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On the Monotonicity of the Volume of Hyperbolic Convex Polyhedra

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Abstract. We give a proof of the monotonicity of the volume of nonobtuse-angled compact convex polyhedra in terms of their dihedral angles. More exactly we prove the following. Let P and Q be nonobtuse-angled compact convex polyhedra of the same simple combinatorial type in hyperbolic 3-space. If each (inner) dihedral angle of Q is at least as large as the corresponding (inner) dihedral angle of P, then the volume of P is at least as large as the volume of Q. Moreover, we extend this result to nonobtuse-angled hyperbolic simplices of any dimension.

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1. Introduction

A compact convex polyhedron in hyperbolic 3-space $\mathbb{H}^3$ is the convex hull of finitely many points with nonempty interior or equivalently is the bounded intersection of finitely many closed halfspaces with nonempty interior. The different dimensional faces of a compact convex polyhedron are called vertices, edges and faces. Any two faces meeting along an edge are called adjacent faces. Finally, a compact convex polyhedron in $\mathbb{H}^3$ is called nonobtuse-angled if the inner dihedral angles at all edges do not exceed $\frac{\pi}{2}$. In a sequence of highly influential papers Andreev gave a description of the geometry of nonobtuse-angled compact convex

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polyhedra in $\mathbb{H}^3$. (For more details on the Koebe-Andreev-Thurston theorem see for example [6].) In [3] he proved that the planes of any two nonadjacent faces of a nonobtuse-angled compact convex polyhedron do not intersect in $\mathbb{H}^3$. It is easy to see that the combinatorial type of any nonobtuse-angled compact convex polyhedron is simple that is at any vertex exactly 3 faces (resp. 3 edges) meet. More importantly it is shown in [1] that if P and Q are nonobtuse-angled compact convex polyhedra of the same simple combinatorial type in $\mathbb{H}^3$ such that the corresponding inner dihedral angles of P and Q are equal, then P and Q are congruent. All these results have been extended to higher dimensions by Andreev in [1] and [3]. However, the most striking result of Andreev is a 3-dimensional one. Namely, for a given simple combinatorial type of a polyhedron, not a tetrahedron, Andreev [1] determined necessary and sufficient conditions on the inner dihedral angles under which there exists a compact convex polyhedron with the given dihedral angles not greater than $\frac{\pi}{2}$ and with the given simple combinatorial type. ([2] extends this result to finite-volume convex polyhedra in $\mathbb{H}^3$.)

Based on this, in this note we prove the monotonicity of the volume of 3-dimensional nonobtuse-angled compact convex polyhedra in terms of their inner dihedral angles. More exactly we prove the following:

Theorem 1. *Let P and Q be nonobtuse-angled compact convex polyhedra of the same simple combinatorial type in $\mathbb{H}^3$. If each inner dihedral angle of Q is at least as large as the corresponding inner dihedral angle of P, then the volume of P is at least as large as the volume of Q.*

The author believes that the conclusion of Theorem 1 fails to hold if one drops the assumption that P and Q are nonobtuse-angled. However, it is highly possible that the following conjecture holds as a natural extension of Theorem 1 to higher dimensions. A proof of that, however, would require fundamentally new ideas as well.

Conjecture 1. *Let P and Q be nonobtuse-angled compact convex polytopes of the same simple combinatorial type in $\mathbb{H}^d$, $d \geq 4$. If each inner dihedral angle of Q is at least as large as the corresponding inner dihedral angle of P, then the d-dimensional hyperbolic volume of P is at least as large as that of Q.*

Remark 1. Theorem 2 of the proof of Theorem 1 shows that Conjecture 1 holds when P and Q are nonobtuse-angled hyperbolic simplices of any dimension.

Remark 2. It is not hard to see via proper limit procedure (for details see [2]) that Theorem 1 extends to nonobtuse-angled convex polyhedra of finite volume in $\mathbb{H}^3$.

2. Proof of Theorem 1

Case 1. P and Q are simplices.

Let X^n be the spherical, Euclidean or hyperbolic space $\mathbb{S}^n$, $\mathbb{E}^n$ or $\mathbb{H}^n$ of constant curvature $+1$, 0, -1, and of dimension $n \geq 2$. By an n-dimensional simplex Δ^n in X^n we mean a compact subset with nonempty interior which can be expressed as an intersection of $n+1$ closed halfspaces. (In case of spherical space we require that Δ^n lies on an open hemisphere.) Let $F_0, F_1, \dots, F_n$ be the $(n-1)$-dimensional faces of the simplex Δ^n. Each $(n-2)$-dimensional face can be described uniquely as an intersection $F_{ij} = F_i \cap F_j$. We will identify the collection of all inner dihedral angles of the simplex Δ^n with the symmetric matrix $\alpha = [\alpha_{ij}]$ where α_{ij} is the inner dihedral angle between F_i and F_j for $i \neq j$, and where the diagonal entries α_{ii} are set equal to π by definition. Then the Gram matrix $G(\Delta^n) = [g_{ij}(\Delta^n)]$ of the simplex $\Delta^n \subset X^n$ is the $(n+1) \times (n+1)$ symmetric matrix defined by $g_{ij}(\Delta^n) = -\cos \alpha_{ij}$. Note that all diagonal entries $g_{ii}(\Delta^n)$ are equal to one. Finally, let

$$\begin{aligned} G^n_+ &= \{G(\Delta^n) \mid \Delta^n \text{ is an } n\text{-dimensional simplex is } \mathbb{S}^n\}, \\ G^n_0 &= \{G(\Delta^n) \mid \Delta^n \text{ is an } n\text{-dimensional simplex in } \mathbb{E}^n\}, \\ G^n_- &= \{G(\Delta^n) \mid \Delta^n \text{ is an } n\text{-dimensional simplex in } \mathbb{H}^n\} \quad \text{and} \\ G^n &= G^n_+ \cup G^n_0 \cup G^n_-. \end{aligned}$$

The following lemma summarizes some of the major properties of the sets G^n_+, G^n_0, G^n_- and G^n that have been studied on several occasions including the papers of Coxeter [4], Milnor [7] and Vinberg [8].

Lemma 1. (1) *The determinant of $G(\Delta^n)$ is either positive or zero or negative depending on whether the simplex Δ^n is spherical or Euclidean or hyperbolic.*

(2) *G^n is a convex open set in $\mathbb{R}^N$ with $N = \frac{n(n+1)}{2}$. (Note that the affine space consisting of all symmetric unidiagonal $(n+1) \times (n+1)$ matrices has dimension $N = \frac{n(n+1)}{2}$.)*

(3) *G^n_0 is an $(N-1)$-dimensional topological cell that cuts G^n into two open subcells G^n_+ and G^n_-.*

(4) *G^n_+ (resp., $G^n_+ \cup G^n_0$) is a convex open (resp. convex closed) set in $\mathbb{R}^N$.*

We will need the following property for our proof of Theorem 1 that seems to be a new property of G^n_+ (resp., $G^n_+ \cup G^n_0$) not yet mentioned in the literature. It is useful to introduce the notations $\mathbb{R}^N_{<0} = \{(x_1, x_2, \dots, x_N) \mid x_i < 0 \text{ for all } 1 \leq i \leq N\}$ and $\mathbb{R}^N_{\leq 0} = \{(x_1, x_2, \dots, x_N) \mid x_i \leq 0 \text{ for all } 1 \leq i \leq N\}$.

Lemma 2. *$G^n_+ \cap \mathbb{R}^N_{<0}$ (resp., $\left(G^n_+ \cup G^n_0\right) \cap \mathbb{R}^N_{\leq 0}$) is a convex corner i.e. if $g = (g_1, g_2, \dots, g_N) \in G^n_+ \cap \mathbb{R}^N_{<0}$ (resp., $g \in \left(G^n_+ \cup G^n_0\right) \cap \mathbb{R}^N_{\leq 0}$), then for any $g' = (g'_1, g'_2, \dots, g'_N)$ with $g_1 \leq g'_1 < 0, \dots, g_N \leq g'_N < 0$ (resp., $g_1 \leq g'_1 \leq 0, \dots, g_N \leq g'_N \leq 0$) we have that $g' \in G^n_+ \cap \mathbb{R}^N_{<0}$ (resp. $g' \in \left(G^n_+ \cup G^n_0\right) \cap \mathbb{R}^N_{\leq 0}$).*

Proof. Due to Lemma 1 it is sufficient to check the claim of Lemma 2 for the set $G^n_+ \cap \mathbb{R}^N_{<0}$ only.

Let $g = (g_1, g_2, \dots, g_N) \in G^n_+ \cap \mathbb{R}^N_{<0}$. Then it is sufficient to show that for any $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_N$ with $g_1 \leq \varepsilon_1 < 0, g_2 \leq \varepsilon_2 < 0, \dots, g_N \leq \varepsilon_N < 0$ we have that

$$\begin{aligned} g^1 &= (\varepsilon_1, g_2, \dots, g_N) \in G^n_+ \cap \mathbb{R}^N_{<0}, \\ g^2 &= (g_1, \varepsilon_2, g_3, \dots, g_N) \in G^n_+ \cap \mathbb{R}^N_{<0}, \\ &\;\;\vdots \\ g^N &= (g_1, \dots, g_{N-1}, \varepsilon_N) \in G^n_+ \cap \mathbb{R}^N_{<0}. \end{aligned} \tag{5}$$

(Namely, it is easy to see that (5) and the convexity of $G_+^n \cap \mathbb{R}^N_{<0}$ imply that $G_+^n \cap \mathbb{R}^N_{<0}$ is indeed a convex corner. Although it is not needed here, for the sake of completeness we note that the origin of $\mathbb{R}^N$ is in fact, an interior point of G_+^n.) Let Δ^n be the n-dimensional simplex of $\mathbb{S}^n$ whose Gram matrix $G(\Delta^n) = [g_{ij}(\Delta^n)]$ corresponds to $g = (g_1, g_2, \dots, g_N)$ i.e.

$$(g_1, g_2, \dots, g_N) = (-\cos\alpha_{01}, -\cos\alpha_{02}, \dots, -\cos\alpha_{0n}, -\cos\alpha_{12}, \dots, -\cos\alpha_{(n-1)n}).$$

As $g \in G_+^n \cap \mathbb{R}^N_{<0}$ we have that $0 < \alpha_{01} < \frac{\pi}{2}, 0 < \alpha_{02} < \frac{\pi}{2}, \dots, 0 < \alpha_{0n} < \frac{\pi}{2}, 0 < \alpha_{12} < \frac{\pi}{2}, \dots, 0 < \alpha_{(n-1)n} < \frac{\pi}{2}$. In order to show that $g^1 = (\varepsilon_1, g_2, \dots g_N) \in G_+^n \cap \mathbb{R}^N_{<0}$ we have to show the existence of an n-dimensional simplex Δ_1^n of $\mathbb{S}^n$ with dihedral angles $\arccos(-\varepsilon_1), \alpha_{02}, \dots, \alpha_{0n}, \alpha_{12}, \dots, \alpha_{(n-1)n}$. (As the task left for the remaining parts of (5) is the same we do not give details of that here.) We will show the existence of Δ_1^n via polarity. Let ${}^*\Delta^n = \{x \in \mathbb{S}^n \mid x \cdot y \leq 0 \text{ for all } y \in \Delta^n\}$ be the spherical polar of Δ^n, where $x \cdot y$ denotes the inner product of the unit vectors x and y. As it is well-known ${}^*\Delta^n$ is an n-dimensional simplex of $\mathbb{S}^n$ with edgelength $\pi - \alpha_{01}, \pi - \alpha_{02}, \dots, \pi - \alpha_{0n}, \pi - \alpha_{12}, \dots, \pi - \alpha_{(n-1)n}$ each being larger than $\frac{\pi}{2}$.

Let F be the $(n-2)$-dimensional face of ${}^*\Delta^n$ disjoint from the edge of length $\pi - \alpha_{01}$ of ${}^*\Delta^n$. Let v_0 and v_1 be the endpoints of the edge of length $\pi - \alpha_{01}$ of ${}^*\Delta^n$. By assumption $\frac{\pi}{2} < \pi - \arccos(-\varepsilon_1) \leq \pi - \alpha_{01} < \pi$. Now, rotate v_1 towards v_0 about the $(n-2)$-dimensional greatsphere $\mathbb{S}^{n-2}$ of F in $\mathbb{S}^n$ until the rotated image $\bar{v}_1$ of v_1 becomes a point of the $(n-1)$-dimensional greatsphere $\mathbb{S}^{n-1}$ of the facet of ${}^*\Delta^n$ disjoint from v_1. Obviously, the above rotation about $\mathbb{S}^{n-2}$ decreases the (spherical) distance v_0v_1 in a continuous way. We claim via continuity that there is a rotated image say, v_{01} of v_1 such that the spherical distance v_0v_{01} is equal to $\pi - \arccos(-\varepsilon_1)$. Namely, the $n+1$ points formed by $v_0, \bar{v}_1$ and the vertices of F all belong to an open hemisphere of $\mathbb{S}^{n-1}$ with the property that all pairwise spherical distances different from $v_0\bar{v}_1$ are larger than $\frac{\pi}{2}$. (Here we assume that v_0 and $\bar{v}_1$ are distinct since if they coincide, then the existence of v_{01} is trivial.) But, then a theorem of Davenport and Hajós [5] implies that $v_0\bar{v}_1 \leq \frac{\pi}{2}$ and so, the existence of v_{01} follows. Thus, the spherical polar of the n-dimensional simplex of $\mathbb{S}^n$ spanned by v_0, v_{01} and F gives us Δ_1^n. This completes the proof of Lemma 2. □

Now, we are in a position to show that $G_-^n \cap \mathbb{R}^N_{\leq 0}$ is monotone-path connected.

Lemma 3. *$G_-^n \cap \mathbb{R}^N_{\leq 0}$ is monotone-path connected in the following strong sense: if $g = (g_1, \dots, g_N) \in G_-^n \cap \mathbb{R}^N_{\leq 0}$ and $g' = (g'_1, \dots, g'_N) \in G_-^n \cap \mathbb{R}^N_{\leq 0}$ with $g'_1 \leq g_1, \dots, g'_N \leq g_N$, then $[\lambda g' + (1-\lambda)g] \in G_-^n \cap \mathbb{R}^N_{\leq 0}$ for all $0 \leq \lambda \leq 1$.*

Proof. Lemma 1 implies that $[\lambda g' + (1-\lambda)g] \in G^n$ for all $0 \leq \lambda \leq 1$ and so it is sufficient to prove that $[\lambda g' + (1-\lambda)g] \notin G_+^n \cup G_0^n$ for all $0 \leq \lambda \leq 1$. As $g \notin G_+^n \cup G_0^n$ and $G_+^n \cup G_0^n$ is convex moreover, $(G_+^n \cup G_0^n) \cap \mathbb{R}^N_{\leq 0}$ is a convex corner (Lemma 2) therefore there exists a supporting hyperplane H in $\mathbb{R}^N$ that touches $G_+^n \cup G_0^n$ at some point $h \in G_0^n \cap \mathbb{R}^N_{\leq 0}$ and is disjoint from g and separates g from $G_+^n \cup G_0^n$. In fact, using again the convex corner property of $(G_+^n \cup G_0^n) \cap \mathbb{R}^N_{\leq 0}$ we get that H separates $h + \mathbb{R}^N_{\leq 0}$ from $G_+^n \cup G_0^n$ and therefore H separates $g + \mathbb{R}^N_{\leq 0}$ from $G_+^n \cup G_0^n$ as well. Finally, notice that $g' \in g + \mathbb{R}^N_{\leq 0}$ and $g + \mathbb{R}^N_{\leq 0}$ is disjoint from H and therefore $g + \mathbb{R}^N_{\leq 0}$ is disjoint from $G_+^n \cup G_0^n$. This finishes the proof of Lemma 3. □

Now, we are ready to give a proof of the following volume monotonicity property of hyperbolic simplices.

Theorem 2. *Let P and Q be nonobtuse-angled n-dimensional hyperbolic simplices. If each inner dihedral angle of Q is at least as large as the corresponding inner dihedral angle of P, then the n-dimensional hyperbolic volume of P is at least as large as that of Q.*

Proof. By moving to the space of Gram matrices of n-dimensional hyperbolic simplices and then applying Lemma 3 we get that there exists a smooth one-parameter family $P(t)$, $0 \leq t \leq 1$ of nonobtuse-angled n-dimensional hyperbolic simplices with the property that $P(0) = P$ and $P(1) = Q$ moreover, if $\alpha_{01}(t), \alpha_{02}(t), \dots, \alpha_{0n}(t), \alpha_{12}(t), \dots, \alpha_{(n-1)n}(t)$ denote the inner dihedral angles of $P(t)$, then $\alpha_{ij}(t)$ is a monotone increasing function of t for all $0 \leq i < j \leq n$. Now, Schläfli's classical differential formula [7] yields that

$$\frac{d}{dt}\mathrm{Vol}_n(P(t)) = \frac{-1}{n-1} \sum_{0 \leq i < j \leq n} \mathrm{Vol}_{n-2}(F_{ij}(t)) \cdot \frac{d}{dt}\alpha_{ij}(t), \tag{6}$$

where $F_{ij}(t)$ denotes the $(n-2)$-dimensional face of $P(t)$ on which the dihedral angle $\alpha_{ij}(t)$ sits and $\mathrm{Vol}_n(\cdot)$, $\mathrm{Vol}_{n-2}(\cdot)$ refer to the corresponding dimensional volume measures. Thus, as $\frac{d}{dt}\alpha_{ij}(t) \geq 0$ (6) implies that $\frac{d}{dt}\mathrm{Vol}_n(P(t)) \leq 0$ and so indeed $P(0) \geq P(1)$, finishing the proof of Theorem 2. □

Case 2. The combinatorial type of P and Q is different from that of a tetrahedron.

First, recall the following classical theorem of Andreev [1].

Andreev Theorem. *A nonobtuse-angled compact convex polyhedron of a given simple combinatorial type, different from that of a tetrahedron and having given inner dihedral angles exists in $\mathbb{H}^3$ if and only if the following conditions are satisfied:*

(1) *if 3 faces meet at a vertex, then the sum of the inner dihedral angles between them is larger than π;*
(2) *if 3 faces are pairwise adjacent but, not concurrent, then the sum of the inner dihedral angles between them is smaller than π;*
(3) *if 4 faces are cyclically adjacent, then at least one of the dihedral angles between them is different from $\frac{\pi}{2}$;*
(4) *(for triangular prism only) one of the angles formed by the lateral faces with the bases must be different from $\frac{\pi}{2}$.*

Second, observe that Andreev theorem implies that the space of the inner dihedral angles of nonobtuse-angled compact convex polyhedra of a given combinatorial type different from that of a tetrahedron in $\mathbb{H}^3$ is an open convex set. As a result we get that if P and Q are given as in Theorem 1 and are different from a tetrahedron, then there exists a smooth one-parameter family $P(t)$, $0 \leq t \leq 1$ of nonobtuse-angled compact convex polyhedra of the same simple combinatorial type as of P and Q with the property that $P(0) = P$ and $P(1) = Q$ moreover, if $\alpha_E(t)$ denotes the inner dihedral angle of $P(t)$ which sits over the edge corresponding to the edge E of P, then $\alpha_E(t)$ is a monotone increasing function of t for all edges E of P. Applying Schläfli's differential formula [7] to the smooth one-parameter family $P(t)$ we get that

$$\frac{d}{dt}\mathrm{Vol}(P(t)) = -\frac{1}{2}\sum_{E_t} \text{length}\,(E_t)\cdot\frac{d}{dt}\alpha_E(t) \tag{7}$$

where E_t denotes the edge of $P(t)$ corresponding to the edge E of P and E (resp., E_t) runs over all edges of P (resp., $P(t)$). Hence, as $\frac{d}{dt}\alpha_E(t)\geq 0$ (7) implies that $\frac{d}{dt}\mathrm{Vol}(P(t))\leq 0$ and so indeed $P(0)\geq P(1)$, completing the proof of Theorem 1. $\square$

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