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Configurations of Lines and General Hyperplane Sections

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Abstract. Let $Y \subset \mathbf{P}^n$ be a finite union of lines and $H \subset \mathbf{P}^n$ a general hyperplane. Here we study the linearly general position of the finite set $Y \cap H$.

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1. Introduction

Let $S \subset \mathbf{P}^N$ be a finite set. S is said to be in linearly general position if for any $S' \subseteq S$ the linear span $\langle S' \rangle$ of S spans a linear subspace of dimension $\min\{N, \mathrm{card}(S') - 1\}$. It is well-known that if $C \subset \mathbf{P}^n$ is an integral non-degenerate curve and the algebraically closed base field K has characteristic zero, then a general hyperplane section of C is in linearly general position ([5], Lemma 1.1). Obviously, this result is not true for an arbitrary reducible curve and the first aim of this paper is to classify exactly when it is true when C is a union of lines. We will also work over an arbitrary algebraically closed field K. We recall that in general the corresponding result for irreducible curve is not true in positive characteristic ([5]). In Section 2 we will prove the following result.

Theorem 1. *Let $X \subset \mathbf{P}^n$ be a finite union of lines. Assume that a general hyperplane section of X is not in linearly general position and let s be the first integer such that $1 \leq s \leq n-2$ and for a general hyperplane H the set $X \cap H$ contains a set of at least $s+2$ points spanning a linear space of dimension s. Let $A_1, \dots, A_x$, $x \geq 1$, be the s-dimensional linear subspaces of H containing at least $s+2$ points of $X \cap H$. Set $S_i := A_i \cap X$. Then there are $(s+1)$-dimensional*

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linear subspaces $M_i \subset \mathbf{P}^n$ *and subcurves* $X_i \subset M_i \cap X$ *with* $\deg(X_i) = \mathrm{card}(S_i) \geq s+2$. *Furthermore,* $S_i = X_i \cap H$ *for a general hyperplane* H.

Theorem 1 is a quantitative and precise version of the following immediate corollary of it.

Corollary 1. *Let* $X \subset \mathbf{P}^n$ *be a finite union of lines. A general hyperplane section of* X *is in linearly general position if and only if there is no positive integer* $m \leq n-2$ *such that at least* $m+2$ *lines of* X *are contained in an* m*-dimensional linear subspace of* $\mathbf{P}^n$.

We will say that a reduced curve $X \subset \mathbf{P}^n$ is a *dismantled curve* or a *configuration of lines* if each irreducible component of X is a line.

Remark 1. Let $X \subset \mathbf{P}^n$ be a non-degenerate dismantled curve. In general, it is not true that $X \cap H$ spans H for a general hyperplane H. For instance take $n = 3$ and X the union of two disjoint lines. However, by Theorem 1 if $X \cap H$ spans a linear subspace of dimension $s \leq n-2$ and X contains no subcurve of degree at least $s+2$ contained in a linear space of dimension at most $s+1$, then $\deg(X) = s+1$.

Remark 2. Fix integers n, d with $n \geq 3$ and $d \geq n$. Set $m := [(d-1)/(n-1)]$, $\epsilon := d-1-m(n-1)$ and $\pi(n,d) := m(m-1)(n-1)/2 + m\epsilon$. Let $X \subset \mathbf{P}^n$ be a degree d non-degenerate reduced curve such that the general hyperplane section of X is in linearly general position and spans the corresponding hyperplane. By Castelnuovo theory (see e.g. [4], Ch. 3, or [3], p. 252) we have $p_a(X) \leq \pi(n,d)$. Fix a hyperplane H not containing any irreducible component of X. From the exact sequence

$$0 \to \mathcal{I}_X \to \mathcal{I}_X(1) \to \mathcal{I}_{X\cap H,H}(1) \to 0 \tag{1}$$

we see that $X \cap H$ spans H if X is connected.

Now we consider the postulation of the subsets of a generic hyperplane section of a configuration of lines with respect to the homogeneous forms of degree $s \geq 2$.

Theorem 2. *Fix integers* $n \geq 3$, $s \geq 2$ *and* $b > \binom{2n-2}{n} s^{2n-2}/(n-1)$ *and a configuration* $Y \subset \mathbf{P}^3$ *of lines. Assume that for a general hyperplane* H *the following conditions are satisfied:*

(a) *for all integers* t *with* $1 \leq t < s$ *and all* $A \subseteq Y \cap H$ *we have* $h^0(\mathbf{P}^n, \mathcal{I}_A(t)) = \max\{0, \binom{n+t}{n} - \mathrm{card}(A)\}$;

(b) *there is* $S \subseteq Y \cap H$ *such that* $\mathrm{card}(S) = b$ *and* $h^0(\mathbf{P}^3, \mathcal{I}_S(s)) \neq 0$.

Then there is a variety $F \subset \mathbf{P}^n$ *such that at least* $b - \binom{2n-2}{n} s^{2n-2}/(n-1)$ *lines of* Y *are contained in* F, F *is a union of lines,* $2 \leq \dim(F) \leq n-1$, *and* $h^0(\mathbf{P}^n, \mathcal{I}_F(s)) \neq 0$.

The integer $\binom{2n-2}{n}/(n-1)$ in the statement of Theorem 2 is the degree of the Grassmannian $G(2, n+1)$ of all lines of $\mathbf{P}^n$ with respect to the Plücker embedding of $G(2, n+2)$ induced by the positive generator $\mathcal{O}_{G(2,n+1)}(1)$ of $\mathrm{Pic}(G(2, n+1))$ ([2], Example 14.7.11). Since $\dim(G(2, n+1)) = 2n-2$, the integer $\binom{2n-2}{n} s^{2n-2}/(n-1)$ is the top self-intersection of $\mathcal{O}_{G(2,n+1)}(1)$. This observation explains the lower bound for b appearing in the statement of Theorem 2. The thesis of Theorem 2 has two parts:

(a) several lines of Y are contained in a degree s hypersurface G lifting the hypersurface of the generic hyperplane H of $\mathbf{P}^n$ containing many points of $Y \cap H$;

(b) G contains infinitely many lines; in particular if $n = 3$, then G is a ruled surface.

In Section 4 we will consider the linear general position of a general section of a double structure Z on a configuration of lines Y. We will allow unreduced curves Z which are reduced at a general point of some of the lines contained in Y (see Theorem 3).

2. Proof of Theorem 1

Proof of Theorem 1. For every $P \in S_i$ let $L_P \subset X$ be the line such that $\{P\} = L_P \cap H$. Set $X_i := \cup_{P \in S_i} L_P$ and $V_i := \langle X_i \rangle$, $1 \leq i \leq x$. Since the set of all hyperplanes of $\mathbf{P}^n$ is irreducible, while X has only finitely many irreducible components, the integer x and the subcurves X_i, $1 \leq i \leq x$, of X do not depend upon the choice of the sufficiently general hyperplane H. Hence the linear spaces M_i, $1 \leq i \leq x$, do not depend upon the choice of the sufficiently general hyperplane H. Notice that X_i is the maximal subcurve of X such that $X_i \cap H$ is contained in M_i. By the minimality of s for every $S' \subset S_i$ with $\mathrm{card}(S') = s + 1$ we have $A_i = \langle S' \rangle$, i.e. each S_i is in linearly general position in its linear span. To prove Theorem 1 it is sufficient to show that $\dim(V_i) = s + 1$ for every i. We assume that Theorem 1 fails for some dismantled curve and we take n minimal with this property. If $n > s + 2$ a general projection of Y into $\mathbf{P}^{s+2}$ gives a counterexample to Theorem 1. Hence by the minimality of n and s we obtain $n = s + 2$ and $\dim(M_i) = s + 2$ for some index i, say for $i = 1$. The dismantled curve X_1 gives a counterexample to Theorem 1. Hence, we reduced to the case $X = X_1$, $n = s + 2$. Let Y be a degree $s + 1$ subcurve of X_1. By the minimality of s either $\dim(\langle Y \rangle) = s + 1$ or $\dim(\langle Y \rangle) = s + 2$. We saw that $Y \cap H$ spans A_1. For every line $T \subset Y$ choose a general $P \in T$ and call B the union of these $s + 1$ points. Since $\dim(\langle Y \rangle) \geq s + 1$ and the points of B are sufficiently general, we have $\dim(\langle B \rangle) = s$. By the generality of B a general hyperplane H of $\mathbf{P}^{s+2}$ containing B may be considered as a general hyperplane. Fix any general hyperplane H containing B and any line D of X_1 with D not in Y. By the definition of A_1 and X_1 and the generality of H we have $H \cap D \in \langle B \rangle$. Now move H among all hyperplanes containing B. We obtain $D \subseteq \langle B \rangle$. Thus $X_1 \backslash Y \subset \langle B \rangle$. Moving each point of B in the corresponding line of Y we easily obtain a contradiction. $\square$

3. Proof of Theorem 2

Proof of Theorem 2. Let $G(2, n{+}1)$ be the Grassmannian of all lines of $\mathbf{P}^n$ and $\mathcal{O}_{G(2,n+1)}(1)$ the positive generator of $\mathrm{Pic}(G(2, n + 2))$, i.e. the line bundle inducing the Plücker embedding of $G(2, n + 1)$. For any hyperplane $M \subset \mathbf{P}^n$ and any degree s hypersurface T of M, set $G(T, M) := \{D \in G(2, n + 1) : T \cap D \neq \emptyset\}$. $G(T, M)$ is the zero-locus of a non-zero section of $\mathcal{O}_{G(2,n+1)}(s)$. $G(2, n + 1)$ has degree $\binom{2n-2}{n}/(n - 1)$ with respect to the Plücker embedding and $\dim(G(2, n{+}1)) = 2n{-}2$. Thus Bezout's theorem implies that the intersection of at least $2n - 2$ hypersurfaces $G(T_i, M_i)$ either is infinite or it contains at most $\binom{2n-2}{n} s^{2n-2}/(n - 1)$ points. Taking as M_i all general hyperplanes of $\mathbf{P}^n$ and as T_i the corresponding degree s hypersurface containing at least b points of Y, we obtain the existence of an irreducible variety $N \subset \mathbf{P}^n$ with $\dim(N) \geq 2$, N union of lines, such that for a general hyperplane

H we have $h^0(H, \mathcal{I}_{N\cap H,H}(s)) \neq 0$ and $N \cap H$ contained in a degree s hypersurface of H containing $Y \cap H$. If $\mathrm{card}(N \cap Y \cap H) \leq b - \binom{2n-2}{n}s^{2n-2}/(n-1) - 1$, then we may iterate this construction. Since the intersection of all $G(T, M)$ has only finitely many irreducible components, after finitely many steps we find the ruled variety F with all the properties claimed by Theorem 2. □

4. Double structures

In this section we consider the linearly general position of general hyperplane sections of double structures on configurations of lines. The definition of linearly general position makes sense even for zero-dimensional subschemes of projective spaces ([1]).

Theorem 3. *Let $Z \subset \mathbf{P}^n$ be a non-degenerate purely one-dimensional locally Cohen-Macaulay scheme such that $Y := Z_{red}$ is a configuration of lines and for each line $T \subseteq Y$ the multiplicity of Z at a general point of T is one or two. Assume that a general hyperplane section of Y is in linearly general position in its linear span. Then the general hyperplane section of Z is in linearly general position in its linear span.*

Proof. Fix a general hyperplane H and assume that the result is not true. Let s be the minimal integer such that $1 \leq s < \dim(\langle Z\cap H\rangle)$ and there is an s-dimensional linear subspace V of H with $\mathrm{length}(Z \cap V) \geq s + 2$. By the minimality of s V is spanned by $V \cap Z$. Let $Y' \subseteq Y$ be the union of all lines of Y intersecting V and $Z' \subseteq Z$ the maximal locally Cohen-Macaulay subscheme of Z such that $Z'_{red} = Y'$. Since the general hyperplane section of Y is in linearly general position, we have $Z'\cap V \neq Y'\cap V$. By the generality of H, H is transversal to Y. Fix $P \in Y' \cap V$ such that Z' is not reduced at P, i.e. it is unreduced at a general point of the line T containing P, and the connected component of $Z \cap H$ supported by P is contained in V. Let A be the union of the connected components of the zero-dimensional scheme $Z \cap V$ supported by $Y \cap V\backslash\{P\}$. Hence $\mathrm{length}(W) = \mathrm{length}(Z \cap W) - 2$. Thus $\mathrm{length}(W \cup \{P\}) = \mathrm{length}(Z \cap V) - 1$. By the minimality of s we have $\dim(\langle W \cup \{P\}\rangle \geq s$ and hence $\langle W\cup\{P\}\rangle = V$. Now we move H among all hyperplanes containing M and call H' a general such hyperplane. Since H is general, the length two subscheme of $H'\cap Z$ supported by P must be contained in V. Hence, varying H' we see that V contains T. Since $V \subset H$ and $Z \cap H$ is a zero-dimensional scheme, we have obtained a contradiction. □

References

[1] Eisenbud, D.; Harris, J.: *Finite projective schemes in linearly general position.* J. Algebr. Geom. **1** (1992), 15–30. Zbl 0804.14002

[2] Fulton, W.: *Intersection Theory.* Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, Bd. 2, Springer-Verlag, Berlin Heidelberg New York 1984. Zbl 0541.14005

[3] Griffiths, Ph.; Harris, J.: *Principles of algebraic geometry.* Pure and Applied Mathematics, John Wiley & Sons, 1978. Zbl 0408.14001

[4] Harris, J.: *Curves in projective space.* With the collaboration of David Eisenbud. Les Presses de l'Université de Montréal, Montréal 1982. Zbl 0511.14014

[5] Rathmann, J.: *The uniform position principle for curves in characteristic p.* Math. Ann. **276** (1987), 565–579. Zbl 0595.14041

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