Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 46 (2005), No. 2, 363-376.

Extensions of Radical Operations to Fractional Ideals

Ali Benhissi

Department of Mathematics, Faculty of Sciences
5000 Monastir, Tunisia

0. Introduction

In this paper, A will denote a commutative ring with identity. The notion of radical operations is a natural generalization of the usual radical of ideals, it was introduced and studied by the author in [6] and [7]. In the first section of this paper, we study the $*$-primes; i.e., the prime ideals of A which are $*$-ideals. In the second section, we introduce radical operations of finite character and we prove that to any radical operation $*$, we can associate a radical operation of finite character $*_s$. Then $*_s$ is used to characterize the acc property for the $*$-ideals of A. Among many other things, we prove that if $*$ is of finite character and if each member in the class $\mathcal{P}$ of minimal $*$-primes over a given ideal is $*$-finite, then $\mathcal{P}$ is finite. In the third section, we associate to each radical operation $*$ on a domain A a multiplicative system N^* of the formal power series ring $A[[X]]$ and we study the quotient ring $A[[X]]_{N^*}$. The analog for the polynomial ring case is given in the fourth section. The fifth section is devoted to the Kronecker functions domain. In the rest of the paper, we extend the notion of radical operations to fractional ideals of integral domains. This generalization allows us to define the $*$-invertibility of ideals. We then give a characterization for a fractional ideal to be $*$-invertible. These results are applied to the contents of polynomials.

There are some similarities between radical operations and star operations. But the two concepts are very different. Indeed, the first notion concerns ideals in general commutative rings and the second deals with fractional ideals of integral domains. And even when we restrict ourselves to integral ideals of a domain the difference subsists. For example, in $\mathbb{Z}$, the usual radical of ideals is not a star operation and the v-operation is not a radical operation. The good references for star operations are the books of Jaffard [10], Gilmer [8] and Halter-Koch [9]. The paper [2] of D. D. Anderson deals with star operations satisfying the relation $(I \cap J)^* = I^* \cap J^*$, which makes them more close to radical operations. The paper [4] of D. F. Anderson concerns the $*$-invertibility for $*$-operations. The analog of this notion for radical operations will be introduced in our paper. It is natural that a good knowledge of star operations and their classical properties helps in the study of radical operations. In different places of this paper, we adapt many results coming essentially from Gilmer's book for general properties of radical operations and their associated Kronecker function domains, from the papers [11] of Kang for links between radical operations and polynomials ring, [3] of Anderson - Zafrullah for $*$-primes containing an ideal and [14] of Zafrullah for ACC on $*$-ideals.

1. Generalities on radical operations

1.1. Definition. *A radical operation on a ring A assigns to each ideal I of A an ideal I^* of A, subject of the following conditions*

(i) $I \subset I^*$; $I^{**} = I^*$.
(ii) $(I \cap J)^* = I^* \cap J^* = (IJ)^*$.

Examples. 1) For any radical operation $*$, since $A \subseteq A^* \subseteq A$, then $A^* = A$.

2) The trivial radical operation on a ring A is defined by $I^* = A$, for any ideal I.

1.2. Lemma. [6] *The following properties hold for each radical operation on the ring A and each pair of ideals I and J of A:*

(iii) $I \subseteq J \Longrightarrow I^* \subseteq J^*$.
(iv) $(I + J)^* = (I + J^*)^* = (I^* + J^*)^*$.
(v) $(IJ)^* = (IJ^*)^* = (I^*J^*)^*$.
(vi) $I^* = \sqrt{I^*}$.

1.3. Definition. *An ideal I is said a $*$-ideal if $I^* = I$.*

1.4. Lemma. *I is a $*$-ideal if and only if $I = J^*$, for some ideal J of A.*

Proof. $\Longrightarrow$ Clear.
$\Longleftarrow$ $I^* = J^{**} = J^* = I$.

1.5. Proposition. *For any family $(I_\alpha)_{\alpha\in\Lambda}$ of ideals of A, $\big(\sum_{\alpha\in\Lambda} I_\alpha\big)^* = \big(\sum_{\alpha\in\Lambda} I_\alpha^*\big)^*$ and $\bigcap_{\alpha\in\Lambda} I_\alpha^* = \big(\bigcap_{\alpha\in\Lambda} I_\alpha^*\big)^*$. If each I_α is a $*$-ideal, then so is $\bigcap_{\alpha\in\Lambda} I_\alpha$.*

Proof. (1) By (iii), for each $\beta \in \Lambda$, $I_\beta^* \subseteq \big(\sum_{\alpha\in\Lambda} I_\alpha\big)^*$; so $\sum_{\alpha\in\Lambda} I_\alpha^* \subseteq \big(\sum_{\alpha\in\Lambda} I_\alpha\big)^*$, and then $\big(\sum_{\alpha\in\Lambda} I_\alpha^*\big)^* \subseteq \big(\sum_{\alpha\in\Lambda} I_\alpha\big)^*$. The reverse inclusion is clear.
(2) For each $\beta \in \Lambda$, $\bigcap_{\alpha\in\Lambda} I_\alpha^* \subseteq I_\beta^*$; so $\big(\bigcap_{\alpha\in\Lambda} I_\alpha^*\big)^* \subseteq I_\beta^*$, and then $\big(\bigcap_{\alpha\in\Lambda} I_\alpha^*\big)^* \subseteq \bigcap_{\alpha\in\Lambda} I_\alpha^*$. The reverse inclusion is clear. The last assertion follows from the second equality.

Notation. A $*$-ideal which is prime is said to be a $*$-prime.

1.6. Corollary. *Let I be any ideal of A and P a $*$-prime containing I. Then P can be shrunk to a $*$-prime minimal among the $*$-primes containing I.*

Proof. The set $\mathcal{F}$ of the $*$-primes containing I is nonempty since $P \in \mathcal{F}$. It is inductive for the containment relation by the preceding proposition. A maximal element contained in P is the desired ideal.

Notation. A maximal element in the set of proper $*$-ideals of A is called $*$-maximal. We denote by $*$-Max A the set of $*$-maximal ideals of A.

1.7. Proposition. *Any $*$-maximal ideal is $*$-prime; i.e., $*-Max\ A \subseteq spec\ A$.*

Proof. Suppose that P is a $*$-maximal ideal of A which is not prime, and let $a, b \in A \setminus P$ such that $ab \in P$. Let $I = P + aA$ and $J = P + bA$. Then $P \subset I \subseteq I^*$ and $P \subset J \subseteq J^*$. By maximality, $I^* = J^* = A$; so $(IJ)^* = (I^*J^*)^* = (AA)^* = A^* = A$. On the other hand, $IJ = (P+aA)(P+bA) = P^2 + aP + bP + abA \subseteq P$; so $A = (IJ)^* \subseteq P^* = P$, then $A = P$, which is impossible.

2. Radical operations of finite character

2.1. Proposition. *For any radical operation $*$ on a ring A, the map $*_s$ defined by $I^{*_s} = \bigcup\{J^* : J$ finitely generated sub-ideal of $I\}$ is a radical operation on A. Moreover, $I^{*_s} \subseteq I^*$ and if I is a finitely generated ideal of A, then $I^{*_s} = I^*$.*

Proof. First of all, we prove that I^{*_s} is an ideal of A. Let $a \in A$ and $x, y \in I^{*_s}$. Then there are two finitely generated sub-ideals J and L of I such that $x \in J^*$ and $y \in L^*$. Thus $x + y \in J^* + L^* \subseteq (J + L)^* \subseteq I^{*_s}$ and $ax \in aJ^* \subseteq (aJ^*)^* = (aJ)^* \subseteq I^{*_s}$.

(i) It is clear that $I \subseteq I^{*_s}$ for any ideal I and if $J \subseteq L$ are ideals, then $J^{*_s} \subseteq L^{*_s}$. In particular, $I^{*_s} \subseteq I^{*_s *_s}$. For the reverse inclusion, let $J = (a_1, \ldots, a_n)$ be a finitely generated sub-ideal of I^{*_s}. For each i, there is a finitely generated sub-ideal J_i of I such that $a_i \in J_i^*$. Then $L = J_1 + \cdots + J_n$ is a finitely generated sub-ideal of I with $J \subseteq J_1^* + \cdots + J_n^* \subseteq (J_1 + \cdots + J_n)^* = L^* \subseteq I^{*_s}$, so $I^{*_s *_s} \subseteq I^{*_s}$.

(ii) Let I and J be ideals of A. Since $IJ \subseteq I \cap J$, then $(IJ)^{*_s} \subseteq (I \cap J)^{*_s}$. Since $I \cap J \subseteq I$, then $(I \cap J)^{*_s} \subseteq I^{*_s}$. By the same argument $(I \cap J)^{*_s} \subseteq J^{*_s}$; so $(I \cap J)^{*_s} \subseteq I^{*_s} \cap J^{*_s}$. We have only to prove that $I^{*_s} \cap J^{*_s} \subseteq (IJ)^{*_s}$. Let $x \in I^{*_s} \cap J^{*_s}$. There are a finitely generated sub-ideal I_1 of I and a finitely generated sub-ideal J_1 of J such that $x \in I_1^*$ and $x \in J_1^*$. Thus $x^2 \in I_1^* J_1^* \subseteq (I_1^* J_1^*)^* = (I_1 J_1)^*$, and hence $x \in \sqrt{(I_1 J_1)^*} = (I_1 J_1)^* \subseteq (IJ)^{*_s}$.

For the last assertion, if J is a finitely generated sub-ideal of an ideal I, then $J^* \subseteq I^*$. Thus $I^{*_s} \subseteq I^*$ and the reverse inclusion is true if I is finitely generated.

2.2. Definition. *A radical operation $*$ on a ring A is said to be of finite character if $*_s = *$; this means that for any ideal I of A, $I^* = \bigcup\{J^* :\ J$ finitely generated sub-ideal of*

$I\}$, which is also equivalent to saying that for any ideal I of A and any $x \in I^*$, there is a finitely generated sub-ideal J of I, such that $x \in J^*$.

Examples. 1) The trivial radical operation on any ring is of finite character.

2) For any radical operation $*$ on a ring A, the radical operation $*_s$ is of finite character. Indeed, if I is any ideal of A, $I^{*_s} = \bigcup\{J^* : J$ finitely generated sub-ideal of $I\}$, but since J is finitely generated, $J^* = J^{*_s}$. We call $*_s$ the radical operation of finite character associated to $*$.

3) The usual radical is of finite character.

2.3. Lemma. *Let $*$ be a radical operation of finite character on a ring A and $(I_\alpha)_{\alpha\in\Lambda}$ be a totally ordered family of ideals of A, then $\bigcup\limits_{\alpha\in\Lambda} I_\alpha^* = \Big(\bigcup\limits_{\alpha\in\Lambda} I_\alpha\Big)^*$. Moreover, if each I_α is a $*$-ideal, then so is $\bigcup\limits_{\alpha\in\Lambda} I_\alpha$.*

Proof. For each $\beta \in \Lambda$, $I_\beta^* \subseteq \Big(\bigcup\limits_{\alpha\in\Lambda} I_\alpha\Big)^*$; so $\bigcup\limits_{\alpha\in\Lambda} I_\alpha^* \subseteq \Big(\bigcup\limits_{\alpha\in\Lambda} I_\alpha\Big)^*$. For the reverse inclusion, let J be a finitely generated sub-ideal of $\bigcup\limits_{\alpha\in\Lambda} I_\alpha$, since the family is totally ordered, $J \subseteq I_{\alpha_0}$ for some $\alpha_0 \in \Lambda$, then $J^* \subseteq I_{\alpha_0}^* \subseteq \bigcup\limits_{\alpha\in\Lambda} I_\alpha^*$. Since $*$ is of finite character, then $\Big(\bigcup\limits_{\alpha\in\Lambda} I_\alpha\Big)^* \subseteq \bigcup\limits_{\alpha\in\Lambda} I_\alpha^*$.

2.4. Proposition. *Let $*$ be a non trivial radical operation of finite character on a ring A. Then any proper $*$-ideal of A is contained in a $*$-maximal ideal of A.*

Proof. Since $*$ is non trivial, the set $\mathcal{F}$ of proper $*$-ideals of A is nonempty. By the preceding lemma, it is inductive for the inclusion, since for any totally ordered family $(I_\alpha)_{\alpha\in\Lambda}$ of elements of $\mathcal{F}$, $\bigcup\limits_{\alpha\in\Lambda} I_\alpha$ is a proper $*$-ideal. Zorn's lemma applied.

Notation. Let $*$ be a radical operation on a ring A. An ideal I of A is $*$-finite if there is a finitely generated ideal F of A such that $I^* = (F)^*$. If $F \subseteq I$, we say that I is strictly $*$-finite. Note that if $*$ is of finite character, any $*$-finite ideal I is strictly $*$-finite. Indeed, let $F = (a_1, \dots, a_n)$ be such that $I^* = F^*$, then $a_i \in F_i^*$, with F_i a finitely generated sub-ideal of I, $1 \leq i \leq n$. Let $F' = F_1 + \cdots + F_n \subseteq I$, F' is finitely generated and $F \subseteq F_1^* + \cdots + F_n^* \subseteq (F')^*$. Since $I^* = F^* \subseteq (F')^{**} = (F')^* \subseteq I^*$, then $I^* = (F')^*$.

2.5. Proposition. *Let $*$ be a radical operation of finite character on a ring A and I a proper ideal of A. Let $\mathcal{P}$ be the class of minimal elements in the set of $*$-primes of A containing I. If each element of $\mathcal{P}$ is $*$-finite, then $\mathcal{P}$ is finite.*

Proof. Since $*$-primes containing I and I^* respectively are the same, we can suppose that I is a proper $*$-ideal. Let $\mathcal{S} = \{P_1 \dots P_n;\ n \in \mathbb{N}^*, P_i \in \mathcal{P}\}$. If there is some $C = P_1 \dots P_n \in \mathcal{S}$ such that $C \subseteq I$, then for each $P \in \mathcal{P}$, $P_i \subseteq P$, for some i, and by minimality, $P = P_i$; so $\mathcal{P} = \{P_1, \dots, P_n\}$ is finite. Hence we may assume $C \not\subseteq I$, for each $C \in \mathcal{S}$. The set $\mathcal{T}$ of all the $*$-ideals J of A containing I such that for each $C \in \mathcal{S}$, $C \not\subseteq J$, is then nonempty. It is inductive for the inclusion relation. Indeed, let (J_i) be a totally ordered family of elements of $\mathcal{T}$. By Lemma 2.3, $J = \bigcup J_i$ is a $*$-ideal of A containing I. Suppose that there is some $C = P_1 \dots P_n \in \mathcal{S}$ such that $C \subseteq J$. By the hypothesis, for each j, $1 \leq j \leq n$, there is some finite subset F_j of P_j such that $P_j = (F_j)^*$. Since $\big((F_1) \dots (F_n)\big)^* = \big((F_1)^* \dots (F_n)^*\big)^* = (P_1 \dots P_n)^* \subseteq J^* = J$, then $(F_1) \dots (F_n) \subseteq J_i$, for some i. So $C = P_1 \dots P_n \subseteq (P_1 \dots P_n)^* = \big((F_1) \dots (F_n)\big)^* \subseteq J_i^* = J_i$, which is impossible. Thus $J \in \mathcal{T}$. By Zorn's lemma, $\mathcal{T}$ admits a maximal element M. It is a $*$-ideal of A containing I and for each $C \in \mathcal{S}$, $C \not\subseteq M$. Suppose that M is not prime, there are $a, b \in A \setminus P$, such that $ab \in M$. By the maximality of M in $\mathcal{T}$, there are $P_1, \dots, P_n, Q_1, \dots, Q_s \in \mathcal{P}$ such that $P_1 \dots P_n \subseteq (M + aA)^*$ and $Q_1 \dots Q_s \subseteq (M + bA)^*$. So $P_1 \dots P_n Q_1 \dots Q_s \subseteq (M + aA)^* \cap (M + bA)^* = \big((M + aA)(M + bA)\big)^* = (M^2 + aM + bM + abA)^* \subseteq M^* = M$, which is impossible. By Corollary 1.6, there is $P \in \mathcal{P}$ such that $P \subseteq M$, but this contradicts the definition of $\mathcal{T}$.

2.6. Proposition. *If $*$ is a radical operation of finite character, then any minimal prime over a $*$-ideal is a $*$-prime.*

Proof. Let P be a minimal prime over the $*$-ideal I in the ring A. Then PA_P is the only minimal prime over the ideal IA_P in the ring A_P; so $\sqrt{IA_P} = PA_P$. If $x \in P$, then $x^n \in IA_P$, for some $n \in \mathbb{N}^*$; so $sx^n \in I$, for some $s \in A \setminus P$. If J is a finitely generated sub-ideal of P, we can find $n \in \mathbb{N}^*$ and $s \in A \setminus P$ such that $sJ^n \subseteq I$, then $s(J^n)^* \subseteq \big(s(J^n)^*\big)^* = (sJ^n)^* \subseteq I^* = I \subseteq P$. Since P is prime and $s \notin P$, then $(J^n)^* \subseteq P$. By (v), $(J^*)^n \subseteq \big((J^*)^n\big)^* = (J^n)^* \subseteq P$, then $J^* \subseteq P$. Since $*$ is of finite character, $P^* = \bigcup\{J^* :\ J \text{ finitely generated sub-ideal of } P\} \subseteq P$.

2.7. Proposition. *Let $*$ be a radical operation on a ring A and $*_s$ the radical operation of finite character associated. The following statements are equivalent:*

(1) *Each $*_s$-ideal of A is $*_s$-finite.*
(2) *Each $*_s$-prime ideal of A is $*_s$-finite.*
(3) *A satisfies the ascending chain condition on $*_s$-ideals.*
(4) *A satisfies the ascending chain condition on $*$-ideals.*
(5) *Each ideal of A is strictly $*$-finite.*

Moreover, if any of the above equivalent statements hold then $* = *_s$.

Proof. (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3) follow from [6; Lemma 3.4 and Corollary 3.6].

(3) $\Longrightarrow$ (4) Let $(I_n)_{n \in \mathbb{N}}$ be an ascending chain of $*$-ideals of A. For each $n \in \mathbb{N}$, $I_n \subseteq I_n^{*_s} \subseteq I_n^* = I_n$; so $I_n^{*_s} = I_n$. By (3), this chain is finite.

(4) $\Longrightarrow$ (5) Suppose that I is an ideal of A not strictly $*$-finite. If $a_1 \in I$, then $(a_1)^* \subset I^*$, there is $a_2 \in I \setminus (a_1)^*$; so $(a_1)^* \subset (a_1, a_2)^* \subset I^*$. If $a_3 \in (a_1, a_2)^* \setminus I$, then $(a_1)^* \subset$

$(a_1, a_2)^* \subset (a_1, a_2, a_3)^* \subset I^*, \ldots$. By this way, we construct an infinite ascending chain of $*$-ideals of A, which is impossible.

(5) $\Longrightarrow$ (1) Let I be a $*_s$-ideal of A. By (5), there is a finite subset F of I such that $I^* = (F)^*$. Since $I \subseteq I^* = (F)^* = (F)^{*_s} \subseteq I^{*_s} = I$, then $I = (F)^{*_s}$.
For the last statement, let I be any ideal of A, by (5), there is a finite subset F of I such that $I^* = (F)^*$. Since $I^* = (F)^* \subseteq I^{*_s} \subseteq I^*$, then $I^* = I^{*_s}$.

2.8. Remark. Since the ascending chain condition on $*$-ideals implies $* = *_s$, then the hypothesis (vii) in [6, Theorem 4.5] is superfluous.

3. The radical operations and the formal power series ring

Notation. If A is a ring and $f \in A[[X]]$, we denote by A_f the content of f; i.e., the ideal of A generated by the coefficients of f.

3.1. Definition. *Let $*$ be a radical operation on a ring A, we define $N^* = \{f \in A[[X]];\ A_f^* = A\}$.*

3.2. Proposition. *If $*$ is a radical operation of finite character on a ring A, then $N^* = A[[X]] \setminus \bigcup\{M[[X]];\ M \in *\text{-}Max\ A\}$.*

Proof. The result is clear for the trivial operation. Suppose $*$ not trivial and let $f \in A[[X]]$. By Proposition 2.4, $f \in N^* \Longleftrightarrow A_f^* = A \Longleftrightarrow \forall M \in *\text{-}Max\ A,\ A_f^* \nsubseteq M \Longleftrightarrow \forall M \in *\text{-}Max\ A,\ A_f \nsubseteq M \Longleftrightarrow \forall M \in *\text{-}Max\ A,\ f \notin M[[X]] \Longleftrightarrow f \in A[[X]] \setminus \bigcup\{M[[X]];\ M \in *\text{-}Max\ A\}$.

3.3. Proposition. *If $*$ is a radical operation of finite character on a ring A, then N^* is a saturated multiplicative subset of $A[[X]]$.*

Proof. By Proposition 1.7, any $M \in *\text{-}Max\ A$ is prime in A; so $M[[X]]$ is prime in $A[[X]]$. By Proposition 3.2, $N^* = A[[X]] \setminus \bigcup\{M[[X]];\ M \in *\text{-}Max\ A\}$ is a multiplicative subset of $A[[X]]$. Let f and $g \in A[[X]]$ such that $fg \in N^*$, then $A_{fg}^* = A$. Since $A_{fg} \subseteq A_f$, then $A = A_{fg}^* \subseteq A_f^* \subseteq A$; so $A_f^* = A$ and $f \in N^*$.

3.4. Proposition. *Let $*$ be a radical operation of finite character on a domain A and I an ideal of A. Then $IA[[X]]_{N^*} \cap A \subseteq I[[X]]_{N^*} \cap A \subseteq I^*$. Moreover, if I is a $*$-ideal, then $IA[[X]]_{N^*} \cap A = I[[X]]_{N^*} \cap A = I$.*

Proof. The first inclusion is clear. Let $x \in I[[X]]_{N^*} \cap A$, $x \in A$ and $x = \frac{f}{g}$, with $f \in I[[X]]$ and $g \in N^*$. Since $xg = f$, then $xA_g = A_{xg} = A_f$; so $(xA_g)^* = A_f^*$. By (v), $(xA_g^*)^* = A_f^*$; so $(xA)^* = A_f^* \subseteq I^*$ and $x \in I^*$. If I is a $*$-ideal, then $I^* = I \subseteq IA[[X]]_{N^*} \cap A$.

3.5. Lemma. [5] *Let A be a domain and S a multiplicative set of A. If P is a prime ideal of A disjoint from S, then $(A_S)_{PA_S} = A_P$.*

3.6. Proposition. *Let $*$ be a radical operation of finite character on a domain A and M a $*$-maximal ideal of A. Then $(A[[X]]_{N^*})_{M[[X]]_{N^*}} = A[[X]]_{M[[X]]}$.*

Proof. Since $N^* = A[[X]] \setminus \bigcup\{M'[[X]];\ M' \in *\text{-}Max\ A\}$ is a multiplicative subset of $A[[X]]$ disjoint with $M[[X]]$, we can use the preceding lemma.

4. The radical operations and the polynomials ring

4.1. Definition. *Let $*$ be a radical operation on a ring A, we define $\mathcal{N}^* = \{f \in A[X];\ A_f^* = A\}$.*

4.2. Proposition. *Let A be a ring, $*$ a radical operation on A and $*_s$ the radical operation of finite character associated. Then $\mathcal{N}^{*_s} = \mathcal{N}^*$ is a saturated multiplicative subset of $A[X]$.*

Proof. By Proposition 2.1, for any finitely generated ideal I of A, $I^{*_s} = I^*$, then $\mathcal{N}^{*_s} = \{f \in A[X];\ A_f^{*_s} = A\} = \{f \in A[X];\ A_f^* = A\} = \mathcal{N}^*$. By Proposition 3.3, $\mathcal{N}^{*_s}$ is a saturated multiplicative subset of $A[X]$. We can also furnish a direct proof based on the Dedekind-Mertens lemma [13]. Let $f, g \in \mathcal{N}^*$, then $A_f^* = A_g^* = A$. There is $m \in \mathbb{N}^*$ such that $A_f^m A_{fg} = A_f^{m+1} A_g$; so $(A_f^m A_{fg})^* = (A_f^{m+1} A_g)^*$. By (v), $\big((A_f^*)^m A_{fg}\big)^* = \big((A_f^*)^{m+1} A_g\big)^*$, then $A_{fg}^* = A_g^* = A$.

4.3. Proposition. *Let $*$ be a radical operation on a ring A and $*_s$ the associated radical operation of finite character, then $\mathcal{N}^* = A[X] \setminus \bigcup\{M[X]; M \in *_s\text{-}Max\ A\}$.*

Proof. Since $*_s$ is of finite character, we can use Proposition 3.2.

4.4. Proposition. *Let $*$ be a radical operation on a domain A and I an ideal of A, then $I[X]_{\mathcal{N}^*} \cap A \subseteq I^*$. Moreover, if I is a $*$-ideal, then $I[X]_{\mathcal{N}^*} \cap A = I$.*

Proof. Let $x \in I[X]_{\mathcal{N}^*} \cap A$, $x \in A$ and $x = \frac{f}{g}$, with $f \in I[X]$ and $g \in \mathcal{N}^*$. Since $xg = f$, then $xA_g = A_{xg} = A_f$; so $(xA_g)^* = A_f^*$. By (v), $(xA_g^*)^* = A_f^*$; so $(xA)^* = A_f^* \subseteq I^*$ and $x \in I^*$. If I is a $*$-ideal, then $I^* = I \subseteq I[X]_{\mathcal{N}^*} \cap A$.

4.5. Corollary. *Let $*$ be a radical operation on a domain A, $*_s$ the associated radical operation of finite character and I a $*_s$-ideal of A, then $I[X]_{\mathcal{N}^*} \cap A = I$.*

Proof. By the preceding proposition, $I = I[X]_{\mathcal{N}^{*_s}} \cap A = I[X]_{\mathcal{N}^*} \cap A$.

4.6. Proposition. *Let $*$ be a radical operation on a domain A and $*_s$ the associated radical operation of finite character. Then $Max(A[X]_{\mathcal{N}^*}) = \{M[X]_{\mathcal{N}^*} :\ M \in *_s\text{-}Max\ A\}$.*

Proof. Let $P \in spec(A[X]_{\mathcal{N}^*})$, there is $Q \in spec(A[X])$ such that $Q \cap \mathcal{N}^* = \emptyset$ and $P = Q_{\mathcal{N}^*}$. The set I of the coefficients of elements in Q is an ideal of A. Suppose that $1 \in I^{*_s}$, there is a finitely generated ideal $J = (a_1, \ldots, a_k) \subseteq I$ such that $1 \in J^*$, a_i is a coefficient of some $f_i \in Q$, let $n_i = deg\ f_i$, $1 \leq i \leq k$, and $f = f_1 + X^{n_1+1} f_2 + X^{n_1+n_2+2} f_3 +$

$\cdots + X^{n_1+n_2+\cdots+n_{k-1}+k-1} f_k \in Q$. Since $a_1, \ldots, a_k$ are coefficients of f, then $A_f^* = A$; so $f \in \mathcal{N}^*$, which is impossible because $\mathcal{N}^* \cap Q = \emptyset$. By Proposition 2.4, there is $M \in *_s$-$Max\ A$ such that $I \subseteq I^{*_s} \subseteq M$, then $Q \subseteq M[X]$ and $P = Q_{\mathcal{N}^*} \subseteq M[X]_{\mathcal{N}^*}$. Since by Proposition 4.3, the $M[X]_{\mathcal{N}^*}$ are prime, to conclude that they are the maximal ideals of $A[X]_{\mathcal{N}^*}$, it is sufficient to prove that they are incomparable. But if $M[X]_{\mathcal{N}^*} \subseteq M'[X]_{\mathcal{N}^*}$, with $M, M' \in *_s$-$Max\ A$, by Corollary 4.5, $M \subseteq M'$. By maximality, $M = M'$ and $M[X]_{\mathcal{N}^*} = M'[X]_{\mathcal{N}^*}$.

4.7. Proposition. *Let $*$ be a radical operation on a domain A and $*_s$ the associated radical operation of finite character. Then for each $M \in *_s$-$aMax\ A$, $\left(A[X]_{\mathcal{N}^*}\right)_{M[X]_{\mathcal{N}^*}} = A[X]_{M[X]}$.*

Proof. By Proposition 4.3, $M[X] \cap \mathcal{N}^* = \emptyset$, the equality follows by Lemma 3.5.

5. Kronecker function domains

In this section, $*$ is a radical operation on a domain A, with quotient field K, satisfying the property: for any ideals I, J, L of A, with L finitely generated, the inclusion $(IL)^* \subseteq (JL)^*$ implies $I^* \subseteq J^*$.

5.1. Lemma. *If $f, g \in A[X]$, then $A_{fg}^* = (A_f A_g)^*$.*

Proof. By Dedekind-Mertens lemma [13], there is $m \in \mathbb{N}^*$ such that $A_f^m A_{fg} = A_f^{m+1} A_g$, then $(A_f^m A_{fg})^* = (A_f^{m+1} A_g)^*$; so $A_{fg}^* = (A_f A_g)^*$.

5.2. Theorem. *The set $A_* = \left\{ \frac{f}{g} :\ f, g \in A[X], g \neq 0, A_f^* \subseteq A_g^* \right\}$ is a Bezout overring of $A[X]$.*

Proof. First of all, we prove that A_* is well defined.; i.e., if $f, g, s, t \in A[X]$, with $gt \neq 0$, $\frac{f}{g} = \frac{s}{t}$ and $A_f^* \subseteq A_g^*$, then $A_s^* \subseteq A_t^*$. Since $ft = gs$, by the Lemma 5.1, $(A_g A_s)^* = A_{gs}^* = A_{ft}^* = (A_f A_t)^* = (A_f^* A_t)^* \subseteq (A_g^* A_t)^* = (A_g A_t)^*$; so $A_s^* \subseteq A_t^*$.
It is clear that $A[X] \subseteq A_* \subseteq K(X)$. We will prove that A_* is a sub ring of $K(X)$, let $\frac{f}{g}, \frac{s}{t} \in A_*$, with $f, g, s, t \in A[X]$, $gt \neq 0$, $A_f^* \subseteq A_g^*$ and $A_s^* \subseteq A_t^*$, then $\frac{f}{g} - \frac{s}{t} = \frac{ft-gs}{gt}$ and $\frac{f}{g}\frac{s}{t} = \frac{fs}{gt}$. Since $A_{ft-gs} \subseteq A_{ft} + A_{gs}$, then $A_{ft-gs}^* \subseteq (A_{ft} + A_{gs})^* = (A_{ft}^* + A_{gs}^*)^* = \left((A_f A_t)^* + (A_g A_s)^*\right)^* = \left((A_f^* A_t^*)^* + (A_g^* A_s^*)^*\right)^* \subseteq \left((A_g^* A_t^*)^* + (A_g^* A_s^*)^*\right)^* = \left((A_g^* A_t^*)^*\right)^* = (A_g^* A_t^*)^* = (A_g A_t)^* = A_{gt}^*$. We have also $A_{fs}^* = (A_f A_s)^* = (A_f^* A_s^*)^* \subseteq (A_g^* A_t^*)^* = (A_g A_t)^* = A_{gt}^*$. Then $\frac{f}{g} - \frac{s}{t}, \frac{f}{g}\frac{s}{t} \in A_*$.
Let $\alpha = \frac{f}{h}$ and $\beta = \frac{g}{h} \in A_*$, with $f, g, h \in A[X]$, $h \neq 0$. Let $n > deg\ f$ an integer and $\gamma = \alpha + X^n \beta \in (\alpha, \beta)$. Since $\frac{\alpha}{\gamma} = \frac{f}{f+X^n g}$ and $\frac{\beta}{\gamma} = \frac{g}{f+X^n g} \in A_*$, because $A_f \subseteq A_{f+X^n g}$ and $A_g \subseteq A_{f+X^n g}$, then $(\alpha, \beta) = \gamma$ and A_* is a Bezout domain.

5.3. Proposition. *If V is a valuation overring of A_*, then V is the trivial extension of $V \cap K$ to $K(X)$.*

Proof. Let v the valuation associated with V and w its restriction to K. We show, for each $0 \neq f = a_0 + a_1X + \cdots + a_nX^n \in K[X]$, that $v(f) = min\{w(a_i) : \; 0 \leq i \leq n\}$. Since $X, X^{-1} \in A_* \subseteq V$, $v(X) = 0$; so $v(a_iX^i) = w(a_i)$ and $v(f) \geq min\{w(a_i) : \; 0 \leq i \leq n\}$. On the other hand, for $0 \leq i \leq n$, $(a_i) \subseteq A_f$; so $\frac{a_i}{f} \in A_* \subseteq V$, then $w(a_i) \geq v(f)$. It follows that $v(f) = min\{w(a_i) : \; 0 \leq i \leq n\}$.

6. Extension of radical operations to fractional ideals

Let A be a domain with quotient field K. A fractional ideal of A is a sub A-module I of K, such that $dI \subseteq A$ for some nonzero element $d \in A$. To avoid any confusion, when $I \subseteq A$, we say that I is an integral ideal. In the sequel, $\mathcal{F}(A)$ will be the set of fractional ideals of A. We will extend the notion of radical operations to $\mathcal{F}(A)$. Its restriction to integral ideals induces the usual radical operation. It turns out that by treating radical operations on fractional ideals, we achieve not only a simplicity of argument but also the introduction of new tools such as $*$-invertibility. The problem of how and when can a radical operation defined only on integral ideals be extended to fractional ideals is not considered in this paper.

6.1. Definition. *A radical operation on a domain A assigns to each element $I \in \mathcal{F}(A)$ an element $I^* \in \mathcal{F}(A)$ such that the set of integral ideals is closed under $*$ and the following conditions are satisfied for each $I, J \in \mathcal{F}(A)$:*
(i) $I \subset I^*$; $I^{**} = I^*$.
(ii) $(I \cap J)^* = I^* \cap J^* = (IJ)^*$.

Note that the restriction of $*$ to the integral ideals is a radical operation on A in the usual sense. The following lemma can be proved as in [6].

6.2. Lemma. *The following properties hold for each radical operation on the domain A and each pair of elements $I, J \in \mathcal{F}(A)$:*
(iii) $I \subseteq J \Longrightarrow I^* \subseteq J^*$.
(iv) $(I + J)^* = (I + J^*)^* = (I^* + J^*)^*$.
(v) $(IJ)^* = (IJ^*)^* = (I^*J^*)^*$.
(vi) *I^* is strongly radical; i.e, if $x \in K$ and $x^n \in I^*$ for some $n \in \mathbb{N}^*$, then $x \in I^*$.*

The following result can be proved in the same way as Proposition 2.1.

6.3. Proposition. *For any radical operation $*$ on a domain A, the map $*_s$ defined by $I^{*_s} = \bigcup\{J^* : \; J$ finitely generated sub-fractional ideal of $I\}$ is a radical operation on A. Moreover, $I^{*_s} \subseteq I^*$ and if I is a finitely generated fractional ideal of A, then $I^{*_s} = I^*$. Also $*_s$ is of finite character.*

7. The $*$-invertibility of fractional ideals

7.1. Definition. *Let $*$ be a radical operation on a domain A and $I \in \mathcal{F}(A)$. We say that I is $*$-invertible if $(IJ)^* = A$ for some $J \in \mathcal{F}(A)$.*

7.2. Proposition. *Let $*$ be a radical operation on a domain A and $I \in \mathcal{F}(A)$. Then I is $*$-invertible if and only if I^* is $*$-invertible.*

Proof. I is $*$-invertible $\Longleftrightarrow (IJ)^* = A$ for some $J \in \mathcal{F}(A)$. Since by (v), $(IJ)^* = (I^*J)^*$, then we have the result.

Notation. If $I \in \mathcal{F}(A)$, then $I^{-1} = \{x \in K : xI \subseteq A\}$ is a fractional ideal of A. As usual, we put $(I^{-1})^{-1} = I_v$. Note that $II^{-1} \subseteq A$ and if the equality holds we say that I is invertible.

7.3. Proposition. *Let $*$ be a radical operation on a domain A and $I, J \in \mathcal{F}(A)$. If $(IJ)^* = A$, then $J^* = I^{-1}$.*

Proof. Since $IJ^* \subseteq (IJ^*)^* = (IJ)^* = A$, then $J^* \subseteq I^{-1}$. On the other hand, $I^{-1} \subseteq (I^{-1})^* = (I^{-1}A)^* = (I^{-1}(IJ)^*)^* = (I^{-1}IJ)^* \subseteq (AJ)^* = J^*$.

7.4. Corollary. *Let $*$ be a radical operation on a domain A and $I \in \mathcal{F}(A)$. If I is $*$-invertible, then I^{-1} is a $*$-ideal.*

Proof. If $(IJ)^* = A$, then $I^{-1} = J^*$ is a $*$-ideal.

7.5. Corollary. *Let $*$ be a radical operation on a domain A and $I \in \mathcal{F}(A)$. Then I is $*$-invertible if and only if $(II^{-1})^* = A$.*

Proof. If $(IJ)^* = A$, for some $J \in \mathcal{F}(A)$, then by Proposition 7.3, $I^{-1} = J^*$; so $A = (IJ)^* = (IJ^*)^* = (II^{-1})^*$.

7.6. Corollary. *Let $*$ be a radical operation on a domain A and $I \in \mathcal{F}(A)$. If I is $*$-invertible, then I^{-1} is $*$-invertible.*

7.7. Corollary. *Let $*$ be a radical operation on a domain A. Then for any $0 \neq x \in K$, $(x)^* = (x)$.*

Proof. Since $(\frac{1}{x})(x) = A$, then $\left((\frac{1}{x})(x)\right)^* = A$ and $(\frac{1}{x})$ is $*$-invertible. By Corollary 7.4, (x) is a $*$-ideal.

7.8. Corollary. *Let $*$ be a radical operation on a domain A and $I \in \mathcal{F}(A)$. If I is $*$-invertible, then $I^* = I_v$. In particular, if every nonzero fractional ideal is $*$-invertible, then $* = v$.*

Proof. Since $(II^{-1})^* = A$, by Proposition 7.3, $I^* = (I^{-1})^{-1} = I_v$.

7.9. Corollary. *Let $*$ be a radical operation on a domain A. If every nonzero fractional ideal is $*$-invertible, then A is completely integrally closed.*

Proof. Let $0 \neq x \in K$ an almost integral element over A and $I = (x^i;\ i \in \mathbb{N})$, then $xI \subseteq I \Longrightarrow xI + I = I \Longrightarrow (x,1)I = I \Longrightarrow \big((x,1)II^{-1}\big)^* = (II^{-1})^* = A \Longrightarrow \big((x,1)(II^{-1})^*\big)^* = A \Longrightarrow (x,1)^* = A \Longrightarrow x \in A$.

7.10. Lemma. *Let $*$ be a radical operation on a domain A, $*_s$ the associated radical operation of finite character and $I, J \in \mathcal{F}(A)$. Then $(IJ)^{*_s} = \big\{(I_0J_0)^* :\ I_0 \subseteq I,\ J_0 \subseteq J$ finitely generated fractional ideals $\big\}$.*

Proof. Let $L = (a_0, \dots, a_n)$ be a finitely generated sub-ideal of IJ, for $1 \leq i \leq n$, $a_i = b_{i,1}c_{i,1} + \cdots + b_{i,m_i}c_{i,m_i}$, with $b_{i,j} \in I$ and $c_{i,j} \in J$. If $I_0 = \big(b_{i,j},\ 1 \leq i \leq n,\ 1 \leq j \leq m_i\big) \subseteq I$ and $J_0 = \big(c_{i,j},\ 1 \leq i \leq n,\ 1 \leq j \leq m_i\big) \subseteq J$, then $L \subseteq I_0J_0$; so $L^* \subseteq (I_0J_0)^*$ and the first containment is proved. The second one is clear.

Example. If $*$ is a radical operation of finite character on A and $I, J \in \mathcal{F}(A)$, then $(IJ)^* = \big\{(I_0J_0)^* :\ I_0 \subseteq I,\ J_0 \subseteq J$ finitely generated fractional ideals$\big\}$.

We recall from [12, Theorems 58 and 59] that if a fractional ideal I is invertible, then it is finitely generated. Moreover, if A is local then I is principal.

7.11. Theorem. *Let $*$ be a radical operation of finite character on a domain A and $(0) \neq I \in \mathcal{F}(A)$. Then I is $*$-invertible if and only if it is $*$-finite and $*$-locally principal; i.e., for each $M \in *$-Max A, IA_M is a principal fractional ideal of A_M.*

Proof. $\Longrightarrow$ Let $J \in \mathcal{F}(A)$ such that $(IJ)^* = A$. By the preceding example, there are two finitely generated ideals $I_0 \subseteq I$ and $J_0 \subseteq J$ such that $1 \in (I_0J_0)^*$. Hence $A \subseteq (I_0J_0)^* \subseteq (IJ)^* = A$; so $(I_0J_0)^* = A$. By Proposition 7.3, $I_0^* = J_0^{-1}$ and $I^* = J^{-1}$. Since $J_0 \subseteq J$, then $J^{-1} \subseteq J_0^{-1}$; so $I^* \subseteq I_0^*$, but the reverse containment is clear; so $I^* = I_0^*$ and I is $*$-finitely generated.

Let $M \in *$-*Max* A, if $II^{-1} \subseteq M$, then $A = (II^{-1})^* \subseteq M^* = M$, so $M = A$, which is impossible because M is a prime ideal, by Proposition 1.7. Hence $II^{-1} \nsubseteq M$ and $II^{-1}A_M = (IA_M)(I^{-1}A_M) = A_M$. Since IA_M is invertible, it is principal, by the preceding remark.

$\Longleftarrow$ Since I is $*$-finite and $*$ is of finite character, there is a finitely generated fractional ideal $J \subseteq I$ such that $I^* = J^*$. Suppose that $(II^{-1})^* \neq A$, by Proposition 2.4, there is $M \in *$-*Max* A such that $(II^{-1})^* \subseteq M$. Since IA_M is principal, $IA_M = aA_M$, with $0 \neq a \in I$. Since J is finitely generated and $\frac{1}{a}J \subseteq \frac{1}{a}I \subseteq A_M$, then $\frac{s}{a}J \subseteq A$, for some $s \in A \setminus M$. Hence $\frac{s}{a}I \subseteq (\frac{s}{a}I)^* = (\frac{s}{a}I^*)^* = (\frac{s}{a}J^*)^* = (\frac{s}{a}J)^* \subseteq A^* = A$. We conclude that $\frac{s}{a} \in I^{-1}$; so $s \in aI^{-1} \subseteq II^{-1} \subseteq (II^{-1})^* \subseteq M$, which is impossible.

7.12. Corollary. *Let $*$ be a radical operation of finite character on a domain A and I a proper nonzero integral ideal of A. Let $\mathcal{P}$ be the class of minimal elements in the set of the $*$-primes of A containing I. If each element of $\mathcal{P}$ is $*$-invertible, then $\mathcal{P}$ is finite.*

Proof. By the preceding theorem, each $*$-invertible ideal is $*$-finite, we use Proposition 2.5.

8. Application of the $*$-invertibility to polynomial contents

Notation. For a domain A, the set $A(X) = \left\{\frac{f}{g};\ f, g \in A[X], A_g = A\right\}$ is an overring of $A[X]$.

8.1. Proposition. [5] *If P is a prime ideal of a domain A, then $A_P(X) = A[X]_{P[X]}$.*

8.2. Corollary. *Let A be a domain, $*$ a radical operation on A and $*_s$ the associated radical operation of finite character. Then for each $M \in *_s$-$Max\ A$, $A_M(X) = A[X]_{M[X]} = \left(A[X]_{\mathcal{N}^*}\right)_{M[X]_{\mathcal{N}^*}}$.*

Proof. The first equality follows from Proposition 8.1 and the second from Proposition 4.7.

8.3. Theorem. [1] *Let A be a domain and $0 \neq f \in A[X]$, the following assertions are equivalent:*

(1) *A_f is locally principal.*
(2) *$fA(X) = A_fA(X)$.*
(3) *$fA(X) = IA(X)$, for some integral ideal of A.*

8.4. Corollary. *Let A be a local domain and $0 \neq f \in A[X]$, the following assertions are equivalent:*

(1) *A_f is principal.*
(2) *$fA(X) = A_fA(X)$.*
(3) *$fA(X) = IA(X)$, for some integral ideal of A.*

8.5. Lemma. *Let $*$ be a radical operation of finite character on a domain A and $0 \neq f \in A[X]$. Then A_f is $*$-invertible if and only if $fA[X]_{\mathcal{N}^*} = A_fA[X]_{\mathcal{N}^*}$.*

Proof. Since A_f is finitely generated, by Theorem 7.11, A_f is $*$-invertible if and only if A_f is locally principal; i.e., for each $M \in *$-$Max\ A$, $(A_f)_M = (A_M)_f$ is a principal ideal. Since A_M is a local domain, by Corollary 8.4, $(A_M)_f$ is a principal ideal of the domain A_M if and only if $fA_M(X) = A_fA_M(X)$. By Corollary 8.2, $A_M(X) = \left(A[X]_{\mathcal{N}^*}\right)_{M[X]_{\mathcal{N}^*}}$; so the equality $fA_M(X) = A_fA_M(X)$ becomes $f\left(A[X]_{\mathcal{N}^*}\right)_{M[X]_{\mathcal{N}^*}} = A_f\left(A[X]_{\mathcal{N}^*}\right)_{M[X]_{\mathcal{N}^*}}$. But by Proposition 4.6, $Max(A[X]_{\mathcal{N}^*}) = \{M[X]_{\mathcal{N}^*};\ M \in *\text{-}Max\ A\}$, hence A_f is $*$-invertible if and only if $fA[X]_{\mathcal{N}^*} = A_fA[X]_{\mathcal{N}^*}$.

8.6. Theorem. *Let $*$ be a radical operation of finite character on a domain A and $0 \neq f \in A[X]$, the following assertions are equivalent:*

(1) *A_f is $*$-locally principal.*
(2) *$fA[X]_{\mathcal{N}^*} = A_fA[X]_{\mathcal{N}^*}$.*
(3) *$fA[X]_{\mathcal{N}^*} = IA[X]_{\mathcal{N}^*}$, for some integral ideal I of A.*

Proof. (1) $\Longrightarrow$ (2) By hypothesis, A_f is $*$-locally principal. Since A_f is finitely generated, it is $*$-finitely generated. By Theorem 7.11, A_f is $*$-invertible and by the preceding lemma, $fA[X]_{\mathcal{N}^*} = A_fA[X]_{\mathcal{N}^*}$.

$(2) \Longrightarrow (3)$ Take $I = A_f$.

$(3) \Longrightarrow (1)$ By Proposition 4.6, for any $M \in *\text{-}Max\ A$, $M[X]_{\mathcal{N}^*} \in spec\big(A[X]_{\mathcal{N}^*}\big)$. By localizing the equality $fA[X]_{\mathcal{N}^*} = IA[X]_{\mathcal{N}^*}$, we obtain $f\big(A[X]_{\mathcal{N}^*}\big)_{M[X]_{\mathcal{N}^*}} = I\big(A[X]_{\mathcal{N}^*}\big)_{M[X]_{\mathcal{N}^*}}$. By Corollary 8.2, $\big(A[X]_{\mathcal{N}^*}\big)_{M[X]_{\mathcal{N}^*}} = A_M(X)$; so $fA_M(X) = IA_M(X)$. By Corollary 8.4, $(A_M)_f = (A_f)_M$ is a principal ideal. Hence A_f is $*$-locally principal.

8.7. Lemma. *Let $*$ be a radical operation on a domain A and I a nonzero fractional ideal of A. Then $\big(I[X]_{\mathcal{N}^*}\big)^{-1} = I^{-1}[X]_{\mathcal{N}^*}$.*

Proof. $\supseteq$ Since $I^{-1}[X]_{\mathcal{N}^*}I[X]_{\mathcal{N}^*} = \big(I^{-1}[X]I[X]\big)_{\mathcal{N}^*} \subseteq (I^{-1}I)[X]_{\mathcal{N}^*} \subseteq A[X]_{\mathcal{N}^*}$, then $I^{-1}[X]_{\mathcal{N}^*} \subseteq \big(I[X]_{\mathcal{N}^*}\big)^{-1}$.

$\subseteq$ Let $u \in \big(I[X]_{\mathcal{N}^*}\big)^{-1}$, then $uI[X]_{\mathcal{N}^*} \subseteq A[X]_{\mathcal{N}^*}$; in particular, $uI \subseteq A[X]_{\mathcal{N}^*}$. Let $0 \neq a \in I$ be a fixed element, since $ua \in A[X]_{\mathcal{N}^*}$, then $u \in \frac{1}{a}A[X]_{\mathcal{N}^*} \subseteq K[X]_{\mathcal{N}^*}$, where K is the quotient field of A. Put $u = \frac{f}{h}$, with $f \in K[X]$ and $h \in \mathcal{N}^* \subseteq A[X]$, then $f = uh \in \big(I[X]_{\mathcal{N}^*}\big)^{-1}$; so $fI[X]_{\mathcal{N}^*} \subseteq A[X]_{\mathcal{N}^*}$ and in particular $fI \subseteq A[X]_{\mathcal{N}^*}$. For each $b \in I$, there is some $g \in \mathcal{N}^*$ such that $bfg \in A[X]$. By Dedekind-Mertens theorem [13], $A_g^m A_{bfg} = A_g^{m+1}A_{bf}$ for some $m \in \mathbb{N}^*$. Hence $\big(A_g^{m+1}A_{bf}\big)^* = \big(A_g^m A_{bfg}\big)^* \Longrightarrow \big((A_g^*)^{m+1}A_{bf}\big)^* = \big((A_g^*)^m A_{bfg}\big)^*$. Since $A_g^* = A$, then $A_{bf}^* = A_{bfg}^* \subseteq A^* = A$; in particular, $bA_f \subseteq A$, for each $b \in I$. Then $IA_f \subseteq A \Longrightarrow A_f \subseteq I^{-1} \Longrightarrow f \in I^{-1}[X] \Longrightarrow u = \frac{f}{h} \in I^{-1}[X]_{\mathcal{N}^*}$.

8.8. Theorem. *Let $*$ be a radical operation of finite character on a domain A and I a nonzero integral ideal of A. Then I is $*$-invertible in A if and only if $I[X]_{\mathcal{N}^*}$ is invertible in $A[X]_{\mathcal{N}^*}$.*

Proof. $\Longrightarrow$ Since $(II^{-1})^* = A$, for each $M \in *\text{-}Max(A)$, $II^{-1} \not\subseteq M$. If $(II^{-1})[X]_{\mathcal{N}^*} \subseteq M[X]_{\mathcal{N}^*}$, by Proposition 4.4, $II^{-1} \subseteq (II^{-1})[X]_{\mathcal{N}^*} \cap A \subseteq M[X]_{\mathcal{N}^*} \cap A = M$, which is impossible. But by Proposition 4.6, $Max(A[X]_{\mathcal{N}^*}) = \{M[X]_{\mathcal{N}^*};\ M \in *\text{-}Max\ A\}$, then $(II^{-1})[X]_{\mathcal{N}^*} = A[X]_{\mathcal{N}^*}$. By the preceding lemma, $A[X]_{\mathcal{N}^*} = I[X]_{\mathcal{N}^*}I^{-1}[X]_{\mathcal{N}^*} = I[X]_{\mathcal{N}^*}\big(I[X]_{\mathcal{N}^*}\big)^{-1}$.

$\Longleftarrow$ Since $I[X]_{\mathcal{N}^*}$ is invertible, by Lemma 8.7, $A[X]_{\mathcal{N}^*} = I[X]_{\mathcal{N}^*}\big(I[X]_{\mathcal{N}^*}\big)^{-1} = I[X]_{\mathcal{N}^*} I^{-1}[X]_{\mathcal{N}^*} = (II^{-1})[X]_{\mathcal{N}^*}$. By Proposition 4.4, $A = A[X]_{\mathcal{N}^*} \cap A = (II^{-1})[X]_{\mathcal{N}^*} \cap A \subseteq (II^{-1})^* \subseteq A$, hence $(II^{-1})^* = A$.

Acknowledgments. I thank the referee for several helpful suggestions. These suggestions have contributed to the improvement of this paper a great deal.

References

[1] Anderson, D. D.: *Some remarks on the ring $R(X)$*. Comment. Math. Univ. St. Pauli **26**(2) (1978), 137–140. Zbl 0373.13003

[2] Anderson, D. D.: *Star operations induced by overrings.* Commun. Algebra **16**(12) (1988), 2535–2553. Zbl 0672.13001

[3] Anderson, D. D.; Zafrullah, M.: *On t-invertibility, IV.* In: D. D. Anderson (ed.): Factorization in integral domains. Lect. Notes Pure Appl. Math. **189**, 221–225. Marcel Dekker 1997. Zbl 0882.13001

[4] Anderson, D. F.: *A general theory of class groups.* Commun. Algebra **16** (1988), 805–847. Zbl 0648.13002

[5] Arnold, J. T.: *On the ideal theory of the Kronecker function ring and the domain D(X).* Can. J. Math. **21** (1969), 558–563. Zbl 0181.05102

[6] Benhissi, A.: *Hilbert's basis theorem for $*$-radical ideals.* J. Pure Appl. Algebra **161**(2) (2001), 245–253. Zbl 0988.13001

[7] Benhissi, A.: *On radical operations.* In: M. Fontana; S. Kabbaj; Wiegand, S. (eds.): Commutative ring theory and applications. Lect. Notes Pure Appl. Math. **231**, 51–59. Marcel Dekker 2003. Zbl pre02084920

[8] Gilmer, R.: *Multiplicative ideal theory.* Pure and Applied Mathematics **12**. Marcel Dekker 1972. Zbl 0248.13001

[9] Halter-Koch, F.: *Ideal systems. An introduction to ideal theory.* Pure and Applied Mathematics **211**. Marcel Dekker 1998. Zbl 0953.13001

[10] Jaffard, P.: *Les systémes d'idéaux.* Dunod, Paris 1960. Zbl 0101.27502

[11] Kang, B. G.: *Prüfer v-multiplication domains and the ring $R[X]_{N_v}$.* J. Algebra **123**(1) (1989), 151–170. Zbl 0668.13002

[12] Kaplansky, I.: *Commutative rings.* Allyn and Bacon, Boston 1970. Zbl 0203.34601

[13] Northcott, D. G.: *A generalization of a theorem on the content of polynomials.* Proc. Camb. Philos. Soc. **55** (1959), 282–288. Zbl 0103.27102

[14] Zafrullah, M.: *Ascending chain conditions and star operations.* Commun. Algebra **17**(6) (1989), 1523–1533. Zbl 0691.13013

Received February 4, 2003