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The Maximum of the Smallest Maximal Coordinate of the Minimum Vectors in 6-lattices Equals 1

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Abstract. This paper is related to the question of Á. G. Horváth [3]: How to find a basis of any n-lattice in $\mathbb{E}^n$ such that the maximal coordinate belonging to the minima of this lattice are "small as possible". We prove that in the 6-dimensional case, in every lattice there exists a basis for which all the coordinates of the minima are $-1, 0, 1$.

Introduction

Let $\mathbb{E}^n\,(\mathbf{0}, \mathrm{V}^n\,(\mathbb{R}, \langle\ ,\ \rangle))$ be the Euclidean n-space with a distinguished origin $\mathbf{0}$, with the n-vector space V^n, over the real numbers $\mathbb{R}$, of the translation of $\mathbb{E}^n$ and with the positive definite symmetric scalar product $\langle\ ,\ \rangle : \mathrm{V}^n \times \mathrm{V}^n \to \mathbb{R}, (\mathbf{x}, \mathbf{y}) \to \langle \mathbf{x}, \mathbf{y}\rangle$. Let $\mathrm{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_n\} = \{\mathbf{a}_i\}$ be a basis of V^n with the Gramian $G := (a_{ij}) := (\langle \mathbf{a}_i, \mathbf{a}_j\rangle)$ that defines also the length $|\mathbf{v}|$ of any vector $\mathbf{v} := \sum_{i=1}^{n} v_i \mathbf{a}_i$ by $|\mathbf{v}| = \sqrt{\sum_{i,j=1}^{n} a_{ij} v_i v_j} = \sqrt{\mathbf{v}^T G \mathbf{v}}$.

A $\mathbb{Z}$-lattice to the basis A is defined as

$$\Lambda(\mathrm{A}, \mathbb{Z}) = [\mathbf{a}_1, \dots, \mathbf{a}_n] = \left\{ \sum_{i=1}^{n} x_i \mathbf{a}_i : x_i \in \mathbb{Z} \ \text{ for any } \ i = 1, 2, \dots, n \right\}.$$

As usual, $\mathbb{Z}$ is the set of integer numbers. The minimum $m(\Lambda)$ of the lattice Λ is defined by

$$m(\Lambda) \in \mathbb{R}^+ : m(\Lambda) = |\mathbf{m}| \le |\mathbf{v}| \ \text{for an} \ \mathbf{m} \in \Lambda \setminus \{\mathbf{0}\} =: \dot{\Lambda} \ \text{and for any} \ \mathbf{v} \in \dot{\Lambda}.$$

We may assume (by similarity of $\mathbb{E}^n$) that $m(\Lambda) = 1$. The set of minimum vectors is called minima of Λ and denoted by

$$\mathrm{M}(\Lambda) := \{\mathbf{m} \in \Lambda : |\mathbf{m}| =: m(\Lambda) =: 1\}.$$

The maximal A-coordinate of the minima of Λ is defined by

$$L\,(\mathrm{A}) := \max\left\{ x_i \in \mathbb{Z} : \sum_{i=1}^{n} x_i \mathbf{a}_i = \mathbf{m}, \mathbf{m} \in \mathrm{M}(\Lambda) \right\} \in \mathbb{N}.$$

Consider the minimum of these maximal A-coordinates of the minima of Λ by changing the basis A in Λ, i.e. define

$$L(\Lambda) := \min\{L(\mathrm{A}) \in \mathbb{N} : \mathrm{A} \text{ is any basis of} \Lambda\}.$$

Finally, vary the lattices Λ in $\mathbb{E}^n$. Then

$$L_n := L(\mathbb{E}^n) := \max\{L(\Lambda) \in \mathbb{N} : \Lambda \text{ is any lattice of } \Lambda \in \mathbb{E}^n\}$$

has to be determined as a "max-min-max-min problem". The problem is solved for $n \leq 5$ in [3] where the unique existence and increasing of L_n by n are also discussed. Our result is that $L_6 = 1$, our conjecture is that $L_7 = 1$, too. However, Á. G. Horváth conjectures that $L_8 = 2$, he has proved $L_8 > 1$ by the famous lattice E_8 in $\mathbb{E}^8$.

For the proof we need a clear strategy. In details: $L_n \geq L(\Lambda)$ for any lattice Λ in $\mathbb{E}^n$, however there is a lattice $\Lambda_0 \subset \mathbb{E}^n$ for which $L_n = L(\Lambda_0)$. $L(\Lambda_0) \leq L(\mathrm{A})$ for any basis A in Λ_0, but there is a basis $\mathrm{A}_0 \subset \Lambda_0$ for which $L(\Lambda_0) = L(\mathrm{A}_0)$. $L(\mathrm{A}_0) \geq x_i$ for any x_i where $\mathrm{A}_0 = \{\mathbf{a}_i\}$, $\sum\limits_{i=1}^{n} x_i \mathbf{a}_i =: \mathbf{m} \in \mathrm{M}(\Lambda_0)$, $1 = |\mathbf{m}|$ and $1 \leq |\mathbf{l}|$ for any $\mathbf{l} \in \Lambda_0$, but there is x_{ik} in a $\sum\limits_{i=1}^{n} x_{ik} \mathbf{a}_i =: \mathbf{m}_k \in \mathrm{M}(\Lambda_0)$ for which $L(\mathrm{A}_0) = x_{ik}$.

By Ryshkov's observation to the Minkowski reduction of lattice bases, we may assume that any lattice $\Lambda \subset \mathbb{E}^n$ considered has n independent minima of Λ. These minima $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ span a sublattice $\bar{\Lambda} \subset \Lambda$ such that Λ is an admissible extension of $\bar{\Lambda}$, i.e. $m(\Lambda) = m(\bar{\Lambda})$, the minimum does not decrease under the extension. Centering is also used instead of extension, but we use the term extension in this paper, because a centering – in the newer terminology of higher dimensional crystallography – keeps the symmetries of a geometric crystal class (rational, i.e. $\mathbb{Q}$-class) of the starting lattice. Our extension is not related with such a symmetry assumption. The index of the admissible extension is defined by the number $\mathrm{ind}\left(\Lambda/\bar{\Lambda}\right) = v\left(\bar{\Lambda}\right)/v\left(\Lambda\right)$, where $v\left(\Lambda\right)$ is the volume of a basic parallelepiped in the lattice Λ (see [3], [4], [5]). In other words the factor group $\Lambda \setminus \bar{\Lambda}$ is a finite Abelian group of order p and this order is called the index of the extension. Thus, studying the admissible extensions of lattices in $\mathbb{E}^n$ (as by S. S. Ryshkov [9] and N. V. Zaharova-Novikova [10] up to $n \leq 8$) and the basis changes of Λ, we shall solve our problem completely. Namely, S. S. Ryshkov's Theorem 10 from [9] guarantees these balanced bases. Moreover, Z. Major's concept of non-isomorphic lattice extension by finite Abelian groups makes a clearer situation for studying lattice bases with the optimally small maximal coordinate of all minimum lattice vectors [6].

1. The theorem and its proving strategy

Theorem 1. *L_6 is equal to one, i.e., speaking in the above sense, to every Euclidean 6-lattice Λ there is a basis in which the maximal coordinate of all the minimum vectors of Λ is equal to 1, at most.*

In a few words we sketch the strategy of the proof. In the following we suppose that $\Lambda \subset \mathbb{E}^6$ is a lattice possessing 6 independent minima according to the introduction. Let $\{\mathbf{a}_1, \dots, \mathbf{a}_6\}$ be a basis of the sublattice $\bar{\Lambda} \subset \Lambda$, where $\mathbf{a}_i$ are minimum vectors of length 1. The lattice Λ is an admissible extension of the lattice $\bar{\Lambda}$. Consider the lattice $\bar{\Lambda}$, for which $\operatorname{ind}\left(\Lambda/\bar{\Lambda}\right)$ is maximal. The Gramian of the lattice $\bar{\Lambda}$ is denoted by G.

Ryshkov [9] has classified the admissible extensions for $n = 6$, too. There are three admissible extensions with index 2, one with index 3 and one with index 4. The proof of the theorem is divided into three statements according to these indices. In Statement 1 we shall investigate the case in which the index of the admissible extension is four. Up to similarity only one lattice has an admissible extension with index four. This easily follows from [9], but we shall give a new proof for it by estimating the sum of the elements of the Gram matrix. Such estimations will be useful in other cases, too. In this lattice the characteristic matrix, i.e. the system of all different minima of the lattice, can be written easily. This matrix will be denoted by $[\mathbf{m}_1, \dots, \mathbf{m}_\sigma]$ (see [9], [3], [4]) where $\pm\mathbf{m}_1, \dots, \pm\mathbf{m}_\sigma$ are all different minima of the lattice. Finally, changing the basis of the lattice all elements of the characteristic matrix will be $0, -1, +1$, respectively.

In Statement 2 we study lattice extensions with index three and, by estimating the sum of the elements of the Gramian, we get very interesting conditions. In accordance with these conditions, we order the lattices into two classes. In the first case, writing all possible minima of each lattice of this class, we can change the basis of these lattices such that the coordinates of all possible minima of these lattices are $0, -1, +1$. In the second case, we prove that for any lattice Λ in this class $\mathrm{M}(\Lambda) \subseteq \mathrm{M}(E_6)$ holds, where $\mathrm{M}(\Lambda)$ denotes all the minima of Λ. Á. G. Horváth proved the following

Theorem 2. [4] $L(C_n) = L(A_n) = L(D_n) = L(E_6) = L(E_7) = 1$ *and* $L(E_8) = 2$.

In this theorem the lattices $C_n, A_n, D_n, E_6, E_7, E_8$ are the famous special lattices with a lot of minima (see [2], [4]). Using this statement and the properties of the lattice E_6, this case will also be solved. In the proof we give a new construction of E_6, namely, as the special extension of a lattice in the role of above $\bar{\Lambda}$. This construction stresses the geometric and symmetry properties of E_6. We remark that lattices with maximal symmetry in E^6 were given by W. Plesken and M. Pohst [8]. In the proof we use that the lattice E_6 has maximal minimum vectors and maximal symmetry.

In Statement 3 we examine admissible extensions with index two. The proof is divided into three parts, according to the three admissible extensions of index two. The basic idea of the proof is similar to that of [4]. In all the three cases we write all possible minima of every lattice in E^6 and we prove that, with respect to a suitable basis of Λ, the coordinates of the minima of are $\pm 1, 0$.

2. The proof

Statement 1. *Let $\Lambda \subset \mathbb{E}^6$ and $\bar{\Lambda} \subset \Lambda$ be any lattice having 6 independent minima. If* $\operatorname{ind}\left(\Lambda/\bar{\Lambda}\right) = 4$, *then with respect to a suitable basis of Λ, the coordinates of the minima of Λ are $\pm 1, 0$, respectively.*

Lemma 2.1. *If* $\operatorname{ind}\left(\Lambda/\bar{\Lambda}\right) = 4$, *then the lattice $\bar{\Lambda} = [\mathbf{a}_1, \dots, \mathbf{a}_6]$ is the 6-dimensional cube lattice and the minima of lattice Λ can be written with respect to the basis $\{\mathbf{a}_1, \dots, \mathbf{a}_6\}$ in the characteristic matrix form:*

$$\begin{bmatrix} \pm 1 & 0 & 0 & 0 & 0 & 0 & \pm\frac{1}{2} & 0 & \pm\frac{1}{2} \\ 0 & \pm 1 & 0 & 0 & 0 & 0 & \pm\frac{1}{2} & 0 & \pm\frac{1}{2} \\ 0 & 0 & \pm 1 & 0 & 0 & 0 & \pm\frac{1}{2} & \pm\frac{1}{2} & 0 \\ 0 & 0 & 0 & \pm 1 & 0 & 0 & \pm\frac{1}{2} & \pm\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \pm 1 & 0 & 0 & \pm\frac{1}{2} & \pm\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & \pm 1 & 0 & \pm\frac{1}{2} & \pm\frac{1}{2} \end{bmatrix}$$

A basis of the lattice Λ is $\{\mathbf{m}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5, \mathbf{m}_2\}$, where $\mathbf{m}_1\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0\right)^T$ and $\mathbf{m}_2\left(0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)^T$.

Proof. We shall distinguish two main cases.

1. Suppose that the lattice Λ has two sublattices $\Lambda_1^{d_1} \subset \Lambda$ and $\Lambda_2^{d_2} \subset \Lambda$, where any above dimension $d_j < 6$. Let $\overline{\Lambda_j^{d_j}} \subset \Lambda_j^{d_j}$ be such that the lattice $\Lambda_j^{d_j}$ is an admissible extension of the lattice $\overline{\Lambda_j^{d_j}}$ and $\operatorname{ind}\left(\Lambda_j^{d_j}/\overline{\Lambda_j^{d_j}}\right) = 2$. Then the dimension $d_j > 3$ so we have the following cases.

In the first subcase $d_1 = d_2 = 4$. The sublattices Λ_1^4 and Λ_2^4 are the well-known four-dimensional space-centered cubic lattices, see e.g. [9], and the 4-lattice $[\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4] = \overline{\Lambda_1^4} \subset \Lambda_1^4$ is the 4-dimensional cubic lattice. Thus $\mathbf{m}_1 = \frac{1}{2}\mathbf{a}_1 + \frac{1}{2}\mathbf{a}_2 + \frac{1}{2}\mathbf{a}_3 + \frac{1}{2}\mathbf{a}_4 \in \Lambda_1^4$. If $[\mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5, \mathbf{a}_6] = \overline{\Lambda_2^4} \subset \Lambda_2^4$, then $\mathbf{m}_2 = \frac{1}{2}\mathbf{a}_3 + \frac{1}{2}\mathbf{a}_4 + \frac{1}{2}\mathbf{a}_5 + \frac{1}{2}\mathbf{a}_6 \in \Lambda_2^4$. Thus $|\mathbf{m}_1 - \mathbf{m}_2| = 1$. The lattice Λ spanned by $\{\mathbf{m}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5, \mathbf{m}_2\}$ is an admissible extension of a 6-dimensional cubic lattice with index 4. The intersection of the lattice Λ with the space of sublattice $\overline{\Lambda_3^4} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_5, \mathbf{a}_6]$ is also a 4-dimensional space-centered cubic lattice, since $\frac{1}{2}\mathbf{a}_1 + \frac{1}{2}\mathbf{a}_2 + \frac{1}{2}\mathbf{a}_5 + \frac{1}{2}\mathbf{a}_6 = \mathbf{m}_1 + \mathbf{m}_2 - \mathbf{a}_3 - \mathbf{a}_4 \in \Lambda$. Thus $\mathbf{m}_3\left(\frac{1}{2}, \frac{1}{2}, 0, 0, \frac{1}{2}, \frac{1}{2}\right)^T \in \Lambda_3^4$. The minima of the lattice can be expressed with respect to the basis $\{\mathbf{a}_i\}$ like in the statement.

If $[\mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5] = \overline{\Lambda_4^4} \subset \Lambda_4^4$, then $\mathbf{m}_4 = \frac{1}{2}\mathbf{a}_2 + \frac{1}{2}\mathbf{a}_3 + \frac{1}{2}\mathbf{a}_4 + \frac{1}{2}\mathbf{a}_5 \in \Lambda_4^4$. Then $|\mathbf{m}_1 - \mathbf{m}_4| = \frac{\sqrt{2}}{2}$ would lead to a contradiction.

If $d_1 = 4$, $d_2 = 5$, then $\mathbf{m}_1\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0\right)^T \in \Lambda_1^4$ and $\mathbf{m}_2\left(0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)^T \in \Lambda_2^5$ hold in the basis $\{\mathbf{a}_i\}$, hence $|\mathbf{m}_1 - \mathbf{m}_2| < 1$. Analogously, if $d_1 = d_2 = 5$, then $\mathbf{m}_1\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0\right)^T \in \Lambda_1^5$ and $\mathbf{m}_2\left(0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)^T \in \Lambda_2^5$ hold in the basis $\{\mathbf{a}_i\}$, hence $|\mathbf{m}_1 - \mathbf{m}_2| < 1$. Both are impossible.

2. Suppose that the lattice Λ has a sublattice $\Lambda^d \subset \Lambda$ and Λ^d is an admissible extension of the lattice $\overline{\Lambda^d}$. It is well-known that there is no lattice Λ^d for which $d < 6$ and $\operatorname{ind}\left(\Lambda^d/\overline{\Lambda^d}\right) = 4$. (We can prove this by analogy with case $d = 6$.) Therefore, $d = 6$ and $\operatorname{ind}\left(\Lambda/\bar{\Lambda}\right) = 4$ hold

for lattice $\bar{\Lambda} \subset \Lambda$. Then there is a lattice vector $\mathbf{m} = \frac{1}{4}\mathbf{a}_1 + \frac{1}{4}\mathbf{a}_2 + \frac{1}{4}\mathbf{a}_3 + \frac{1}{4}\mathbf{a}_4 + \frac{1}{4}\mathbf{a}_5 + \frac{1}{4}\mathbf{a}_6 \in \Lambda$. Thus

$$|\mathbf{m}|^2 = \mathbf{m}^T G \mathbf{m} \geq 1$$

means

$$\frac{3}{8} + \frac{1}{8} \sum_{i<j} a_{ij} \geq 1,$$

hence

$$\sum_{i<j} a_{ij} \geq 5.$$

On the other hand, clearly $|\mathbf{m} - \mathbf{a}_k|^2 \geq 1$ holds, where $k = 1, \dots, 6$.

$$|\mathbf{m} - \mathbf{a}_1|^2 = \frac{7}{8} - \frac{3}{8} \sum_{j=2}^{6} a_{1j} + \frac{1}{8} \sum_{1<i<j} a_{ij} \geq 1,$$

$$\vdots$$

$$|\mathbf{m} - \mathbf{a}_6|^2 = \frac{7}{8} - \frac{3}{8} \sum_{i=1}^{5} a_{i6} + \frac{1}{8} \sum_{i<j<6} a_{ij} \geq 1.$$

Summing up these inequalities, we get

$$\frac{21}{4} - \frac{1}{4} \sum_{i<j} a_{ij} \geq 6,$$

hence

$$\sum_{i<j} a_{ij} \leq -3.$$

This is a contradiction. $\square$

Proof of Statement 1. It follows from Lemma 2.1, that the characteristic matrix of the minima of Λ can be expressed with respect to the basis $\{\mathbf{m}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, \mathbf{m}_2\}$ in the following way:

$$\begin{array}{c}
\begin{array}{rrrrrrrrrrrrrrr} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & m_1 & & & & & & & & m_2 \end{array} \\
\left[\begin{array}{rrrrrrrrrrrrrrr}
2 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 & 1 & 0 \\
-1 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 \\
-1 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & -1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right.
\end{array}$$

$$
\begin{array}{c}
\begin{array}{ccccccccccccccc}
 & & & & & & & m_3 & & & & & & & \\
\end{array}\\
\left.\begin{array}{ccccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 & 1\\
1 & 0 & 0 & -1 & 1 & 1 & 0 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 1\\
1 & 0 & -1 & 0 & 1 & 0 & 1 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 1\\
1 & -1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & -1 & 1\\
-1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & -1
\end{array}\right]
\end{array}
$$

Linear operation with the rows of the characteristic matrix is equivalent to a basis-change of the lattice (see [3], [4]). Adding the second row to the first one and the fifth row to the last one, we get that the elements of the characteristic matrix are $0, 1, -1$. Thus, in the suitable basis, the minima of the lattice can be written with coordinates $0, 1, -1$.

Statement 2. *Let $\Lambda \subset \mathbb{E}^6$ and $\bar{\Lambda} \subset \Lambda$ be any lattice possessing 6 independent minima. If $\operatorname{ind}\left(\Lambda/\bar{\Lambda}\right) = 3$, then with respect to a suitable basis of Λ, the coordinates of the minima of Λ are $\pm 1, 0$, respectively.*

In the following lemmas we investigate the properties of the 6-lattices with index 3. In Lemma 2.2 we estimate the sum of the elements of the Gram matrix. In Lemma 2.3 we describe the vectors that can be the minima of the lattice. By the last two lemmas we order the 6-lattices with index 3 into two types. Lattices of the first type do not contain a 4-dimensional space-centered cubic lattice. Those of the other type have a 4-dimensional space-centered cubic lattice.

Lemma 2.2. *Let $\{\mathbf{a_1}, \dots, \mathbf{a_6}\}$ of $\mathbb{E}^6$ be a basis of the lattice $\bar{\Lambda}$ and $|\mathbf{a}_i| = 1 = |\overrightarrow{OA_i}|$. Let $\operatorname{ind}\left(\Lambda/\bar{\Lambda}\right)$ be equal to 3. Consider the points $B_j\left(\frac{j}{6}, \dots, \frac{j}{6}\right)$ with respect to the basis $\mathbf{a}_i$, then $\overrightarrow{OB_2}, \overrightarrow{OB_4} \in \Lambda$. If $G = (a_{ij})$ is the Gram matrix of $\mathbf{a}_i$, then we have:*

$$
\frac{3}{2} \leq \sum_{i<j} a_{ij} \leq \frac{15}{4} \tag{1}
$$

and the length of the segment $\overrightarrow{A_iB_2}$ is equal to 1 (Fig. 1).

Proof. Let $\overrightarrow{OB_2} = \mathbf{b}_2\left(\frac{1}{3}, \dots, \frac{1}{3}\right)^T$. Clearly

$$
|\mathbf{b}_2|^2 = \mathbf{b}_2^T G \mathbf{b}_2 \geq 1
$$

holds. Thus

$$
\frac{2}{9}\sum_{i<j} a_{ij} + \frac{2}{3} \geq 1,
$$

$$
\sum_{i<j} a_{ij} \geq \frac{3}{2}\,. \tag{2}
$$

Consider now the points C_{kl} with $\overrightarrow{OC_{kl}} := \mathbf{a}_k + \mathbf{a}_l$, where $1 \le k < l \le 6$. $|\overrightarrow{C_{kl}B_2}|^2 \ge 1$ for every pair of indices k, l. Since the extension is admissible we have:

$$|\overrightarrow{C_{12}B_2}|^2 = \frac{8}{9}a_{12} - \frac{4}{9}\sum_{i=1}^{2}\sum_{j=3}^{6} a_{ij} + \frac{2}{9}\sum_{2<i<j} a_{ij} + \frac{4}{3} \ge 1.$$

$$\vdots$$

$$|\overrightarrow{C_{56}B_2}|^2 = \frac{8}{9}a_{56} - \frac{4}{9}\sum_{i=1}^{4}\sum_{j=5}^{6} a_{ij} + \frac{2}{9}\sum_{i<j<5} a_{ij} + \frac{4}{3} \ge 1.$$

Summing up the $\binom{6}{2} = 15$ inequalities $|C_{kl}B_2|^2 \ge 1$, we get the following:

$$-\frac{4}{3}\sum_{i<j} a_{ij} + \frac{4}{3}\cdot 15 \ge 15$$

$$\sum_{i<j} a_{ij} \le \frac{15}{4}. \tag{3}$$

The inequality (1) follows from (2) and (3).

Analogously, we have $|\overrightarrow{A_kB_2}|^2 \ge 1$ for every $k = 1, \dots, 6$.

$$|\overrightarrow{A_1B_2}|^2 = 1 - \frac{4}{9}\sum_{j=2}^{6} a_{1j} + \frac{2}{9}\sum_{1<i<j} a_{ij} \ge 1.$$

$$\vdots$$

$$|\overrightarrow{A_6B_2}|^2 = 1 - \frac{4}{9}\sum_{j=1}^{5} a_{6j} + \frac{2}{9}\sum_{i<j<6} a_{ij} \ge 1.$$

Again, by summing up the above inequalities we get

$$|\overrightarrow{A_1B_2}|^2 + \dots + |\overrightarrow{A_6B_2}|^2 = 6 + \left(-\frac{4}{9}\cdot 2 + \frac{2}{9}\cdot 4\right)\sum_{i<j} a_{ij} = 6 \ge 6.$$

The equality holds if and only if $|\overrightarrow{A_kB_2}|^2 = 1$ for every k, as we stated. □

Corollary. *The points A_i, $i = 1, \dots, 6$ and point B_1 above lie in the same 5-dimensional hyperplane. We will denote this hyperplane by H_1. Since $|OB_1| = |B_1B_2|$ and $|OA_i| = 1 = |A_iB_2|$, the hyperplane H_1 is perpendicular to the segment OB_2. We denote by H_i the hyperplane orthogonal to the line OB_i through B_i (Fig. 1).*

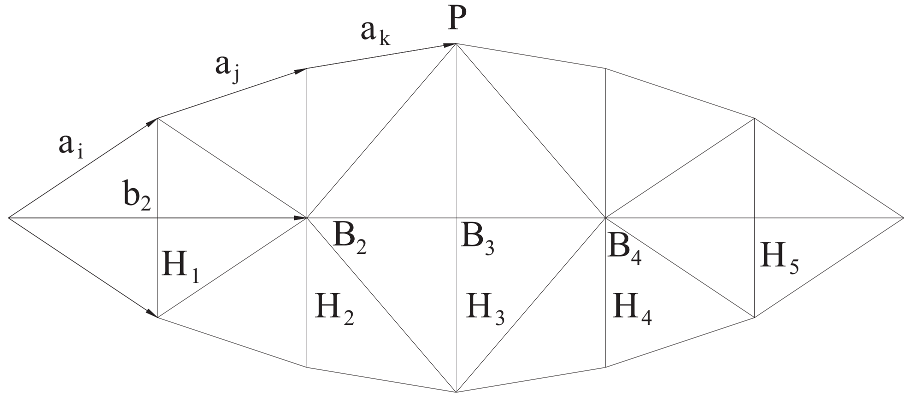

Figure 1

Lemma 2.3. *Let* $\{\mathbf{a_1},\dots,\mathbf{a_6}\}$ *of* $\mathbb{E}^6$ *be a basis of the lattice* $\bar{\Lambda}$, $|\mathbf{a}_i| = 1 = |\overrightarrow{OA_i}|$ *and consider that* $\operatorname{ind}\left(\Lambda/\bar{\Lambda}\right)$ *be equal to* 3. *Then the following vectors are the possible minima of the lattice* Λ*:* $\pm\mathbf{a}_i$, $\pm\mathbf{b}_2$, $\pm(\mathbf{a}_i-\mathbf{a}_j)$, $\pm(\mathbf{a}_i-\mathbf{b}_2)$, $\pm((\mathbf{a}_i+\mathbf{a}_j)-\mathbf{b}_2)$, $\pm((\mathbf{a}_i+\mathbf{a}_j+\mathbf{a}_k)-\mathbf{b}_2)$, *where* $i,j,k = 1,\dots,6$ *and* i,j,k *are pairwise distinct indices, as above* $\mathbf{b}_2$ *is* $\frac{1}{3}(\mathbf{a}_1+\mathbf{a}_2+\mathbf{a}_3+\mathbf{a}_4+\mathbf{a}_5+\mathbf{a}_6)$ *(Fig. 1).*

Proof. Let π be a parallelepiped spanned by the vectors $\{\mathbf{a_1},\dots,\mathbf{a_6}\}$. We show that the minima of the lattice $\bar{\Lambda}$ can be represented by vectors corresponding to certain diagonals and the edges of the parallelepiped π. In fact, if a minimum vector $\mathbf{m}$ from the origin properly intersects e.g. the facet of π parallel to that spanned by the vectors $\{\mathbf{a_1},\dots,\mathbf{a_5}\}$ of π, then $v(\pi') > v(\pi)$, where π' is the parallelepiped of $\{\mathbf{a_1},\dots,\mathbf{a_5},\mathbf{m}\}$. It would be $\operatorname{ind}(\Lambda/\Lambda') > 3$ for $\Lambda' = [\mathbf{a}_1,\dots,\mathbf{a}_5,\mathbf{m}]$. This would be a contradiction.

We will prove that the length of some diagonals of the parallelepiped π is greater than 1. So only other diagonals and edges can be the minima of the parallelepiped π, i.e. the minima of the lattice $\bar{\Lambda}$. It can easily be seen that the other minimum vectors of the lattice Λ can be $\pm\mathbf{b}_2$, $\pm(\mathbf{a}_i-\mathbf{b}_2)$, $\pm((\mathbf{a}_i+\mathbf{a}_j)-\mathbf{b}_2)$, $\pm((\mathbf{a}_i+\mathbf{a}_j+\mathbf{a}_k)-\mathbf{b}_2)$, where $i,j,k = 1,\dots,6$ and i,j,k are pairwise distinct indices.

We prove that $|(\mathbf{a}_i+\mathbf{a}_j)-(\mathbf{a}_k+\mathbf{a}_l)| > 1$, where $i,j,k,l = 1,\dots,6$ and i,j,k,l are pairwise distinct indices. Let $\mathbf{c}_{12} = \mathbf{a}_1+\mathbf{a}_2$, $\mathbf{c}_{34} = \mathbf{a}_3+\mathbf{a}_4$, $\mathbf{c}_{56} = \mathbf{a}_5+\mathbf{a}_6$. Clearly $|\mathbf{c}_{kl}| < 2$. As $\mathbf{c}_{12}+\mathbf{c}_{34}+\mathbf{c}_{56} = \overrightarrow{OB_6}$ with $\mathbf{c}_{kl} = \overrightarrow{OC_{kl}}$, $C_{kl} \in H_2$ and $H_2 \perp \overrightarrow{OB_6}$, the vectors $\mathbf{c}_{kl}$ can be written in the following form: $\mathbf{c}_{kl} = \mathbf{c}_{kl}^{\perp} + \mathbf{c}_{kl}^{\|}$, where $\mathbf{c}_{kl}^{\perp} = \overrightarrow{B_2C_{kl}} \in H_2$ and $\mathbf{c}_{kl}^{\|} = \overrightarrow{OB_2}$. Consequently, $\mathbf{c}_{12}^{\perp}+\mathbf{c}_{34}^{\perp}+\mathbf{c}_{56}^{\perp} = \mathbf{0}$, i.e. $\mathbf{c}_{56}^{\perp} = -\left(\mathbf{c}_{12}^{\perp}+\mathbf{c}_{34}^{\perp}\right)$ therefore $|\mathbf{c}_{56}^{\perp}| = |\mathbf{c}_{12}^{\perp}+\mathbf{c}_{34}^{\perp}|$. If we suppose that $|\mathbf{c}_{12}-\mathbf{c}_{34}| = 1$, then $|\mathbf{c}_{12}^{\perp}-\mathbf{c}_{34}^{\perp}| = 1$. As $|\mathbf{c}_{kl}^{\perp}| \geq 1$, $|\mathbf{c}_{56}^{\perp}| = |\mathbf{c}_{12}^{\perp}+\mathbf{c}_{34}^{\perp}| \geq \sqrt{3}$, thus inequality $|\mathbf{c}_{56}| \geq 2$ follows from $|\mathbf{c}_{56}^{\|}| \geq 1$. This is a contradiction, hence $|\mathbf{c}_{12}-\mathbf{c}_{34}| \neq 1$ and $|(\mathbf{a}_i+\mathbf{a}_j)-(\mathbf{a}_k+\mathbf{a}_l)| > 1$.

Finally, it can be seen that $|(\mathbf{a}_i+\mathbf{a}_j+\mathbf{a}_k)-(\mathbf{a}_l+\mathbf{a}_m+\mathbf{a}_n)| > 1$ where $i,j,k,l,m,n = 1,\dots,6$ and i,j,k,l,m,n are pairwise distinct indices. Denote $\mathbf{a}_1+\mathbf{a}_2+\mathbf{a}_3$ by $\mathbf{c}_{123}$ and $\mathbf{a}_4+\mathbf{a}_5+\mathbf{a}_6$ by $\mathbf{c}_{456}$. Suppose that $|\mathbf{c}_{123}-\mathbf{c}_{456}| = 1$. Since point C_{456} with $\overrightarrow{C_{456}B_6} = \mathbf{c}_{456}$ is the reflected image of C_{123} with $\overrightarrow{OC_{123}} = \mathbf{c}_{123}$ in the point B_3, we have $|\overrightarrow{B_3C_{123}}| = |\overrightarrow{B_3C_{456}}| = \frac{1}{2}$.

On the other hand,

$$|\overrightarrow{B_2B_3'}|^2 = |\overrightarrow{OB_1'}|^2 = |\mathbf{b}_1|^2 = \mathbf{b}_1^T G \mathbf{b}_1 = \frac{1}{18}\sum_{i<j} a_{ij} + \frac{1}{6}.$$

It follows from (3) that

$$\sum_{i<j} a_{ij} \leq \frac{15}{4}.$$

Hence

$$|\overrightarrow{B_2B_3'}|^2 \leq \frac{1}{18}\cdot\frac{15}{4} + \frac{1}{6} = \frac{3}{8}, \qquad \text{i.e.}$$

$$|\overrightarrow{B_2B_3'}| \leq \sqrt{\frac{3}{8}}.$$

Since $\overrightarrow{B_2B_3'}$ is perpendicular to $\overrightarrow{B_3'C_{123}}$, $|\overrightarrow{B_2C_{123}}| \leq \sqrt{\frac{5}{8}}$. This is impossible, because $|\overrightarrow{B_2C_{123}}| \geq 1$. This proves Lemma 2.3.

Lemma 2.4. *Consider the lattices Λ and $\bar{\Lambda}$ possessing 6 independent minima. Let $\{\mathbf{a_1}, \dots, \mathbf{a_6}\}$ of $\mathbb{E}^6$ be a basis of $\bar{\Lambda}$ and $|\mathbf{a}_i| = 1$. Let $\operatorname{ind}\left(\Lambda/\bar{\Lambda}\right)$ be equal to 3. If $|\overrightarrow{PB_2'}| > 1$ for every $P \in H_3$ for which $\overrightarrow{OP} \in \Lambda$, then the possible minima of lattice Λ can be found (in the basis $\{\mathbf{a}_i\}$ choosing an appropriate order of vectors $\{\mathbf{a}_i\}$) in the following blocks of coordinates:*

$$\begin{array}{c}
\begin{array}{ccccccccccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & b_2 & m_1 & m_2 & m_3 & m_4 & m_5 & m_6 \end{array} \\
\left[\begin{array}{ccccccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 1 & 0 & 0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{3} & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{3} & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & \frac{1}{3} & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & \frac{1}{3} & 0 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} & 0 & 0 & 0 & 0 & -1 & 1
\end{array}\right.
\end{array}$$

$$\begin{array}{c}
\begin{array}{cccccc} m_7 & \dots & m_{12} & m_{13} & m_{14} & \dots & m_{27} \end{array} \\
\left.\begin{array}{ccccccc}
-\frac{2}{3} & \dots & \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} & \dots & \frac{1}{3} \\
\frac{1}{3} & \dots & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & \dots & \frac{1}{3} \\
\frac{1}{3} & \dots & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & \dots & \frac{1}{3} \\
\frac{1}{3} & \dots & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \dots & \frac{1}{3} \\
\frac{1}{3} & \dots & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \dots & -\frac{2}{3} \\
\frac{1}{3} & \dots & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \dots & -\frac{2}{3}
\end{array}\right]
\end{array}$$

We remark that a special lattice Λ, in general, does not contain all types of the above minimum vectors, however every type of minimum vectors of a lattice Λ can be found among the above vectors.

Proof. Clearly, $\mathbf{a}_i$ are minimum vectors, where $i = 1, \dots, 6$. If $|\overrightarrow{OB_2}| = 1$, then $\mathbf{b}_2$ is also a minimum vector. First we prove that at most two elements a_{ij} of a row of the Gramian G can be equal to $\frac{1}{2}$. Suppose that in the first row we have three $\frac{1}{2}$ elements. By Lemma 2.2

$$|\overrightarrow{A_1B_2}|^2 = 1 - \frac{4}{9}(a_{12} + \cdots + a_{16}) + \frac{2}{9}(a_{23} + \cdots + a_{56}) = 1,$$

thus

$$2(a_{12} + \cdots + a_{16}) = (a_{23} + \cdots + a_{56}).$$

But we have $(a_{12} + \cdots + a_{16}) \geq \frac{3}{2}$ by assumption, therefore $(a_{23} + \cdots + a_{56}) \geq \frac{6}{2}$ and $(a_{12} + \cdots + a_{56}) \geq \frac{9}{2}$. This leads to a contradiction by the inequality (3) in Lemma 2.2. Therefore, the matrix G can be written in the following form:

$$G = \begin{bmatrix} 1 & \frac{1}{2} & a_{13} & a_{14} & a_{15} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} & a_{24} & a_{25} & a_{26} \\ a_{13} & \frac{1}{2} & 1 & \frac{1}{2} & a_{35} & a_{36} \\ a_{14} & a_{24} & \frac{1}{2} & 1 & \frac{1}{2} & a_{46} \\ a_{15} & a_{25} & a_{35} & \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & a_{26} & a_{36} & a_{46} & \frac{1}{2} & 1 \end{bmatrix} \tag{4}$$

(Indeed, if at most two elements a_{ij} of a row of the matrix G are equal to $\frac{1}{2}$ and G is not in the above form (4), then it can easily be seen that we get the above form by changing the order of vectors $\mathbf{a}_i$.) Thus $a_{12} = a_{23} = a_{34} = a_{45} = a_{56} = a_{16} = \frac{1}{2}$, therefore vectors $\mathbf{a}_1 - \mathbf{a}_2$, $\mathbf{a}_2 - \mathbf{a}_3$, $\mathbf{a}_3 - \mathbf{a}_4$, $\mathbf{a}_4 - \mathbf{a}_5$, $\mathbf{a}_5 - \mathbf{a}_6$, $\mathbf{a}_6 - \mathbf{a}_1$ are minima, too.

Secondly, by Lemma 2.2 $|A_iB_2| = 1$ in this lattice Λ, thus $\mathbf{m}_i$, where $i = 7, \dots, 12$ are minimum vectors. If by the notation of Lemma 2.2 $|C_{ij}B_2| = 1$ holds in a lattice Λ, then vectors $\mathbf{m}_i$, where $i = 13, \dots, 27$ are also minima of the lattice Λ. From Lemma 2.3 and the condition of $|\overrightarrow{PB_2}| > 1$ it is clear that other minimum vectors do not exist in Λ. We have verified the statement.

Lemma 2.5. *Considering the lattices Λ and $\bar{\Lambda}$ possessing 6 independent minima, let $\{\mathbf{a}_1, \dots, \mathbf{a}_6\}$ of $\mathbb{E}^6$ be a basis of $\bar{\Lambda}$ where $|\mathbf{a}_i| = 1$. Let $\operatorname{ind}(\Lambda/\bar{\Lambda})$ be equal to 3. If $\exists P$ for which $|\overrightarrow{PB_2}| = 1$, where $P \in H_3$ and $\overrightarrow{OP} \in \Lambda$, then $\mathrm{M}(\Lambda) \subseteq \mathrm{M}(E_6)$.*

Proof. Because $|B_2B_3| = |B_3B_4|$ and hyperplane H_3 is perpendicular to B_2B_4, $|PB_2| = |PB_4| = 1$. Let $\overrightarrow{PB_4} = \mathbf{m}_1$ and $\overrightarrow{B_2P} = \mathbf{m}_2$, respectively $\bar{\Lambda}_1^4 = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{m}_1]$, $\Lambda_1^4 = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{b}_2]$ $\bar{\Lambda}_2^4 = [\mathbf{a}_4, \mathbf{a}_5, \mathbf{a}_6, \mathbf{m}_2]$ and $\Lambda_2^4 = [\mathbf{a}_4, \mathbf{a}_5, \mathbf{a}_6, \mathbf{b}_2]$. If $\mathbf{p} = \mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3$, then $\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 + \mathbf{m}_1 = \frac{2}{3}(\mathbf{a}_1 + \cdots + \mathbf{a}_6) = 2\mathbf{b}_2$ and $\mathbf{a}_4 + \mathbf{a}_5 + \mathbf{a}_6 + \mathbf{m}_2 = \frac{2}{3}(\mathbf{a}_1 + \cdots + \mathbf{a}_6) = 2\mathbf{b}_2$. Hence $\operatorname{ind}(\Lambda_i^4/\bar{\Lambda}_i^4) = 2$, where $i = 1, 2$, namely lattices Λ_1^4 and Λ_2^4 are the well-known space-centered cubic lattices. Therefore the scalar products of vectors are: $\langle \mathbf{b}_2, \mathbf{a}_1 \rangle = \langle \mathbf{b}_2, \mathbf{a}_2 \rangle = \langle \mathbf{b}_2, \mathbf{a}_3 \rangle = \langle \mathbf{b}_2, \mathbf{m}_1 \rangle = \frac{1}{2}$, $\langle \mathbf{a}_1, \mathbf{a}_2 \rangle = \langle \mathbf{a}_1, \mathbf{a}_3 \rangle = \langle \mathbf{a}_1, \mathbf{m}_1 \rangle = \langle \mathbf{a}_2, \mathbf{a}_3 \rangle = \langle \mathbf{a}_2, \mathbf{m}_1 \rangle = \langle \mathbf{a}_3, \mathbf{m}_1 \rangle = 0$, resp. $\langle \mathbf{b}_2, \mathbf{a}_4 \rangle = \langle \mathbf{b}_2, \mathbf{a}_5 \rangle = \langle \mathbf{b}_2, \mathbf{a}_6 \rangle = \langle \mathbf{b}_2, \mathbf{m}_2 \rangle = \frac{1}{2}$, $\langle \mathbf{a}_4, \mathbf{a}_5 \rangle = \langle \mathbf{a}_4, \mathbf{a}_6 \rangle = \langle \mathbf{a}_4, \mathbf{m}_2 \rangle = \langle \mathbf{a}_5, \mathbf{a}_6 \rangle = \langle \mathbf{a}_5, \mathbf{m}_2 \rangle = \langle \mathbf{a}_6, \mathbf{m}_2 \rangle = 0$, $\langle \mathbf{b}_2, \mathbf{b}_2 \rangle = 1$. Multiplied the equalities $\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 + \mathbf{m}_1 =$

$2\mathbf{b}_2$ and $\mathbf{a}_4+\mathbf{a}_5+\mathbf{a}_6+\mathbf{m}_2=2\mathbf{b}_2$ consequently by $\mathbf{m}_1,\mathbf{m}_2,\mathbf{a}_1,\dots,\mathbf{a}_6$, we get the following form of the Gram matrix:

$$G=\begin{bmatrix} 1 & 0 & 0 & a_{14} & a_{15} & a_{16} \\ 0 & 1 & 0 & a_{24} & a_{25} & a_{26} \\ 0 & 0 & 1 & a_{34} & a_{35} & a_{36} \\ a_{14} & a_{24} & a_{34} & 1 & 0 & 0 \\ a_{15} & a_{25} & a_{35} & 0 & 1 & 0 \\ a_{16} & a_{26} & a_{36} & 0 & 0 & 1 \end{bmatrix},$$

where the following equalities hold:

$$\begin{aligned} a_{14}+a_{15}+a_{16} &= \tfrac{1}{2} \\ a_{24}+a_{25}+a_{26} &= \tfrac{1}{2} \\ a_{34}+a_{35}+a_{36} &= \tfrac{1}{2} \\ a_{14}+a_{24}+a_{34} &= \tfrac{1}{2} \\ a_{15}+a_{25}+a_{35} &= \tfrac{1}{2} \\ a_{16}+a_{26}+a_{36} &= \tfrac{1}{2}\,. \end{aligned} \tag{5}$$

It can easily be seen that the minimum vectors are in the lattice Λ: $\pm\mathbf{a}_i$, $\pm(\mathbf{a}_i-\mathbf{b}_2)$, where $i=1,\dots,6$, $\pm\mathbf{b}_2$, $\pm\mathbf{m}_1$, $\pm\mathbf{m}_2$, $\pm(\mathbf{a}_1+\mathbf{a}_2-\mathbf{b}_2)$, $\pm(\mathbf{a}_1+\mathbf{a}_3-\mathbf{b}_2)$, $\pm(\mathbf{a}_2+\mathbf{a}_3-\mathbf{b}_2)$, $\pm(\mathbf{a}_4+\mathbf{a}_5-\mathbf{b}_2)$, $\pm(\mathbf{a}_4+\mathbf{a}_6-\mathbf{b}_2)$, $\pm(\mathbf{a}_5+\mathbf{a}_6-\mathbf{b}_2)$. Thus, if there exists $\overrightarrow{OP}\in\Lambda$ such that $|\overrightarrow{PB_2}|=1$ and $P\in H_3$, then lattice Λ has these minimum vectors. Denote the set of these minimum vectors by $\mathrm{M}(\Lambda)$, as before.

Let $\mathbf{m}_3=\mathbf{a}_1+\mathbf{a}_2+\mathbf{a}_4-\mathbf{b}_2$ and $\mathbf{m}_4=\mathbf{a}_3+\mathbf{a}_5+\mathbf{a}_6-\mathbf{b}_2$, resp. $\bar\Lambda_3^4=[\mathbf{a}_3,\mathbf{a}_5,\mathbf{a}_6,\mathbf{m}_3]$, $\Lambda_3^4=[\mathbf{a}_3,\mathbf{a}_5,\mathbf{a}_6,\mathbf{b}_2]$, $\bar\Lambda_4^4=[\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_4,\mathbf{m}_4]$ and $\Lambda_4^4=[\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_4,\mathbf{b}_2]$. If $|\mathbf{m}_3|=|\mathbf{m}_4|=1$, then $\mathrm{ind}\left(\Lambda_i^4/\bar\Lambda_i^4\right)=2$, where $i=3,4$, namely lattices Λ_3^4 and Λ_4^4 are the well-known space-centered cubic lattices, too. So $a_{14}=a_{24}=a_{35}=a_{36}=0$ and by (5) we get that $a_{34}=\frac{1}{2}$, i.e. $|\mathbf{a}_3-\mathbf{a}_4|=1$. Therefore the new minimum vectors are: $\pm\mathbf{m}_3$, $\pm\mathbf{m}_4$, $\pm(\mathbf{a}_3-\mathbf{a}_4)$, $\pm(\mathbf{a}_1+\mathbf{a}_4-\mathbf{b}_2)$, $\pm(\mathbf{a}_2+\mathbf{a}_4-\mathbf{b}_2)$, $\pm(\mathbf{a}_3+\mathbf{a}_5-\mathbf{b}_2)$, $\pm(\mathbf{a}_3+\mathbf{a}_6-\mathbf{b}_2)$.

Let $\mathbf{m}_5=\mathbf{a}_1+\mathbf{a}_3+\mathbf{a}_5-\mathbf{b}_2$ and $\mathbf{m}_6=\mathbf{a}_2+\mathbf{a}_4+\mathbf{a}_6-\mathbf{b}_2$, resp. $\bar\Lambda_5^4=[\mathbf{a}_2,\mathbf{a}_4,\mathbf{a}_6,\mathbf{m}_5]$, $\Lambda_5^4=[\mathbf{a}_2,\mathbf{a}_4,\mathbf{a}_6,\mathbf{b}_2]$, $\bar\Lambda_6^4=[\mathbf{a}_1,\mathbf{a}_3,\mathbf{a}_5,\mathbf{m}_6]$ and $\Lambda_6^4=[\mathbf{a}_1,\mathbf{a}_3,\mathbf{a}_5,\mathbf{b}_2]$. If $|\mathbf{m}_5|=|\mathbf{m}_6|=1$, then $\mathrm{ind}\left(\Lambda_i^4/\bar\Lambda_i^4\right)=2$, where $i=5,6$, namely lattices Λ_5^4 and Λ_6^4 are the space-centered cubic 4-lattices, too. Hence $a_{15}=a_{26}=0$ and by (5) we get $a_{25}=\frac{1}{2}$ and finally $a_{16}=\frac{1}{2}$, i.e. $|\mathbf{a}_2-\mathbf{a}_5|=1$ and $|\mathbf{a}_1-\mathbf{a}_6|=1$. We remark that $|\mathbf{m}_7|=|\mathbf{m}_8|=1$ and Λ_7^4 and Λ_8^4 are the space-centered cubic 4-lattices, too, where $\mathbf{m}_7=\mathbf{a}_1+\mathbf{a}_4+\mathbf{a}_5-\mathbf{b}_2$, $\mathbf{m}_8=\mathbf{a}_2+\mathbf{a}_3+\mathbf{a}_6-\mathbf{b}_2$ and $\Lambda_7^4=[\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_6,\mathbf{b}_2]$, $\Lambda_8^4=[\mathbf{a}_1,\mathbf{a}_4,\mathbf{a}_5,\mathbf{b}_2]$. In this case the minimum vectors are: $\pm\mathbf{m}_5$, $\pm\mathbf{m}_6$, $\pm\mathbf{m}_7$, $\pm\mathbf{m}_8$, $\pm(\mathbf{a}_2-\mathbf{a}_5)$, $\pm(\mathbf{a}_1-\mathbf{a}_6)$, $\pm(\mathbf{a}_1+\mathbf{a}_5-\mathbf{b}_2)$, $\pm(\mathbf{a}_2+\mathbf{a}_6-\mathbf{b}_2)$. This lattice has 72 minimum vectors, therefore this is the lattice E_6.

Suppose that the vector $\mathbf{m}$ is a minimum of the lattice Λ, but not a minimum of the lattice E_6. So by Lemma 2.3 the vector $\mathbf{m}$ can be from among the vectors $\mathbf{a}_i-\mathbf{a}_j$ or $\mathbf{a}_i+\mathbf{a}_j-\mathbf{b}_2$ or $\mathbf{a}_i+\mathbf{a}_j+\mathbf{a}_k-\mathbf{b}_2$. On the other hand, by the assumption of the lemma we can prove that minima of Λ belong to the above set of minima of E_6. For example, if the minimum vector of Λ is of the from $\mathbf{m}=\mathbf{a}_i+\mathbf{a}_j+\mathbf{a}_k-\mathbf{b}_2$, where $i,j,k=1,\dots,6$, then we get $\mathbf{m}=\mathbf{m}_i$ or we have this form by changing the order of the vectors $\mathbf{a}_i$. If $\mathbf{m}=\mathbf{a}_i+\mathbf{a}_j-\mathbf{b}_2$ or $\mathbf{m}=\mathbf{a}_i-\mathbf{a}_j$, then it can be seen that by changing the order of the vectors $\mathbf{a}_i$ we get the above minimum vectors. It follows from (5) that there is no other minimum vector. The lemma has been proved.

Proof of Statement 2. If $\operatorname{ind}\left(\Lambda/\bar{\Lambda}\right)$ is equal to 3 and $|\overrightarrow{PB_2}| > 1$ for every $P \in H_3$, $\overrightarrow{OP} \in \Lambda$, then it follows from Lemma 2.4 that the possible minima of the lattice can be written with respect to the basis $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5, \mathbf{b}_2$ in the following form:

$$\begin{array}{c}
\begin{array}{ccccccccccccccccc}
e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & b_2 & m_1 & m_2 & m_3 & m_4 & m_5 & m_6 & m_7 & m_8 & m_9 & m_{10}
\end{array} \\
\left[\begin{array}{rrrrrrrrrrrrrrrrr}
1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & -2 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 & 1 & -1 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & 1 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & 1 & -1 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 3 & 1 & 0 & 0 & 0 & 0 & -3 & 3 & 0 & 1 & 1 & 1
\end{array}\right.
\end{array}$$

$$\left.\begin{array}{rrrrrrrrrrrrrrrrr}
0 & 1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & -1 & 0 & 0 & 0 & 1 & -1 & -1 & -1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 \\
-1 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 1 & 0 & -1 & 1 & -1 & 1 & 0 \\
1 & -2 & 1 & 1 & 1 & 1 & -2 & 1 & 1 & 1 & -2 & 1 & 1 & -2 & 1 & -2 & -2
\end{array}\right]$$

Adding the third and fourth rows to the last row and subtracting the third row from the first and fifth one, the elements of the characteristic matrix are $0, 1, -1$. Thus the coordinates of the minima of the lattice are $0, 1, -1$ in this basis.

If there exists $\overrightarrow{OP} \in \Lambda$ such that $|\overrightarrow{PB_2}| = 1$ and $P \in H_3$, then it follows from Lemma 2.5 that $\mathrm{M}(\Lambda) \subseteq \mathrm{M}(E_6)$, moreover, by Theorem 3 the minima of the lattice E_6 can be written with $0, 1, -1$ coordinates in a suitable basis. We have verified the statement.

Statement 3. *Let $\Lambda \subset \mathbb{E}^6$ and $\bar{\Lambda} \subset \Lambda$ be any lattice possessing* 6 *independent minima. If* $\operatorname{ind}\left(\Lambda/\bar{\Lambda}\right) = 2$, *then with respect to a suitable basis of Λ the coordinates of the minima of Λ are $\pm 1, 0$, respectively.*

Proof. In the paper of Á. G. Horváth [3] we can see a construction of a basis $A = \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ of an n-lattice for the minimum parallelepiped $\pi(\mathbf{m}_1, \dots, \mathbf{m}_n)$ with minimal volume which has the property that the edge vectors $\mathbf{m}_j = \sum\limits_{i=1}^{n} v_{ji}\mathbf{a}_i$ with coordinates v_{ji} satisfy the following inequalities:

$$\begin{array}{ll}
\text{(i)} & v_{jj} > 0, \quad j = 1, \dots, n \quad v_{ji} = 0 \ \text{ for } \ 1 \le j < i \le n \\
\text{(ii)} & 0 \le v_{ji} < v_{jj} \quad \text{for} \quad 1 \le i < j \le n.
\end{array}$$

With respect to this basis each of the minima can be expressed in the form

$$\mathbf{m}_l = \sum_{j=1}^{n} \alpha_{lj}\mathbf{m}_j = \sum_{i=1}^{n}\left(\sum_{j=i}^{n} \alpha_{lj} v_{ji}\right)\mathbf{a}_i, \qquad \text{where } 1 \le l \le \sigma.$$

Then the coordinates of the minimum vectors are:

$$m_{il} = \sum_{j=i}^{n} \alpha_{lj} v_{ji}. \tag{6}$$

Let now be $n = 6$. We distinguish three cases.

3.1. If the lattice Λ does not have a sublattice Λ^d $(d < 6)$ with index 2, then the characteristic matrix can be started in the following way:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & \dots \\ 0 & 1 & 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 1 & 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & 1 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & 1 & 1 & \dots \\ 0 & 0 & 0 & 0 & 0 & 2 & \dots \end{bmatrix}$$

There stand $v_{11} = \dots = v_{55} = v_{6i} = 1$ if $1 \le i < 6$, $v_{66} = 2$ and the other $v_{ji} = 0$ in equation (6). Therefore

$$m_{il} = \alpha_{li} + \alpha_{l6}, \qquad\qquad m_{6l} = 2\alpha_{l6}.$$

Assume that $m_{6l} \ge 0$.

First we investigate that $m_{6l} = 2$ hence $\alpha_{l6} = 1$. Because $|\alpha_{li}| \le 1$, hold $m_{il} = 0, 1, 2$, where $i = 1, \dots, 5$. If there is a zero among m_{il}, for example $m_{1l} = 0$, then the vectors $\mathbf{m}_2, \mathbf{m}_3, \mathbf{m}_4, \mathbf{m}_5, \mathbf{m}_l$ form a minimum system with index 2. This is a contradiction. If there is a 2 among m_{il}, where $i = 1, \dots, 5$, for example $m_{1l} = 2$, then the sublattice Λ^5 spanned by $\mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5, \mathbf{a}_1 + \mathbf{a}_6$ determines the minimum parallelepiped $\pi(\mathbf{m}_2, \mathbf{m}_3, \mathbf{m}_4, \mathbf{m}_5, \mathbf{m}_l)$ with volume 2. This case leads also to a contradiction. Hence if $m_{6l} = 2$, then $m_{il} = 1$, where $i = 1, \dots, 5$, namely $\mathbf{m}_l = \mathbf{m}_6$.

Secondly, we assume $m_{6l} = 1$ and $\alpha_{l6} = \frac{1}{2}$, then the coordinates m_{il}, where $i = 1, \dots, 5$, are equal to zero or one. Denote the matrix of these minimum vectors by A.

Finally, if $m_{6l} = 0$, then the coordinates m_{il}, where $i = 1, \dots, 5$, are $-1, 0, 1$. Let matrix A' contain these minimum vectors.

Thus the characteristic matrix can be written in the following block form:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \\ \\ A \\ \\ \\ \end{bmatrix} \begin{bmatrix} \\ \\ A' \\ \\ \\ \end{bmatrix}$$

Subtracting the first row from the last one, it can easily be seen that every element of the characteristic matrix is $-1, 0, 1$, so coordinates of the minima of the lattice are also $-1, 0, 1$ in a suitable basis.

3.2. If the lattice Λ has a sublattice Λ^5 with index 2, then the characteristic matrix can be written in the following blocks:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} & & \\ & & \\ & A & \\ & & \\ 0 & \dots & 0 \\ 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} & & \\ & & \\ & A' & \\ & & \\ 1 & \dots & 1 \\ 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} & & \\ & B & \\ & & \\ & & \\ & C & \\ 1 & \dots & 1 \end{bmatrix}$$

where elements of the matrix A are $-1, 0, 1$ and elements of A' are $0, 1$ (see [3]). Matrix B has four rows and C is a one-row vector. If an element of C is equal to 2 resp. -2, then C does not have any negative resp. positive coordinate. In fact, for example if $\mathbf{m}_i = (m_{1i}, m_{2i}, m_{3i}, m_{4i}, 2, 1)^T$ and $\mathbf{m}_j = (m_{1j}, m_{2j}, m_{3j}, m_{4j}, -1, 1)^T$, then

$$\det \begin{bmatrix} 1 & 0 & 0 & 0 & m_{1i} & m_{1j} \\ 0 & 1 & 0 & 0 & m_{2i} & m_{2j} \\ 0 & 0 & 1 & 0 & m_{3i} & m_{3j} \\ 0 & 0 & 0 & 1 & m_{4i} & m_{4j} \\ 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} = 3 > 2.$$

If an element of C is equal to 2 resp. -2, then subtracting the last row from, resp. adding the last row to the row containing C, elements of C become $-1, 0, 1$. Therefore, we may assume that the elements of C are $-1, 0, 1$. Let $\mathbf{m} = [x_1, x_2, x_3, x_4, x_5, 1]$ be a minimum vector, where coordinates x_1, x_2, x_3, x_4 are elements of a column of B and x_5 is equal to $-1, 0, 1$. Thus the volume of the parallelepiped $\pi = \pi(\mathbf{m}, \mathbf{m}_2, \mathbf{m}_3, \mathbf{m}_4, \mathbf{m}_5, \mathbf{m}_6)$ is of maximum two. Suppose that B has such an element (for example x_1) whose absolute value is greater then one. Hence

$$v(\pi) = \left|\det \begin{bmatrix} x_1 & 0 & 0 & 0 & 1 & 0 \\ x_2 & 1 & 0 & 0 & 1 & 0 \\ x_3 & 0 & 1 & 0 & 1 & 0 \\ x_4 & 0 & 0 & 1 & 1 & 0 \\ x_5 & 0 & 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\right| = |2x_1 - x_5| > 2,$$

this is a contradiction. For this reason the elements of B are $-1, 0, 1$. Thus every element of the characteristic matrix is $-1, 0, 1$ except of $m_{55} = 2$. The elements of the matrix A are $-1, 0, 1$ and the elements of A' are $0, 1$. Subtracting the first row from the fifth one, we get that the characteristic matrix has elements $-1, 0, 1$ except if $x_5 = -x_1 = \pm 1$. In this case

$$v(\pi) = |2x_1 - x_5| = |3x_1| = 3 > 2.$$

However, this case leads to a contradiction.

3.3. If the lattice Λ has a sublattice Λ^4 with index 2 (this sublattice is a 4-dimensional space-centered cubic lattice), then the characteristic matrix can be written in the following blocks:

$$\begin{bmatrix} 1&0&0&0&1&0&0&0&1&0&0&0&1&1&1\\ 0&1&0&0&1&0&0&0&0&1&0&1&0&1&1\\ 0&0&1&0&1&0&0&0&0&0&1&1&1&0&1\\ 0&0&0&1&2&0&0&1&1&1&1&1&1&1&1\\ 0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0&0&0&0 \end{bmatrix}$$

$$\begin{bmatrix} & A & \\ & & \\ & B & \\ 1 & \dots & 1 \\ 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} & C & \\ & & \\ & D & \\ 0 & \dots & 0 \\ 1 & \dots & 1 \end{bmatrix} \begin{bmatrix} & E & \\ & & \\ & F & \\ 1 & \dots & 1 \\ 1 & \dots & 1 \end{bmatrix}$$

where A, C, E have three rows and B, D, F are one-row vectors.

If an element of B, D or F is equal to 2 resp. -2, then B and F, D and F or F, B and D do not have any negative resp. positive coordinates. In fact, for example if $\mathbf{m}_i\,(m_{1i}, m_{2i}, m_{3i}, 2, 1, 0)^T$ and $\mathbf{m}_j\,(m_{1j}, m_{2j}, m_{3j}, -1, 1, 0)^T$, $\mathbf{m}_k\,(m_{1k}, m_{2k}, m_{3k}, -1, 1, 1)^T$, then

$$\det \begin{bmatrix} 1&0&0&0&m_{1i}&m_{1j}\\ 0&1&0&0&m_{2i}&m_{2j}\\ 0&0&1&0&m_{3i}&m_{3j}\\ 0&0&0&0&2&-1\\ 0&0&0&0&1&1\\ 0&0&0&1&0&0 \end{bmatrix} = \det \begin{bmatrix} 1&0&0&0&m_{1i}&m_{1k}\\ 0&1&0&0&m_{2i}&m_{2k}\\ 0&0&1&0&m_{3i}&m_{3k}\\ 0&0&0&0&2&-1\\ 0&0&0&0&1&1\\ 0&0&0&1&0&1 \end{bmatrix} = 3 > 2.$$

The proof is analogous if an element of D or F is equal to 2 resp. -2.

From this it follows that if an element of B is equal to 2 resp. -2 and an element of D is equal to -2 resp. 2, then F has 0 elements.

If an element of B and D is also equal to 2 resp. -2, then elements of F are 2 resp. -2. For example, let $\mathbf{m}_i\,(m_{1i}, m_{2i}, m_{3i}, 2, 1, 0)^T$, $\mathbf{m}_j\,(m_{1j}, m_{2j}, m_{3j}, 2, 0, 1)^T$, $\mathbf{m}_k\,(m_{1k}, m_{2k}, m_{3k}, x, 1, 1)^T$.

$$\det \begin{bmatrix} 1&0&0&m_{1i}&m_{1j}&m_{1k}\\ 0&1&0&m_{2i}&m_{2j}&m_{2k}\\ 0&0&1&m_{3i}&m_{3j}&m_{3k}\\ 0&0&0&2&2&x\\ 0&0&0&1&0&1\\ 0&0&0&0&1&1 \end{bmatrix} = |-4+x| \le 2$$

This inequality holds for $x = 2$. So every element of F is equal to 2.

We distinguish the following cases:

If B, D, F do not have an element -2. Suppose that B has an element 2, but D does not have an element 2. Then every element of F is $0, 1, 2$. Then subtracting the fifth row from the fourth row, we get that elements of B, D, F are $-1, 0, 1$. Analogously, if D has an element 2 but D does not have an element 2, then subtracting the sixth row from the fourth row, we get that elements of B, D, F are $-1, 0, 1$. If F has an element 2, but B and D do not have an element 2, then each element of B and D is $0, 1$. By subtracting the fifth row from the fourth row, we get that elements of B, D, F are $-1, 0, 1$. If B, D have an element 2, then every element of F is 2 and elements of B, D are positive or zero. Subtracting the fifth and the sixth rows from the fourth row, the elements of B, D, F are $-1, 0, 1$.

If B, D, F do not have an element 2, then the proof is analogous.

Suppose that there is 2 and -2 among the elements of B, D, F. If F has an element 2, then coordinates B and D are positive or zero. Therefore B has an element 2, D has an element -2, so every element of F is 0. Subtracting the fifth row from the fourth row and adding the last row to the fourth row, every element of B, D, F becomes $-1, 0, 1$. If an element of B is -2 and an element of D is 2, then the proof is analogous.

Let either $\mathbf{m}_i = [x_1, x_2, x_3, x_4, 1, 0]^T$ or $\mathbf{m}_j = [x_1, x_2, x_3, x_4, 0, 1]^T$ or $\mathbf{m}_k = [x_1, x_2, x_3, x_4, 1, 1]^T$ be a minimum vector, whose coordinates x_1, x_2, x_3 are elements of a column of A or C or E and x_4 is equal to $-1, 0, 1$. Suppose that A or C or E has such an element (for example x_1) whose absolute value is greater than one. Hence

$$|\det \begin{bmatrix} x_1 & 0 & 0 & 1 & 0 & 0 \\ x_2 & 1 & 0 & 1 & 0 & 0 \\ x_3 & 0 & 1 & 1 & 0 & 0 \\ x_4 & 0 & 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}| = |\det \begin{bmatrix} x_1 & 0 & 0 & 1 & 0 & 0 \\ x_2 & 1 & 0 & 1 & 0 & 0 \\ x_3 & 0 & 1 & 1 & 0 & 0 \\ x_4 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}| =$$

$$= |\det \begin{bmatrix} x_1 & 0 & 0 & 1 & 0 & 0 \\ x_2 & 1 & 0 & 1 & 0 & 0 \\ x_3 & 0 & 1 & 1 & 0 & 0 \\ x_4 & 0 & 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}| = |2x_1 - x_4| > 2.$$

Thus the elements of A, C and E are equal to $-1, 0, 1$. Because of $m_{44} = 2$, subtracting the first row from the fourth row, we get that the characteristic matrix has elements $-1, 0, 1$ except if $x_4 = -x_1 = \pm 1$. In this case

$$|2x_1 - x_4| = |3x_1| = 3 > 2.$$

However, this case leads to a contradiction. We have verified the statement. Our Theorem 1 is completely proved.

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