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Horoball Packings for the Lambert-cube Tilings in the Hyperbolic 3-space

Dedicated to Professor Emil Molnár on the occasion of his 60^{th} birthday

Jenő Szirmai

Institute of Mathematics, Department of Geometry
Budapest University of Technology and Economics, H-1521 Budapest, Hungary
e-mail: szirmai@math.bme.hu

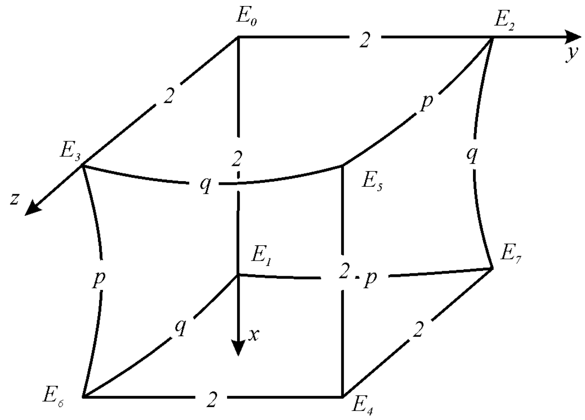
1. Introduction

In this paper those combinatorial (topological) 3-tilings (T, Γ) will be investigated which satisfy the following requirements:

(1) The elements of the tiling T are combinatorial cubes;
(2) The isometry group Γ of the hyperbolic space $\mathbb{H}^3$ acts transitively on the 2-faces of T;
(3) The group Γ is maximal, i.e. $\Gamma \cong AutT$ the group of all bijections preserving the face incidence structure of T;
(4) The reflections in the side planes of the cubes in T are elements of Γ.

These face transitive tilings, called (generalized) Lambert-cube tilings, are the following (Fig. 1):

- (p, q) $(p > 2,\ q = 2)$.
  These infinite tiling series of cubes are the special cases of the classical Lambert-cube tilings. The dihedral angles of the Lambert-cube are $\frac{\pi}{p}$ $(p > 2)$ at the 3 skew edges and $\frac{\pi}{2}$ at the other edges. Their metric realization in the hyperbolic space $\mathbb{H}^3$ is well known. A simple proof was described by E. Molnár in [10]. The volume of this Lambert-cube type was determined by R. Kellerhals in [6] (see 4.4).
- $(p, q) = (4, 4), (3, 6)$.
  In this cases the Lambert-cube types can be realized in the hyperbolic space $\mathbb{H}^3$ as well, because the cubes can be divided into hyperbolic simplices. These cubes have ideal vertices which lie on the absolute quadric of $\mathbb{H}^3$.



(p,q)=(3,3), (3,4), (3,5), (3,6), (4,4), (p>2,q=2)

Figure 1

– $(p, q) = (3, 3), (3, 4), (3, 5)$.

In [12] I proved that these tilings can be realized in $\mathbb{H}^3$.

In an earlier paper [14] I developed an algorithm for computing the volume of a given hyperbolic polyhedron and implemented it for computer. With this method, based on the projective interpretation of the hyperbolic geometry [10], [11], [13], I determined the volumes of the Lambert-cube types W_{pq} with parameters $(p, q)= (3, 3), (3, 4), (3, 5), (3, 6), (4, 4)$. In these cases and for the parameters

$$(p, q),\ p > 2,\ q = 2\ (p = 3, 4, 5, 6, \ldots, 10, \ldots, 100, \ldots, 1000, \ldots, \to \infty)$$

there were described the optimal ball packings as well. I investigated those ball packings, under the symmetry group Γ above, where the ball centres lie either in the Lambert-cubes (the ball has trivial stabilizer in the reflection subgroup of Γ) or in the vertices of the cubes (in this cases there are two classes of the vertices). In each case I gave the volume of the Lambert-cube, the coordinates of the ball centre and the radius, moreover I computed the density of the optimal packing.

In the cases $(p, q) = (3, 6), (4, 4)$, however, the vertices $E_1, E_2, E_3, E_5, E_6, E_7$ lie on the absolute of $\mathbb{H}^3$, therefore these vertices can be centres of some horoballs.

In the first part of this paper I recall some facts concerning the combinatorial and geometric properties of the face transitive cube tilings (Sections 2–5).

In the second part I investigate those horoball packings to these Lambert-cube tilings where only one horoball type is considered (the symmetry group of the horoballs is the reflection subgroup of Γ, Section 6, Fig. 6).

In the third part the optimal horoball packing will be determined for six horoball types (the horoball centres lie in the infinite vertices of the Lambert-cubes and its symmetry group is the reflection subgroup of Γ (Section 7, Fig. 9, Fig. 10)).

In the closing section I shall investigate the optimal horoball packing of Γ.

I shall describe a method – again based on the projective interpretation of the hyperbolic geometry – that determines the data of the optimal horoball and computes the density of the optimal packing.

The densest horosphere packing without any symmetry assumption was determined by Böröczky and Florian in [3]. The general density upper bound is the following:

$$s_0 = (1 + \frac{1}{22} - \frac{1}{42} - \frac{1}{52} + \frac{1}{72} + \frac{1}{82} - \; - \; + \; + \ldots)^{-1} \approx 0.85327609.$$

This limit is achieved by the 4 horoballs touching each other in the ideal regular simplex with Schläfli symbol $\{3, 3, 6\}$, the horoball centres are just in the 4 ideal vertices of the simplex. Beyond of the universal upper bound there are few results in this topic [4], [14]), therefore our method seems to be actual for determining local optimal ball and horoball packings for given hyperbolic tilings. The computations were carried out by Maple V Release 5 up to 20 decimals.

2. On face-transitive hyperbolic cube tilings

In [5] A. W. M. Dress, D. Huson and E. Molnár described the classification of tilings (T, Γ) in the Euklidean space $\mathbb{E}^3$ where a space group Γ acts on the 2-faces of T transitively. Their algoritm and the corresponding computer program, based on the method of D-symbols, led to non-Euclidean tilings as well, mainly in $\mathbb{H}^3$. In general, their method leads to tilings which can be realized in the 8 Thurston geometries:

$$\mathbb{E}^3,\ \mathbb{S}^3,\ \mathbb{H}^3,\ \mathbb{S}^2 \times \mathbb{R},\ \mathbb{H}^2 \times \mathbb{R},\ \widetilde{\mathbb{SL}_2\mathbb{R}},\ \mathbb{N}\mathrm{il},\ \mathbb{S}\mathrm{ol}.$$

In this paper I shall study those combinatorial 3-tilings (T, Γ), which satisfy the properties (1), (2), (3), (4) in the introduction. By the results of [5] there are 2 classes of these tilings. They will be described in subsections 2.1 and 2.2.

2.1. In Fig. 2 we have described the fundamental domain $\mathcal{F}_{pq}$ of the group Γ_{pq} for the cases $(p, q) = (3, 3), (4, 4)$. The generators of the group Γ_{pq} are the following:

$$\begin{aligned} m_1 &: \text{is a plane reflection in the face } A_0^1 A_0^4 A_3, \\ m_2 &: \text{is a plane reflection in the face } A_0^1 A_0^{23} A_3, \\ m_3 &: \text{is a plane reflection in the face } A_0^1 A_0^{23} A_0 4, \\ r &: \text{is a halfturn about the axis } A_3 A_1^{34}. \end{aligned} \tag{2.1}$$

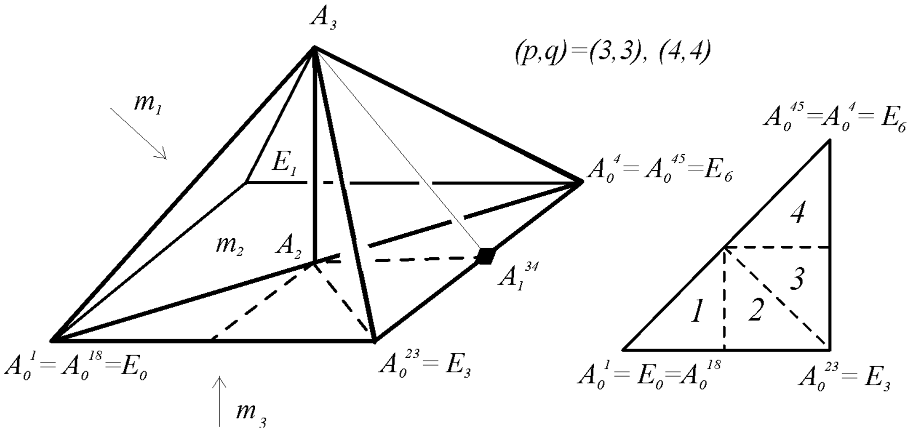


Figure 2

The group Γ_{pq} (now $p = q$) is given by the defining relations:

$$
\begin{aligned}
1 = m_1^2 = m_2^2 = m_3^2 = r^2 = (m_1 m_2)^3 = (m_1 m_3)^2 = \\
= (m_2 m_3)^4 = m_2 r m_1 r = (m_3 r m_3 r)^p, \ (p = 3, 4).
\end{aligned} \tag{2.2}
$$

2.2. The fundamental domain $\mathcal{F}_{pq}$ of the group Γ_{pq} is described in Fig. 3. Now the parameters are $(p, q) = (3, 4), (3, 5), (3, 6), (p > 2, q = 2)$. The generators of the group Γ_{pq} are the following:

$$
\begin{aligned}
&r_1: \text{ is a halfturn about the axis } A_3 A_1^{56}, \\
&r_2: \text{ is a halfturn about the axis } A_3 A_1^{34}, \\
&m: \text{ is a plane reflection in the face } A_0^{18} A_0^{23} A_0^{45} A_0^{67} = E_0 E_3 E_6 E_1, \\
&r: \text{ is a 3-rotation about the axis } A_0^{18} A_3 = E_0 A_3.
\end{aligned} \tag{2.3}
$$

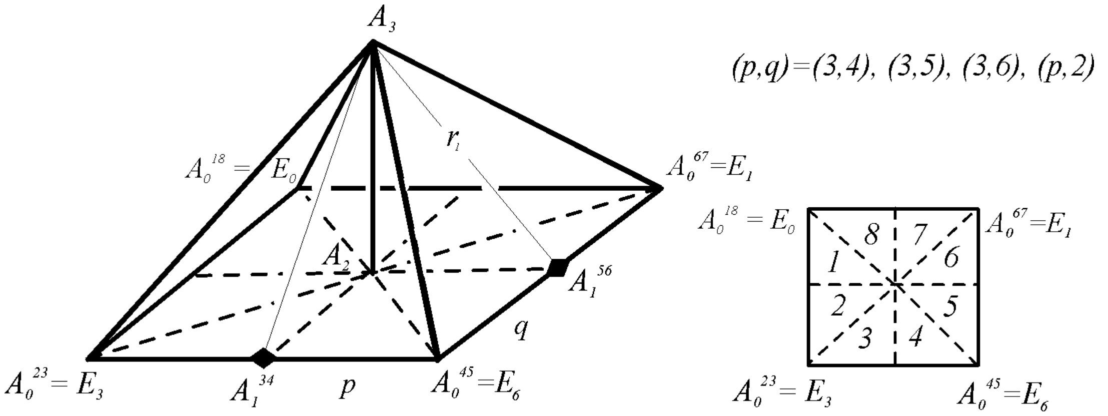


Figure 3

Thus we obtain the group Γ_{pq} by the defining relations:

$$
\begin{aligned}
1 = r_1^2 = r_2^2 = m12 = r^3 = r_1 r_2 r = (m r^{-1} m r)^2 = (r_1 m r_1 m)^q = \\
= (r_2 m r_2 m)^p; \ p = 3, \ q = 4, 5, 6; \ and \ p \geqq 3, \ q = 2.
\end{aligned} \tag{2.4}
$$

From [1], [10], [12] follows that these tilings have metric realizations in the hyperbolic space $\mathbb{H}^3$.

3. The volume of a hyperbolic orthoscheme

Definition 3.1. *Let X denote either the n-dimensional sphere $\mathbb{S}^n$, the n-dimensional Euclidean space $\mathbb{E}^n$ or the hyperbolic space $\mathbb{H}^n$ $n \geq 2$. An orthoscheme $\mathcal{O}$ in X is a simplex bounded by $n+1$ hyperplanes $H_0, \ldots, H_n$ such that* (Fig. 4, [8])

$$H_i \perp H_j, \ \text{for } j \neq i-1, i, i+1.$$

A plane orthoscheme is a right-angled triangle, whose area formula can be expressed by the well known defect formula. For three-dimensional spherical orthoschemes, Ludwig Schläfli about 1850 was able to find the volume differentials depending on differential of the not fixed 3 dihedral angles (Fig. 4). Independently of him, in 1836, Lobachevsky found a volume formula for three-dimensional hyperbolic orthoschemes $\mathcal{O}$ [1].

Theorem. (N. I. Lobachevsky) *The volume of a three-dimensional hyperbolic orthoscheme $\mathcal{O} \subset \mathbb{H}^3$ is expressed with the dihedral angles $\alpha_1, \alpha_2, \alpha_3$ (Fig. 4) in the following form:*

$$\begin{aligned} Vol(\mathcal{O}) = \frac{1}{4}\{&\mathfrak{L}(\alpha_1+\theta) - \mathfrak{L}(\alpha_1-\theta) + \mathfrak{L}(\frac{\pi}{2}+\alpha_2-\theta) + \\ &+ \mathfrak{L}(\frac{\pi}{2}-\alpha_2-\theta) + \mathfrak{L}(\alpha_3+\theta) - \mathfrak{L}(\alpha_3-\theta) + 2\mathfrak{L}(\frac{\pi}{2}-\theta)\}, \end{aligned} \tag{3.1}$$

where $\theta \in (0, \frac{\pi}{2})$ is defined by the following formula:

$$\tan(\theta) = \frac{\sqrt{\cos^2\alpha_2 - \sin^2\alpha_1 \sin^2\alpha_3}}{\cos\alpha_1 \cos\alpha_3} \tag{3.2}$$

and where $\mathfrak{L}(x) := -\int\limits_0^x \log|2\sin t|dt$ denotes the Lobachevsky function.

In [7], [9] there are further results for the volumes of the orthoschemes and simplices in higher dimensions.

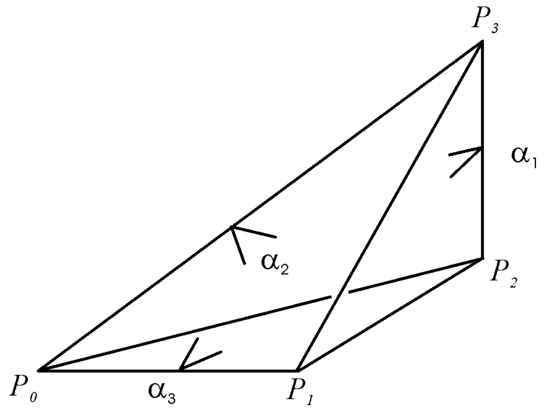


Figure 4

4. On the Lambert-cube types $(\mathbf{p},\mathbf{q}) = (\mathbf{3},\mathbf{3}), (\mathbf{3},\mathbf{4}), (\mathbf{3},\mathbf{5}), (\mathbf{3},\mathbf{6}), (\mathbf{4},\mathbf{4})$, $(\mathbf{p} > \mathbf{2}, \mathbf{q} = \mathbf{2})$

4.1. The projective coordinate system

We consider the real projective 3-space $\mathbb{P}^3(\mathbf{V^4}, V_4^*)$ where the subspaces of the 4-dimensional real vector space $\mathbf{V^4}$ represent the points, lines, planes of $\mathbb{P}^3$. The point $X(\mathbf{x})$ and the plane $\alpha(a)$ are incident iff $\mathbf{x}a = 0$, i.e. the value of the linear form a on the vector $\mathbf{x}$ is equal to zero ($\mathbf{x} \in \mathbf{V^4} \setminus \{0\}$, $a \in V_4^* \setminus \{0\}$). The straight lines of $\mathbb{P}^3$ are characterized by 2-subspaces of $\mathbf{V^4}$ or of V_4^*, i.e. by 2 points or dually by 2 planes, respectively [11]. We introduce a projective coordinate system, by a vector basis $\mathbf{b}_i$, $(i = 0, 1, 2, 3)$ for $\mathbb{P}^3$, with the following coordinates of the vertices of the Lambert-cube types (see Fig. 4).

$$\begin{array}{ll} E_0(1,0,0,0) \sim \mathbf{e}_0 & E_1(1,d,0,0) \sim \mathbf{e}_1 \\ E_2(1,0,d,0) \sim \mathbf{e}_2 & E_3(1,0,0,d) \sim \mathbf{e}_3 \\ E_4(1,\ c,c,\ c) \sim \mathbf{e}_4 & E_5(1,0,x,y) \sim \mathbf{e}_5 \\ E_6(1,y,0,x) \sim \mathbf{e}_6 & E_7(1,x,y,0) \sim \mathbf{e}_7. \end{array} \tag{4.1}$$

Our tilings shall have metric realization in the hyperbolic space whose metric is derived by the following bilinear form

$$\langle \mathbf{x}, \mathbf{y} \rangle = -x^0y^0 + x^1y^1 + x^2y^2 + x^3y^3 \text{ where } \mathbf{x} = x^i\mathbf{b}_i \text{ and } \mathbf{y} = y^j\mathbf{b}_j. \tag{4.2}$$

(The Einstein summation convention will be applied.)

This bilinear form, introduced in $\mathbf{V}_4$ and V_4^*, respectively, allows us to define a bijective polarity between vectors and forms. So we get a geometric polarity between points and planes. A plane (u) and their pole $(\mathbf{u})$ are incident iff

$$0 = \mathbf{u}u = \langle \mathbf{u}, \mathbf{u} \rangle = \langle u, u \rangle. \tag{4.3}$$

Such a point $(\mathbf{u})$ is called end of $\mathbb{H}^3$ (or point on the absolute quadric), its polar (u) is a boundary plane (tangent to the absolute at the end).

In general, the vector $\mathbf{x}$ in the cone

$$\mathcal{C} = \{\mathbf{x} \in \mathbf{V}_4 : \langle \mathbf{x}, \mathbf{x} \rangle < 0\} \tag{4.4}$$

defines the proper point $(\mathbf{x})$ of the hyperbolic space $\mathbb{H}^3$ embedded in $\mathbb{P}^3$.

If $(\mathbf{x})$ and $(\mathbf{y})$ are proper points, with $\langle \mathbf{x}, \mathbf{x} \rangle < 0$ and $\langle \mathbf{y}, \mathbf{y} \rangle < 0$, then their distance d is defined by:

$$\cosh \frac{d}{k} = \frac{-\langle \mathbf{x}, \mathbf{y} \rangle}{\sqrt{\langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle}} \geq 1, \tag{4.5}$$

here the metric constant $k = \sqrt{-K} = 1$ of $\mathbb{H}^3$ is fixed now.

The form u in the exterior cone domain

$$\mathcal{C}^* = \{u \in V_4^* : \langle u, u \rangle > 0\} \tag{4.6}$$

defines the proper plane (u) of the hyperbolic space $\mathbb{H}^3$. Suppose that (u) and (v) are proper planes. They intersect in a proper straight line iff $\langle u, u \rangle \langle v, v \rangle - \langle u, v \rangle^2 > 0$. Their dihedral angle $\alpha(u, v)$ can be measured by

$$\cos \alpha = \frac{\langle u, v \rangle}{\sqrt{\langle u, u \rangle \langle v, v \rangle}}. \tag{4.7}$$

The foot point $Y(\mathbf{y})$ of the perpendicular, dropped from the point $X(\mathbf{x})$ on the plane (u), has the following form:

$$\mathbf{y} = \mathbf{x} - \frac{\langle \mathbf{x}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle}\mathbf{u}, \tag{4.8}$$

where $\mathbf{u}$ represents the pole of the proper plane $u \in V_4^*$. If (u) is a proper plane and $(\mathbf{u})$ is its pole, then the reflection formulas for points and planes are

$$\mathbf{x} \to \mathbf{y} = \mathbf{x} - 2\frac{\langle \mathbf{x}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle}\,\mathbf{u}, \qquad v \to w = v - 2\frac{\langle v, u \rangle}{\langle u, u \rangle}\,u. \tag{4.9}$$

4.2. The coordinates of the vertices and the volumes of the Lambert-cube types

In [12] and [14] the coordinates x, y, c, d in (4.1) have been determined for the parameters

$$(p, q) = (3, 3), (3, 4), (3, 5), (3, 6), (4, 4), (p > 2, q = 2).$$

In the cases $(p > 2, q = 2)$ the parameter d is obtained by the following equation:

$$\cos\frac{\pi}{p} = \frac{d^2}{\sqrt{d^4 - d^2 + 1}}. \tag{4.10}$$

The values of the parameters are summarized in the following table:

Table 1

(p,q)	c	d	x	y
$(3,3)$	0.52915026	0.88191710	0.66143783	0.66143783
$(3,4)$	0.53911695	0.94909399	0.61220809	0.75625274
$(3,5)$	0.54261145	0.98159334	0.58048682	0.80221305
$(3,6)$	0.54427354	1	0.56032419	0.82827338
$(4,4)$	0.54691816	1	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$(p>2, q=2)$	$c = \frac{d}{1+d^2}$	d	d	$d(1-d^2)$

4.3. The volumes of the Lambert-cube types

In [14] the volume of the Lambert-cube was determined for the parameters $(p,q) = (3,3),(3,4),(3,5),(3,6),(4,4)$. In the cases $(p>2, q=2)$ the volume $W_{p,q}$ is obtained by a theorem of R. Kellerhals [6].

$$
\begin{aligned}
&Vol(W_{3,3}) \approx 1.33337239, && Vol(W_{4,2}) \approx 1.33337239,\\
&Vol(W_{3,4}) \approx 2.02767465, && Vol(W_{5,2}) \approx 0.65808153,\\
&Vol(W_{3,5}) \approx 2.57731886, && Vol(W_{6,2}) \approx 0.72992641,\\
&Vol(W_{3,6}) \approx 3.15775682, && Vol(W_{10,2}) \approx 0.84476780,\\
&Vol(W_{4,4}) \approx 3.33769269, && Vol(W_{100,2}) \approx 0.91522568,\\
&Vol(W_{3,2}) \approx 0.32442345, && Vol(W_{1000,2}) \approx 0.91595819,
\end{aligned}
\tag{4.11}
$$

$$Vol(p \to \infty, 2) \approx 0.91596559.$$

5. Description of the horosphere in the hyperbolic space $\mathbb{H}^3$

We shall use the Cayley-Klein ball model of the hyperbolic space $\mathbb{H}^3$ in the Cartesian homogeneous rectangular coordinate system introduced in (4.1) (see Fig. 1.). From the formula (4.9) follows the equation of the horosphere with centre $E_3(1,0,0,1)$ through the point $S(1,0,0,s)$ by Fig. 5:

$$0 = -2s(x^0)^2 - 2(x^3)^2 + 2(s+1)(x^0x^3) + (s-1)((x^1)^2+(x^2)^2) \tag{5.1}$$

in projective coordinates (x^0,x^1,x^2,x^3). In the Cartesian rectangular coordinate system this equation is the following:

$$\frac{2(x^2+y^2)}{1-s} + \frac{4(z-\frac{s+1}{2})^2}{(1-s)^2} = 1, \text{ where } x := \frac{x^1}{x^0}, y := \frac{x^2}{x^0}, z := \frac{x^3}{x^0}. \tag{5.2}$$

The site of this horosphere in the Lambert-cube in the cases $(p,q) = (4,4),(3,6)$ is illustrated in Fig. 6.

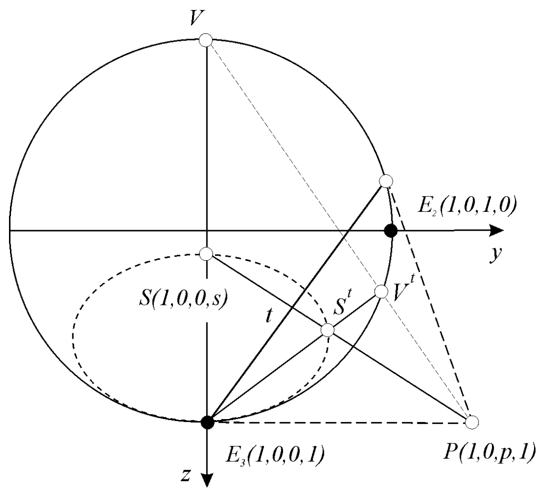


Figure 5

6. The optimal horoball packing with one horoball type for the parameters $(\mathbf{p}, \mathbf{q}) = (\mathbf{3}, \mathbf{6}), (\mathbf{4}, \mathbf{4})$

It is clear that the optimal horosphere has to touch at least one of the faces of the Lambert-cube which does not include the vertex $E_3(1,0,0,1)$ (Fig. 6). Therefore we have determined the foot points $Y_i^{(p,q)}(\mathbf{y}_i^{(p,q)}),\ i = 1,2,3,\ (p,q) = (3,6),(4,4)$ of the perpendicular dropped from the point $E_3(\mathbf{e}_3)$ on the plane (u^i) (by the formula (4.8)) where the planes (u^i) can be expressed from (4.1) by the following forms:

$$E_0E_1E_2 : u^3 \sim \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},\quad E_2E_4E_5 : u^2 \sim \begin{pmatrix} 1 \\ \frac{y-d}{xd} \\ \frac{-1}{d} \\ \frac{x-d}{yd} \end{pmatrix},\quad E_1E_4E_6 : u^1 \sim \begin{pmatrix} 1 \\ \frac{-1}{d} \\ \frac{x-d}{dy} \\ \frac{y-d}{xd} \end{pmatrix},$$

where $d = 1$ now. The coordinates of the foot points

$$Y_i^{(p,q)}(\mathbf{y}_i^{(p,q)}),\ i = 1,2,3,\ Y_i^{(p,q)} \in u^i,\ (p,q) = (3,6),(4,4)$$

are collected in the following tables:

Table 2

$Y_i(\mathbf{y}_i)/(p,q)$	$(3,6)$
$Y_1(\mathbf{y}_1)$	$(1, 0.64861525, 0.34430713, 0.55017056)$
$Y_2(\mathbf{y}_2)$	$(1, 0.17018845, 0.55530533, 0.73946971)$
$Y_3(\mathbf{y}_3)$	$(1, 0, 0, 0)$

Table 3

$Y_i(\mathbf{y}_i)/(p,q)$	$(4,4)$
$Y_1(\mathbf{y}_1)$	$(8-5\sqrt{2}, 2-\sqrt{2}, 3\sqrt{2}-4, 2-\sqrt{2})$
$Y_2(\mathbf{y}_2)$	$(8-5\sqrt{2}, 3\sqrt{2}-4, 2-\sqrt{2}, 2-\sqrt{2})$
$Y_3(\mathbf{y}_3)$	$(1, 0, 0, 0)$

The union of Lambert-cubes with the common vertex $E_3(1,0,0,1)$ forms an infinite polyhedron denoted by $\mathcal{G}$.

In order to find the equation of the optimal horosphere with centre $E_3(1,0,0,1)$ in $\mathcal{G}$ we have substituted the coordinates of the foot points $Y_i^{(p,q)}(\mathbf{y}_i^{(p,q)}),\ i = 1,2,3,\ (p,q) = (3,6),(4,4)$ into (5.1), and we have determined the possible values of the parameters $s_i^{(p,q)}$. The parameters of the optimal horospheres are the following (see Fig. 5):

$$\begin{aligned} s(3,6) &= max\{s_i^{(3,6)}\} \approx 0.26114274, \\ s(4,4) &= s_1^{(4,4)} = s_2^{(4,4)} = s_3^{(4,4)} = 0. \end{aligned} \tag{5.3}$$

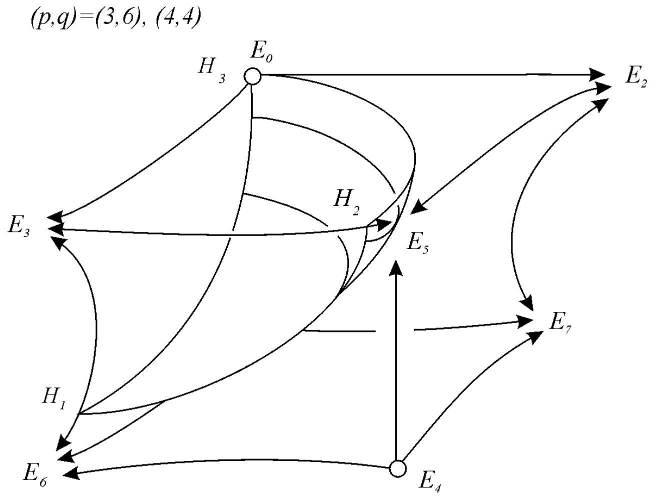


Figure 6

Remark. In the case $(p, q) = (4, 4)$ the optimal horosphere (Fig. 6) contains the centre of the Cayley-Klein model and touches all the three faces of the Lambert-cube which do not include the vertex $E_3(1, 0, 0, 1)$.

The volume of the horoball pieces can be calculated by the classical formula of J. Bolyai. If the area of the figure A on the horosphere is $\mathcal{A}$, the space determined by A and the aggregate of axes drawn from A is equal to

$$V = \frac{1}{2}k\mathcal{A} \quad \text{(we assumed that } k = 1 \text{ here).} \tag{6.1}$$

The Lambert-cubes in $\mathcal{G}$ divide the optimal horosphere into congruent horospherical triangles (see Fig. 6.). The vertices $H_1,\ H_2,\ H_3$ of such a triangle are in the edges $E_3E_6,\ E_3E_5,\ E_3E_0$, respectively, and on the optimal horosphere. Therefore, their coordinates can be determined in the Cayley-Klein model (Tables 4–5,). We shall denote these vertices in Tables 4–5 by $H_i^{(p,q)},\ i = 1, 2, 3,\ (p, q) = (3, 6), (4, 4)$, (see Fig. 6).

Table 4

$H_i(\mathbf{h}_i)/(p,q)$	$(3,6)$
$H_1(\mathbf{h}_1)$	$(1, 0.60227660, 0, 0.68029101)$
$H_2(\mathbf{h}_2)$	$(1, 0, 0.48870087, 0.85022430)$
$H_3(\mathbf{h}_3)$	$(1, 0, 0, 0.26114274)$

Table 5

$H_i(\mathbf{h}_i)/(p,q)$	$(4,4)$
$H_1(\mathbf{h}_1)$	$(1, 0.61678126, 0, 0.74452084)$
$H_2(\mathbf{h}_2)$	$(1, 0, 0.61678126, 0.74452084)$
$H_3(\mathbf{h}_3)$	$(1, 0, 0, 0)$

The distance of two vertices can be calculated by the formula (4.5). The lengths of the sides of the horospherical triangle (they are horocycles) are determined by the classical formula of J. Bolyai (see Fig. 7):

$$l(x) = k \sinh \frac{x}{k} \text{ (at present } k = 1). \tag{6.2}$$

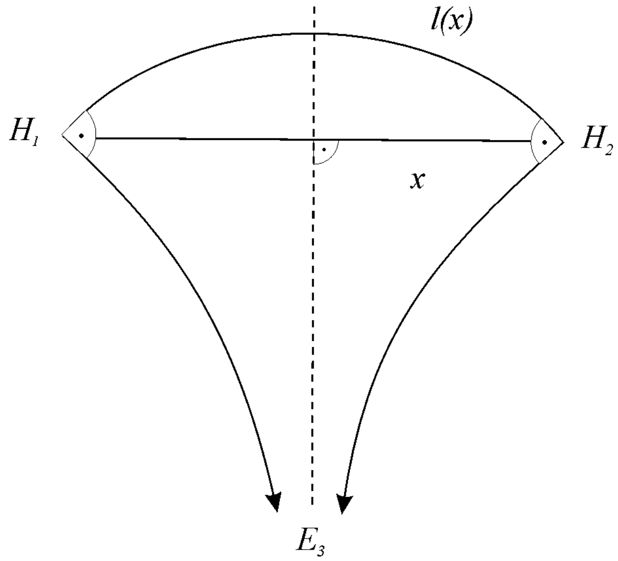


Figure 7

It is well known that the intrinsic geometry of the horosphere is Euclidean, therefore, the area $\mathcal{A}_{pq}$ of the horosperical triangle $H_1^{(p,q)}H_2^{(p,q)}H_3^{(p,q)}$ is obtained by the formula of Heron.

Definition 6.1. *The density of the horoball packing with one horoball type for the Lambert-cube is defined by the following formula:*

$$\delta^{(p,q)} := \frac{\frac{1}{2}k\mathcal{A}_{pq}}{Vol(W_{pq})}. \tag{6.3}$$

In Table 6 we have summarized the results of the optimal horoball packings in the cases $(p,q) = (3,6),\ (4,4)$.

Table 6

(p,q)	$\mathcal{A}_{pq}$	$\frac{1}{2}k\mathcal{A}_{pq}$	$\delta_{opt}^{(p,q)}$
$(3,6)$	1.80056665	0.90028333	0.28510217
$(4,4)$	2.91421356	1.45710678	0.43656110

7. The optimal horoball packing with six horoball types for the parameters $(\mathbf{p}, \mathbf{q}) = (\mathbf{3}, \mathbf{6}), (\mathbf{4}, \mathbf{4})$

In our proof we shall utilize the following two lemmas.

Lemma 1. *Let us two horoballs $\mathcal{B}_1(x)$ and $\mathcal{B}_2(x)$ be given with centres C_1 and C_2 and two congruent trihedra $\mathcal{T}_1$ and $\mathcal{T}_2$ with vertices C_1 and C_2. We assume that these horoballs touch each other in a point $I(x) \in C_1C_2$ and the straight line C_1C_2 is common on the trihedra $\mathcal{T}_1$ and $\mathcal{T}_2$. We define the point of contact $I(0)$ by the following equality for volumes of horoball pieces*

$$V(0) := 2\,Vol(\mathcal{B}_1(0) \cap \mathcal{T}_1) = 2\,Vol(\mathcal{B}_2(0) \cap \mathcal{T}_2). \tag{7.1}$$

If x denotes the hyperbolic distance between $I(0)$ and $I(x)$, then the function

$$V(x) := Vol(\mathcal{B}_1(x) \cap \mathcal{T}_1) + Vol(\mathcal{B}_2(x) \cap \mathcal{T}_2) \tag{7.2}$$

strictly increases in the interval $(0, \infty)$.

Proof. Let ℓ and ℓ' be parallel horocycles with centre C and let A and B be two points on the curve ℓ and $A' := CA \cap \ell'$, $B' := CB \cap \ell'$ (Fig. 8). By the classical formula of J. Bolyai

$$\frac{\mathcal{H}(A'B')}{\mathcal{H}(AB)} = e^{\frac{x}{k}}, \tag{7.3}$$

where the horocyclic distance between A and B is denoted by $\mathcal{H}(A,B)$.

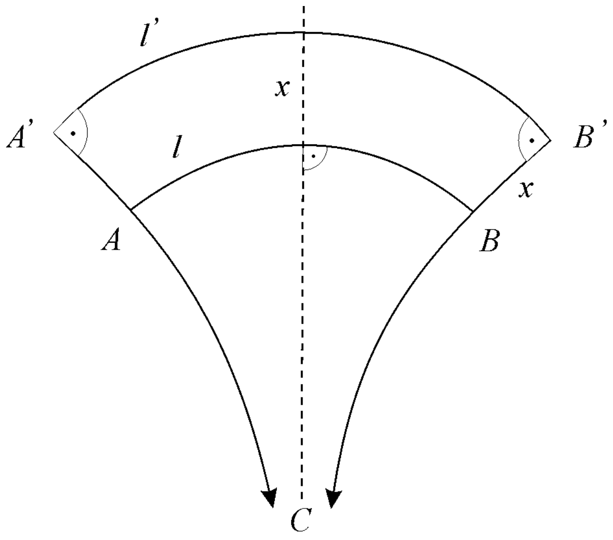


Figure 8

Then by the formulas (6.1), (7.2) and (7.3) we obtain the following volume function:

$$\begin{aligned} V(x) = {} & Vol(\mathcal{B}_1(x) \cap \mathcal{T}_1) + Vol(\mathcal{B}_2(x) \cap \mathcal{T}_2) = \\ & = \frac{1}{2} V(0)(e^{\frac{2x}{k}} + \frac{1}{e^{\frac{2x}{k}}}) = V(0)\cosh(\frac{2x}{k}). \end{aligned} \tag{7.4}$$

It is well known that this function strictly increases in the interval $(0,\infty)$.

Remark. It is easy to see that if the disjoint horoballs $\mathcal{B}_1$ and $\mathcal{B}_2$ with centres C_1 and C_2 do not touch each other then there are contact points $I(x) \in C_1C_2$ where

$$V := Vol(\mathcal{B}_1 \cap \mathcal{T}_1) + Vol(\mathcal{B}_2 \cap \mathcal{T}_2) < V(x).$$

Lemma 2. *Let us two horoball types $\mathcal{B}_1(y)$ and $\mathcal{B}_2(z)$ be given with centre C_1 and a trihedron $\mathcal{T}_1$ with vertex C_1. We assume that the straight line C_1C_2 lies on the trihedron $\mathcal{T}_1$ where the point C_2 is also on the absolute quadric of $\mathbb{H}^3$. We define the points $I(y)$ and $I(z)$ by the following equations:*

$$I(y) := \mathcal{B}_1(y) \cap C_1C_2, \quad I(z) := \mathcal{B}_1(z) \cap C_1C_2$$

where y and z are metric coordinates on C_1C_2 as above. Let Δx denote a distance variable. Then the function

$$\begin{aligned} V(\Delta x) := {} & Vol(\mathcal{B}_1(y+\Delta x) \cap \mathcal{T}_1) + Vol(\mathcal{B}_2(z+\Delta x) \cap \mathcal{T}_1) + \\ & + 2\,Vol(\mathcal{B}_1(y-\Delta x) \cap \mathcal{T}_1) + 2\,Vol(\mathcal{B}_2(z-\Delta x) \cap \mathcal{T}_1) \end{aligned} \tag{7.5}$$

strictly decreases in the interval $(0, \ln \sqrt[4]{2})$ *and strictly increases in the interval* $(\ln \sqrt[4]{2}, \infty)$.

This Lemma can be proved, similarly to the proof of Lemma 1, by derivation.

In the following we shall denote the horoball with centre E_i by $\mathcal{B}_i$, $i = 1,2,3,5,6,7$ (Fig. 9).

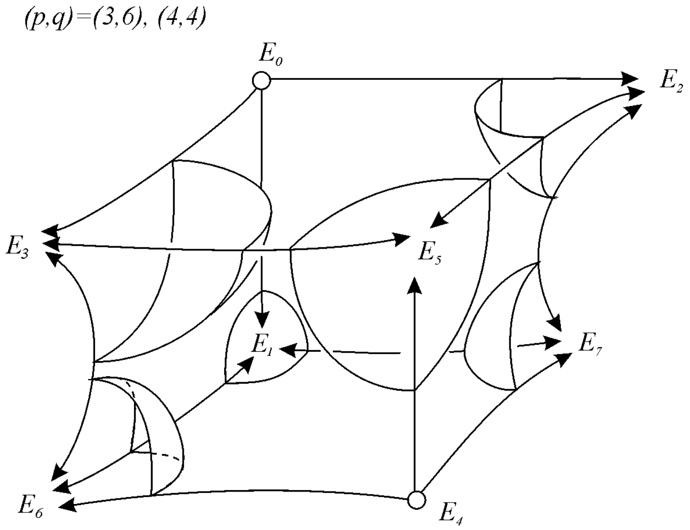


Figure 9

Definition 7.1. *The density of a horoball packing with six (or fewer) horoball types for the Lambert-cube is defined by the following formula:*

$$\delta^{(p,q)} := \frac{\sum\limits_i Vol(\mathcal{B}_i \cap W_{pq})}{Vol(W_{pq})}; \; i = 1,2,3,5,6,7; \; (p,q) = (3,6),(4,4). \tag{7.6}$$

We are interested in determining $\delta_{opt}^{(p,q)}$ as the maximal packing density of the six (or fewer) horoball types in the Lambert-cube. Let $\mathcal{B}_2^a$ and $\mathcal{B}_3^a$ denote the horoballs through the point $M(1,0,\frac{1}{2},\frac{1}{2})$ (with centres E_2 and E_3, respectively). $\mathcal{B}_1^a$ is introduced with touching $\mathcal{B}_3^a$ and $\mathcal{B}_2^a$ by the 3-rotation symmetry of the Lambert-cube about E_0E_4. We denote by $\mathcal{B}_6^a$ the horoball with centre E_6 that touches the horoball $\mathcal{B}_3^a$. The horoballs $\mathcal{B}_5^a$ and $\mathcal{B}_7^a$ are introduced again by the 3-symmetry above (see Fig. 10 in the case $(p,q)=(4,4)$ where e.g. $\mathcal{B}_5^a$ touches $\mathcal{B}_3^a$ and $\mathcal{B}_1^a$).

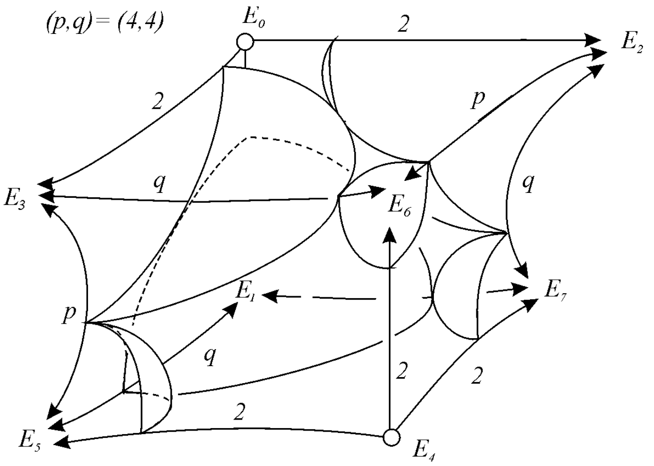


Figure 10

Take the case $(p,q)=(4,4)$. If we blow up the horoball $\mathcal{B}_3^a$ and continuously vary the others, while preserving the packing requirements, then we come to the following horoball set $\{\mathcal{B}_i^b\}$. Let us denote the horoball through the point $E_0(1,0,0,0)$ with centres E_3 by $\mathcal{B}_3^b$ (this was the "optimal horoball" in Section 6). Introduce $\mathcal{B}_1^b$ and $\mathcal{B}_2^b$ touching $\mathcal{B}_3^b$. Introduce horoballs $\mathcal{B}_5^b$, $\mathcal{B}_6^b$ touching $\mathcal{B}_3^b$ (decreasingly), moreover the horoball $\mathcal{B}_7^b$ touching the horoballs $\mathcal{B}_1^b$ and $\mathcal{B}_2^b$.

We consider the horoball set $\{\mathcal{B}_i^b\}$ and blow up $\mathcal{B}_7^b$, preserving the packing requirements, until the horoball does not touch $\mathcal{B}_3^b$. This horoball is denoted by $\mathcal{B}_7^c$. In this case $\mathcal{B}_3^b=\mathcal{B}_3^c$, $\mathcal{B}_5^b=\mathcal{B}_5^c$, $\mathcal{B}_6^b=\mathcal{B}_6^c$, and the horoballs $\mathcal{B}_1^c$, $\mathcal{B}_2^c$ touch $\mathcal{B}_7^c$. Thus we get the horoball set $\{\mathcal{B}_i^c\}$.

Theorem 7.1. *We have obtained the optimal horoball packings $\mathcal{P}_a$, $\mathcal{P}_b$ and $\mathcal{P}_c$ with six horoball types to the Lambert-cube tiling $(p,q)=(4,4)$ by the horoball sets $\{\mathcal{B}_i^a\}$, $\{\mathcal{B}_i^b\}$ and $\{\mathcal{B}_i^c\}$, $i=1,2,3,5,6,7$.*

Proof. The optimal arrangements of the horoballs have to be locally stable, therefore, applying Lemma 1, we obtain the possible optimal horoball sets $\mathcal{P}_a$, $\mathcal{P}_b$ or $\mathcal{P}_c$. If we blow up the horoball $\mathcal{B}_3^a$, preserving the packing requirements by the others, we get the following horoball set

$$\mathcal{B}_3^a(y+\Delta x),\ \mathcal{B}_2^a(y-\Delta x),\ \mathcal{B}_1^a(y-\Delta x),\ \mathcal{B}_7^a(z+\Delta x),\ \mathcal{B}_5^a(z-\Delta x), \mathcal{B}_6^a(z-\Delta x).$$

By Lemma 2 it follows that the volume sum of these horoballs strictly decreases on the interval $(0, \ln\sqrt[4]{2})$ and strictly increases on the interval $(\ln\sqrt[4]{2}, \infty)$. It is easy to see, that

$$\mathcal{B}_3^a \cap E_0E_3 = (1,0,0,\frac{1}{3}), \text{ while } \mathcal{B}_3^b \cap E_0E_3 = (1,0,0,0).$$

Their hyperbolic distance d can be calculated by the formula (4.5), $d = \ln\sqrt{2} = 2\ln\sqrt[4]{2}$, therefore the volume sum of the horoball set $\mathcal{P}_a$ is equal to that of $\mathcal{P}_b$. Analogously follows that the horoball packing $\mathcal{P}_c$ is also an optimal arrangement of the horoballs.

Theorem 7.2. *The optimal horoball packing with six horoball types to the Lambert-cube tilings in the case* $(p,q) = (3,6)$ *can uniquely be realized by the horoballs* $\mathcal{B}_i^a$, $i = 1,2,3,5,6,7$.

This theorem can also be proved by the lemmas as above.

By the projective method we could calculate the density of the optimal horoball packing and the metric data of the horoballs $\mathcal{B}_i^j$, $i = 1,2,3,5,6,7$, $j = a,b,c$.

- $(p,q) = (4,4)$, type $\mathcal{P}_a$.
  There are two types of horoballs in the Lambert-cube:

$$Vol(B_1^a \cap W_{44}) = Vol(B_3^a \cap W_{44}) = Vol(B_2^a \cap W_{44}) \approx 0.72855335,$$

$$Vol(B_5^a \cap W_{44}) = Vol(B_6^a \cap W_{44}) = Vol(B_7^a \cap W_{44}) \approx 0.12500000,$$

$$\delta_{opt}^{(4,4)} := \frac{\sum_i Vol(B_i^a \cap W_{44})}{Vol(W_{44})} \approx 0.76719471; \ i = 1,2,3,5,6,7.$$

- $(p,q) = (4,4)$, type $\mathcal{P}_b$.
  There are three types of horoballs in the Lambert-cube:

$$Vol(B_1^b \cap W_{44}) = Vol(B_2^b \cap W_{44}) \approx 0.36427669,$$

$$Vol(B_5^b \cap W_{44}) = Vol(B_6^b \cap W_{44}) \approx 0.06250000,$$

$$Vol(B_3^b \cap W_{44}) \approx 1.45710678,$$

$$Vol(B_7^b \cap W_{44}) \approx 0.25000000,$$

$$\delta_{opt}^{(4,4)} := \frac{\sum_i Vol(B_i^a \cap W_{44})}{Vol(W_{44})} \approx 0.76719471; \ i = 1,2,3,5,6,7.$$

- $(p,q) = (4,4)$, type $\mathcal{P}_c$.
  There are three types of horoballs in the Lambert-cube:

$$Vol(B_1^b \cap W_{44}) = Vol(B_2^b \cap W_{44}) \approx 0.12500000,$$

$$Vol(B_5^b \cap W_{44}) = Vol(B_6^b \cap W_{44}) \approx 0.06250000,$$
$$Vol(B_3^b \cap W_{44}) \approx 1.45710678,$$
$$Vol(B_7^b \cap W_{44}) \approx 0.72855335,$$
$$\delta_{opt}^{(4,4)} := \frac{\sum_i Vol(B_i^a \cap W_{44})}{Vol(W_{44})} \approx 0.76719471; \; i = 1, 2, 3, 5, 6, 7.$$

– $(p, q) = (3, 6)$.
There are two types of horoballs in $\{\mathcal{B}_i^a\}$:

$$Vol(B_1^a \cap W_{36}) = Vol(B_3^a \cap W_{36}) = Vol(B_2^a \cap W_{36}) \approx 0.76833906,$$
$$Vol(B_5^a \cap W_{36}) = Vol(B_6^a \cap W_{36}) = Vol(B_7^a \cap W_{36}) \approx 0.04228104,$$
$$\delta_{opt}^{(3,6)} := \frac{\sum_i Vol(B_i^a \cap W_{36})}{Vol(W_{36})} \approx 0.77012273; \; i = 1, 2, 3, 5, 6, 7.$$

8. Optimal horoball packings to the face transitive Lambert-cube tilings for the parameters $(\mathbf{p}, \mathbf{q}) = (\mathbf{3}, \mathbf{6}), (\mathbf{4}, \mathbf{4})$

In these cases the horoballs with centres $E_1, E_2, E_3, E_5, E_6, E_7$ are in the same equivalence class by the group Γ_{pq} (see Section 2). Let $\mathcal{B}_3^d$ denote the horoball with centre E_3 through the point $T_{pq}(1, 0, y_1, z_1)$ that is the footpoint of A_3 onto E_3E_5 on a halfturn axis equivalent to $A_3A_1^{56}$ (see Fig. 2, 3, 11, 12). By computations as above (Table 1) we get

$$y_1 = \frac{x(\sqrt{1-3c^2} + 1 - c)(y-1)}{\sqrt{1-3c^2}((y-1)^2 + x^2) + (y-1)^2 + x^2 + cx(c-1-x^2)}$$

$$z_1 = \frac{x^2(\sqrt{1-3c^2} + 1) - c(x - 1 - (y-1)^2 - x(y-1))}{\sqrt{1-3c^2}((y-1)^2 + x^2) + (y-1)^2 + x^2 + cx(c-1-x^2)},$$

and that the horoball $\mathcal{B}_3^d$ induces a horoball packing in $\mathbb{H}^3$ under the group Γ_{pq}. Thus we can formulate the following:

Theorem 8.1. *The optimal horoball packings to the face transitive Lambert-cube tilings for the parameters* $(p, q) = (3, 6), (4, 4)$ *can be realized by the horoballs* $\{\mathcal{B}_i^d\}$, $i = 1, 2, 3, 5, 6, 7$.

By the projective method we can calculate the density of the optimal horoball packing and the metric data of the horoballs $\mathcal{B}_i^d$, $i = 1, 2, 3, 5, 6, 7$.

– $(p, q) = (4, 4)$.

$$Vol(\mathcal{B}_i^d \cap W_{44}) \approx 0.30177669; \; i = 1, 2, 3, 5, 6, 7,$$
$$\delta_{opt}^{(4,4)} := \frac{\sum_i Vol(\mathcal{B}_i^d \cap W_{44})}{Vol(W_{44})} \approx 0.54248858; \; i = 1, 2, 3, 5, 6, 7.$$

– $(p, q) = (3, 6)$.

$$Vol(\mathcal{B}_i^d \cap W_{36}) \approx 0.18023922;\ i = 1, 2, 3, 5, 6, 7;$$

$$\delta_{opt}^{(3,6)} := \frac{\sum_i Vol(\mathcal{B}_i^d \cap W_{36})}{Vol(W_{36})} \approx 0,34246947;\ i = 1, 2, 3, 5, 6, 7.$$

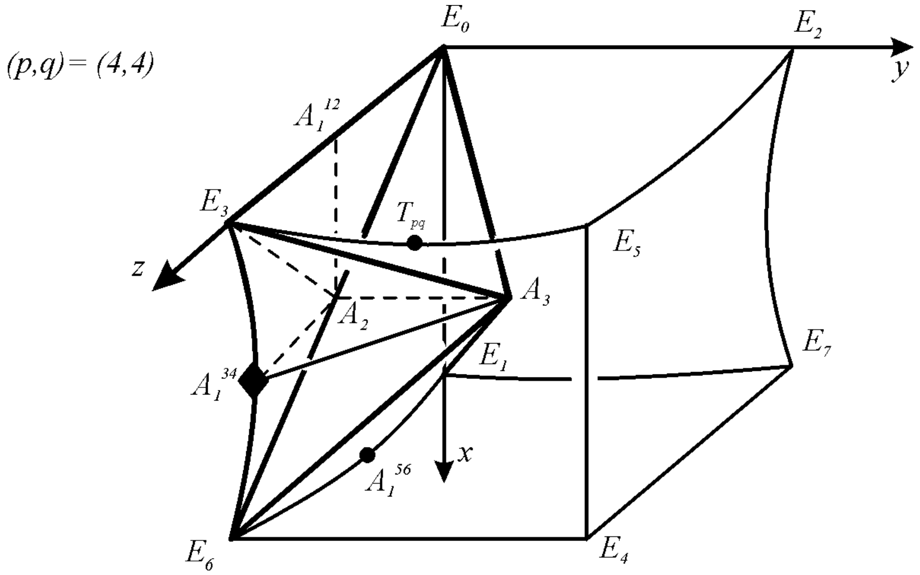


Figure 11

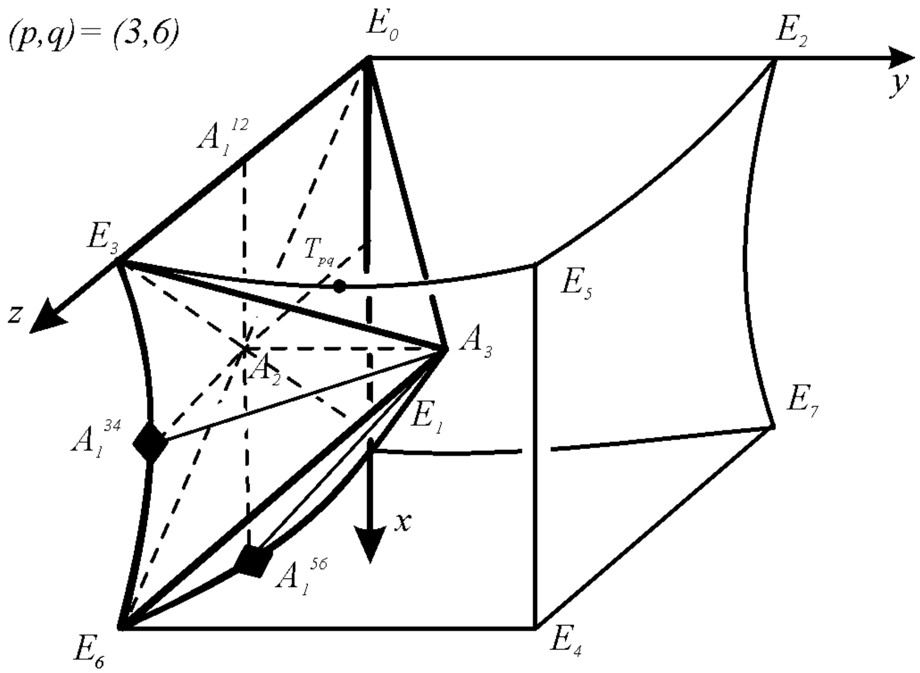


Figure 12

I thank my colleague Emil Molnár for his valuable suggestions.

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