$$\frac{\langle u, v|w_2, \dots, w_p\rangle^2}{\|u, w_2, \dots, w_p\|^2\|v, w_2, \dots, w_p\|^2} = \frac{\langle u_W^\perp, v_W^\perp\rangle^2\|w_2, \dots, w_p\|^4}{\|u_W^\perp\|^2\|v_W^\perp\|^2\|w_2, \dots, w_p\|^4} = \frac{\langle u_W^\perp, v_W^\perp\rangle^2}{\|u_W^\perp\|^2\|v_W^\perp\|^2},$$

as stated.

Suppose now that $u_V = \alpha v + \sum_{k=2}^p \beta_k w_k$. In particular, the scalar α is given by

$$\alpha = \frac{\langle u, v|w_2, \dots, w_p\rangle}{\|v, w_2, \dots, w_p\|^2}.$$

Then, we have

$$\|u_V, w_2, \dots, w_p\|^2 = \langle u, u_V|w_2, \dots, w_p\rangle = \alpha\langle u, v|w_2, \dots, w_p\rangle = \frac{\langle u, v|w_2, \dots, w_p\rangle^2}{\|v, w_2, \dots, w_p\|^2}.$$

Hence, we obtain

$$\frac{\|u_V, w_2, \dots, w_p\|^2}{\|u, w_2, \dots, w_p\|^2} = \frac{\langle u, v|w_2, \dots, w_p\rangle^2}{\|u, w_2, \dots, w_p\|^2\|v, w_2, \dots, w_p\|^2} = \cos^2\theta,$$

as expected.

2.2. An explicit formula for the cosine

Using the standard n-norm (with $n = p$), we define the angle θ between a p-dimensional subspace $U = \mathrm{span}\{u_1, \dots, u_p\}$ and a q-dimensional subspace $V = \mathrm{span}\{v_1, \dots, v_q\}$ of X (with $p \le q$) by

$$\cos^2\theta := \frac{\|\mathrm{proj}_V u_1, \dots, \mathrm{proj}_V u_p\|^2}{\|u_1, \dots, u_p\|^2}, \tag{2.1}$$

where $\mathrm{proj}_V u_i$'s denote the projection of u_i's on V.

The following fact convinces us that our definition makes sense.

Fact. *The ratio on the right hand side of* (2.1) *is a number in* $[0, 1]$ *and is independent of the choice of bases for* U *and* V.

Proof. First note that the projection of u_i's on V is independent of the choice of basis for V. Further, since projections are linear transformations, the ratio is also invariant under any change of basis for U. Indeed, the ratio is unchanged if we (a) swap u_i and u_j, (b) replace u_i by $u_i + \alpha u_j$, or (c) replace u_i by αu_i with $\alpha \neq 0$.

Next, assuming particularly that $\{u_1, \dots, u_p\}$ is orthonormal, we have $\|u_1, \dots, u_p\| = 1$ and $\|\mathrm{proj}_V u_1, \dots, \mathrm{proj}_V u_p\| \leq 1$ because $\|\mathrm{proj}_V u_i\| \leq \|u_i\| = 1$ for each $i = 1, \dots, p$. Therefore, the ratio is a number in $[0, 1]$, and the proof is complete. □

From (2.1), we can derive an explicit formula for the cosine in terms of $u_1, \dots, u_p$ and $v_1, \dots, v_q$, assuming for the moment that $\{v_1, \dots, v_q\}$ is orthonormal. For each $i = 1, \dots, p$, the projection of u_i on V is given by

$$\mathrm{proj}_V u_i = \langle u_i, v_1 \rangle v_1 + \cdots + \langle u_i, v_q \rangle v_q.$$

So, for $i, j = 1, \dots, p$, we have

$$\langle \mathrm{proj}_V u_i, \mathrm{proj}_V u_j \rangle = \sum_{k=1}^{q} \langle u_i, v_k \rangle \langle u_j, v_k \rangle \, .$$

Hence, we obtain

$$\|\mathrm{proj}_V u_1, \dots, \mathrm{proj}_V u_p\|^2 = \det\Big[\sum_{k=1}^{q} \langle u_i, v_k \rangle \langle u_j, v_k \rangle\Big] = \det(MM^{\mathrm{T}})$$

where $M := [\langle u_i, v_k \rangle]$ is a $(p \times q)$ matrix and M^{T} is its transpose. The cosine of the angle θ between U and V is therefore given by the formula

$$\cos^2 \theta = \frac{\det(MM^{\mathrm{T}})}{\det[\langle u_i, u_j \rangle]} \, , \tag{2.2}$$

If $\{u_1, \dots, u_p\}$ happens to be orthonormal, then the formula (2.2) reduces to

$$\cos^2 \theta = \det(MM^{\mathrm{T}}).$$

Further, if $p = q$, then $\det(MM^{\mathrm{T}}) = \det M \cdot \det M^{\mathrm{T}} = \det^2 M$. Hence, from the last formula, we get $\cos \theta = |\det M|$.

2.3. On Risteski and Trančevski's formula

The reader might think that the angle defined by (2.1) is exactly the same as the one formulated by Risteski and Trenčevski ([13], Equation 1.2). But that is not true! They defined the angle θ between two subspaces $U = \mathrm{span}\{u_1, \dots, u_p\}$ and $V = \mathrm{span}\{v_1, \dots, v_q\}$ with $p \le q$ by

$$\cos^2\theta := \frac{\det(MM^{\mathrm{T}})}{\det[\langle u_i, u_j\rangle] \cdot \det[\langle v_k, v_l\rangle]}, \tag{2.3}$$

by first 'proving' the following inequality ([13], Theorem 1.1):

$$\det(MM^{\mathrm{T}}) \le \det[\langle u_i, u_j\rangle] \cdot \det[\langle v_k, v_l\rangle], \tag{2.4}$$

where $M := [\langle u_i, v_k\rangle]$. However, the argument in their proof which says that the inequality is invariant under elementary row operations only allows them to assume that $\{u_1, \dots, u_p\}$ is orthonormal, but not $\{v_1, \dots, v_q\}$, except when $p = q$. As a matter of fact, the inequality (2.4) is only true in the case (a) where $p = q$ (for which the inequality reduces to Kurepa's generalization of the Cauchy-Schwarz inequality, see [9]) or (b) where $\{v_1, \dots, v_q\}$ is orthonormal. Consequently, (2.3) makes sense only in these two cases, for otherwise the value of the expression on the right hand side of (2.3) may be greater than 1.

To show that the inequality (2.4) is false in general, just take for example $X = \mathbb{R}^3$ (equipped with the usual inner product), $U = \mathrm{span}\{u\}$ where $u = (1, 0, 0)$, and $V = \mathrm{span}\{v_1, v_2\}$ where $v_1 = (\frac{1}{2}, \frac{1}{2}, 0)$ and $v_2 = (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$. Acoording to (2.4), we should have

$$\langle u, v_1\rangle^2 + \langle u, v_2\rangle^2 \le \|u\|^2 \|v_1, v_2\|^2.$$

But the left hand side of the inequality is equal to

$$\langle u, v_1\rangle^2 + \langle u, v_2\rangle^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2},$$

while the right hand side is equal to

$$\|u\|^2 \big(\|v_1\|^2 \|v_2\|^2 - \langle v_1, v_2\rangle^2\big) = \frac{3}{8}.$$

This example shows that the inequality is false even in the case where $\{u_1, \dots, u_p\}$ is orthonormal and $\{v_1, \dots, v_q\}$ is orthogonal (which is close to being orthonormal).

2.4. A general formula for $p = 1$ and $q = 2$

Let us consider the case where $p = 1$ and $q = 2$ more closely. For a unit vector u and an orthonormal set $\{v_1, v_2\}$ in X, it follows from our definition of the angle θ between $U = \mathrm{span}\{u\}$ and $V = \mathrm{span}\{v_1, v_2\}$ that

$$\cos^2\theta = \langle u, v_1\rangle^2 + \langle u, v_2\rangle^2 \le 1.$$

Hence, for a nonzero vector u and an orthogonal set $\{v_1, v_2\}$ in X, we have

$$\cos^2\theta = \left\langle \frac{u}{\|u\|}, \frac{v_1}{\|v_1\|} \right\rangle^2 + \left\langle \frac{u}{\|u\|}, \frac{v_2}{\|v_2\|} \right\rangle^2.$$

Thus, for this case, we have

$$\langle u, v_1\rangle^2 \|v_2\|^2 + \langle u, v_2\rangle^2 \|v_1\|^2 \leq \|u\|^2 \|v_1, v_2\|^2,$$

where $\|v_1, v_2\|^2 = \|v_1\|^2\|v_2\|^2$ is the area of the parallelogram spanned by v_1 and v_2.

More generally, suppose that u is a nonzero vector and $\{v_1, v_2\}$ is linearly independent, and we would like to have an explicit formula for the cosine of the angle θ between $U = \text{span}\{u\}$ and $V = \text{span}\{v_1, v_2\}$ in terms of u, v_1 and v_2. Instead of orthogonalizing $\{v_1, v_2\}$ by Gram-Schmidt process, we do the following. Let u_V be the projection of u on V. Then u_V may be expressed as

$$u_V = \frac{\langle u, v_1|v_2\rangle}{\|v_1, v_2\|^2} v_1 + \frac{\langle u, v_2|v_1\rangle}{\|v_1, v_2\|^2} v_2,$$

where $\langle \cdot, \cdot|\cdot\rangle$ is the standard 2-inner product introduced earlier. Now write $u = u_V + u_V^\perp$ where $u_V^\perp$ is the orthogonal complement of u on V. Then

$$\cos^2\theta = \frac{\|u_V\|^2}{\|u\|^2} = \frac{\langle u, u_V\rangle}{\|u\|^2} = \frac{\langle u, v_1\rangle\langle u, v_1|v_2\rangle + \langle u, v_2\rangle\langle u, v_2|v_1\rangle}{\|u\|^2\|v_1, v_2\|^2}. \tag{2.5}$$

Consequently, for any nonzero vector u and linearly independent set $\{v_1, v_2\}$, we have the following inequality

$$\langle u, v_1\rangle\langle u, v_1|v_2\rangle + \langle u, v_2\rangle\langle u, v_2|v_1\rangle \leq \|u\|^2\|v_1, v_2\|^2. \tag{2.6}$$

Here (2.5) and (2.6) serve as corrections for (2.3) and (2.4) for $p = 1$ and $q = 2$.

The inequality (2.6) may be viewed as a generalized Cauchy-Schwarz inequality. The difference between our approach and Risteski and Trenčevski's is that we derive the inequality as a consequence of the definition of the angle between two subspaces, while Risteski and Trenčevski use the 'inequality' to define the angle between two subspaces. As long as $p = q$ or, otherwise, $\{v_1, \ldots, v_q\}$ is orthonormal, their definition makes sense and of course agrees with ours.

2.5. An explicit formula for arbitrary p and q

An explicit formula for the cosine of the angle θ between a p-dimensional subspace $U = \text{span}\{u_1, \ldots, u_p\}$ and a q-dimensional subspace $V = \text{span}\{v_1, \ldots, v_q\}$ of X for arbitrary $p \leq q$ can be obtained as follows.

For each $i = 1, \ldots, p$, the projection of u_i on V may be expressed as

$$\text{proj}_V u_i = \sum_{k=1}^{q} \alpha_{ik} v_k\,,$$

where

$$\alpha_{ik} = \frac{\langle u_i, v_k | v_{i_2(k)}, \ldots, v_{i_q(k)} \rangle}{\|v_1, v_2, \ldots, v_q\|^2}$$

with $\{i_2(k), \ldots, i_q(k)\} = \{1, 2, \ldots, q\} \setminus \{k\},\ k = 1, 2, \ldots, q$. Next observe that

$$\langle \mathrm{proj}_V u_i, \mathrm{proj}_V u_j \rangle = \langle u_i, \mathrm{proj}_V u_j \rangle = \sum_{k=1}^{q} \alpha_{jk} \langle u_i, v_k \rangle$$

for $i, j = 1, \ldots, p$. Hence we have

$$\|\mathrm{proj}_V u_i, \ldots, \mathrm{proj}_V u_p\|^2 = \begin{vmatrix} \sum_{k=1}^{q} \alpha_{1k} \langle u_1, v_k \rangle & \ldots & \sum_{k=1}^{q} \alpha_{pk} \langle u_1, v_k \rangle \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^{q} \alpha_{1k} \langle u_p, v_k \rangle & \ldots & \sum_{k=1}^{q} \alpha_{pk} \langle u_p, v_k \rangle \end{vmatrix} = \frac{\det(M\tilde{M}^{\mathrm{T}})}{\|v_1, \ldots, v_q\|^{2p}},$$

where

$$M := [\langle u_i, v_k \rangle] \quad \text{and} \quad \tilde{M} := [\langle u_i, v_k | v_{i_2(k)}, \ldots, v_{i_q(k)} \rangle] \tag{2.7}$$

with $i_2(k), \ldots, i_q(k)$ as above. (Note that both M and $\tilde{M}$ are $(p \times q)$ matrices, so that $M\tilde{M}^{\mathrm{T}}$ is a $(p \times p)$ matrix.) Dividing by $\|u_1, \ldots, u_p\|^2$, we get the following formula for the cosine:

$$\cos^2 \theta = \frac{\det(M\tilde{M}^{\mathrm{T}})}{\det[\langle u_i, u_j \rangle] \cdot \det^p[\langle v_k, v_l \rangle]}, \tag{2.8}$$

which serves as a correction for Risteski and Trenčevski's formula (2.3). Note that if $\{v_1, \ldots, v_q\}$ is orthonormal, we get the formula (2.2) obtained earlier.

As a consequence of our formula, we have the following generalization of the Cauchy-Schwarz inequality, which can be considered as a correction for (2.4).

Theorem. *For two linearly independent sets $\{u_1, \ldots, u_p\}$ and $\{v_1, \ldots, v_q\}$ in X with $p \leq q$, we have the following inequality*

$$\det(M\tilde{M}^{\mathrm{T}}) \leq \det[\langle u_i, u_j \rangle] \cdot \det^p[\langle v_k, v_l \rangle],$$

where M and $\tilde{M}$ are $(p \times q)$ matrices given by (2.7). Moreover, the equality holds if and only if the subspace spanned by $\{u_1, \ldots, u_p\}$ is contained in the subspace spanned by $\{v_1, \ldots, v_q\}$.

Proof. The inequality follows directly from the definition of the angle between $U = \mathrm{span}\{u_1, \ldots, u_p\}$ and $V = \mathrm{span}\{v_1, \ldots, v_q\}$ as formulated in (2.8). Next, if U is contained in V, then the projection of u_i's on V are the u_i's themselves. Hence the equality holds since the cosine is equal to 1. If at least one of u_i's, say u_{i_0}, is not in V, then, assuming that $\{u_1, \ldots, u_p\}$ and $\{v_1, \ldots, v_q\}$ are orthonormal, the length of the projection of u_{i_0} on V will be strictly less than 1. In this case the cosine will be less than 1, and accordingly we have a strict inequality. □

3. Concluding remarks

As the reader might have realized, the formula (2.1) may also be used to define the angle between a finite p-dimensional subspace U and an infinite dimensional subspace V of X, assuming that the ambient space X is infinite dimensional and complete (that is, X is an infinite dimensional Hilbert space).

In certain cases, an explicit formula for the cosine can be obtained directly from (2.1). For example, take $X = \ell^2$, the space of square summable sequences of real numbers, equipped with the inner product

$$\langle x, y\rangle := \sum_{m=1}^{\infty} x(m)y(m), \quad x = (x(m)),\ y = (y(m)).$$

Let $U = \mathrm{span}\{u_1, u_2\}$ where $u_1 = (u_1(m))$ and $u_2 = (u_2(m))$ are two linearly independent sequences in ℓ^2, and $V := \{(x(m)) \in \ell^2 : x(1) = x(2) = x(3) = 0\}$, which is an infinite dimensional subspace of ℓ^2. Then, for $i = 1, 2$, the projection of u_i on V is

$$\mathrm{proj}_V u_i = (0, 0, 0, u_i(4), u_i(5), u_i(6), \dots).$$

The square of the volume of the parallelogram spanned by $\mathrm{proj}_V u_1$ and $\mathrm{proj}_V u_2$ is

$$\begin{aligned}\|\mathrm{proj}_V u_1, \mathrm{proj}_V u_2\|^2 &= \det[\langle \mathrm{proj}_V u_i, \mathrm{proj}_V u_j\rangle] \\ &= \sum_{m=4}^{\infty} u_1(m)^2 \cdot \sum_{m=4}^{\infty} u_2(m)^2 - \Big(\sum_{m=4}^{\infty} u_1(m)u_2(m)\Big)^2.\end{aligned}$$

Meanwhile, the square of the volume of the parallelogram spanned by u_1 and u_2 is

$$\|u_1, u_2\|^2 = \det[\langle u_i, u_j\rangle] = \sum_{m=1}^{\infty} u_1(m)^2 \cdot \sum_{m=1}^{\infty} u_2(m)^2 - \Big(\sum_{m=1}^{\infty} u_1(m)u_2(m)\Big)^2.$$

Hence, the cosine of the angle θ between U and V is given by

$$\cos^2\theta = \frac{\sum_{m=4}^{\infty} u_1(m)^2 \cdot \sum_{m=4}^{\infty} u_2(m)^2 - \big(\sum_{m=4}^{\infty} u_1(m)u_2(m)\big)^2}{\sum_{m=1}^{\infty} u_1(m)^2 \cdot \sum_{m=1}^{\infty} u_2(m)^2 - \big(\sum_{m=1}^{\infty} u_1(m)u_2(m)\big)^2}.$$

In general, however, in order to obtain an explicit formula for the cosine in terms of the basis vectors for U and V, we need to have an orthonormal basis for V in hand. (Here an orthonormal basis means a maximal orthonormal system; see, for instance, [2].) In such a case, the computations of the projection of the basis vectors for U on V (and then the square of the volume of the p-dimensional parallelepiped spanned by them) can be carried out, and an explicit formula for the cosine in terms of the basis vectors for U and V can be obtained.

As the above example indicated, the formula will involve the determinant of a $(p \times p)$ matrix whose entries are infinite sums of products of two inner products. If desired, this determinant can be expressed as an infinite sum of squares of determinants of $(p\times p)$ matrices, each of which represents the square of the volume of the projected parallelepiped on p-dimensional subspaces of V. See [7] for these ideas.

Acknowledgement. The first and second author are supported by QUE-Project V (2003) Math-ITB. Special thanks go to our colleagues A. Garnadi, P. Astuti, and J. M. Tuwankotta for useful discussions about the notion of angles.

References

[1] Anderson, T. W.: *An Introduction to Multivariate Statistical Analysis*. John Wiley & Sons, Inc., New York 1958. Zbl 0083.14601

[2] Brown, A. L.; Page, A.: *Elements of Functional Analysis*. Van Nostrand Reinhold Co., London 1970. Zbl 0199.17902

[3] Davis, C.; Kahan, W.: *The rotation of eigenvectors by a perturbation. III*. SIAM J. Numer. Anal. **7** (1970), 1–46. Zbl 0198.47201

[4] Fedorov, S.: *Angle between subspaces of analytic and antianalytic functions in weighted L_2 space on a boundary of a multiply connected domain*. In: Operator Theory, System Theory and Related Topics. Beer-Sheva/Rehovot 1997, 229–256. Zbl 0997.46037

[5] Gantmacher, F. R.: *The Theory of Matrices*. Vol. I, Chelsea Publ. Co., New York 1960, 247–256. cf. Reprint of the 1959 translation (1998). Zbl 0927.15002

[6] Gunawan, H.: *On n-inner products, n-norms, and the Cauchy-Schwarz inequality*. Sci. Math. Jpn. **55** (2001), 53–60. Zbl 1009.46011

[7] Gunawan, H.: *The space of p-summable sequences and its natural n-norm*. Bull. Aust. Math. Soc. **64** (2001), 137–147. Zbl 1002.46007

[8] Knyazev, A. V.; Argentati, M. E.: *Principal angles between subspaces in an A-based scalar product: algorithms and perturbation estimates*. SIAM J. Sci. Comput. **23** (2002), 2008–2040. Zbl 1018.65058

[9] Kurepa, S.: *On the Buniakowsky-Cauchy-Schwarz inequality*. Glas. Mat. III. Ser. **1**(21) (1966), 147–158. Zbl 0186.18503

[10] Misiak, A.: *n-inner product spaces*. Math. Nachr. **140** (1989), 299–319. Zbl 0673.46012

[11] Mitrinović, D. S.; Pečarić, J. E.; Fink, A. M.: *Classical and New Inequalities in Analysis*. Kluwer Academic Publishers, Dordrecht 1993, 595–603. Zbl 0771.26009

[12] Rakočević, V.; Wimmer, H. K.: *A variational characterization of canonical angles between subspaces*. J. Geom. **78** (2003), 122–124. Zbl pre02067300

[13] Risteski, I. B.; Trenčevski, K. G.: *Principal values and principal subspaces of two subspaces of vector spaces with inner product*. Beitr. Algebra Geom. **42** (2001), 289–300. Zbl 0971.15001

[14] Wimmer, H. K.: *Canonical angles of unitary spaces and perturbations of direct complements*. Linear Algebra Appl. **287** (1999), 373–379. Zbl 0937.15002

Received December 23, 2003