Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 46 (2005), No. 1, 119-123.

Hajós' Theorem and the Partition Lemma

Sándor Szabó
Department of Mathematics, University of Bahrain
P. O. Box 32038, State of Bahrain

Abstract. If a finite abelian group is a direct product of cyclic subsets, then at least one of the factors must be a subgroup. This result is due to G. Hajós. The purpose of this paper is to show that Hajós' theorem can be proved using the so-called partition lemma.

MSC 2000: 20K01 (primary); 52C22 (secondary)
Keywords: factorization of finite abelian groups, Hajós-Rédei theory

1. Introduction

Let G be a finite abelian group written multiplicatively with identity element e. If $A_1, \ldots, A_n$ are subsets of G such that the product $A_1 \cdots A_n$ is direct and gives G, then we say that the equation $G = A_1 \cdots A_n$ is a *factorization* of G. In the most commonly encountered situation G is a direct product of its subgroups. However, in this paper it will not be assumed that $A_1, \ldots, A_n$ are subgroups of G. A subset A of G is called *cyclic* if it is in the form

$$A = \{e, a, a^2, \ldots, a^{p-1}\}, \tag{1}$$

where p is a prime and $|a| \geq p$. G. Hajós [2] proved that if a finite abelian group is factored into cyclic subsets, then at least one of the factors must be a subgroup.

A subset A of G is *periodic* if there is a $g \in G$ such that $Ag = A$ and $g \neq e$. The element g is called a *period* of A. All the periods of A together with the identity element form a subgroup H of G. Further, there is a subset X of G such that the product XH is direct and is equal to A. If A is a periodic subset of G, $e \in A$ and A contains a prime number of elements, then A is a subgroup of G.

For a character χ and a subset A of G the notation $\chi(A)$ stands for

$$\sum_{a \in A} \chi(a).$$

The set of characters χ of G for which $\chi(A) = 0$ we call the *annihilator* of A and will denote it by $\mathrm{Ann}(A)$. For a character χ and a subgroup H of G $\chi(H) = 0$ if and only if there is an element h of H with $\chi(h) \neq 1$. Equivalently, $\chi(H) = 0$ if and only if $H \not\subset \mathrm{Ker}(\chi)$ or if χ is not the principal character on H.

Suppose that the subset A of G is periodic, that is, $A = XH$, where X is a subset and H is a subgroup of G. Then clearly $\mathrm{Ann}(H) \subset \mathrm{Ann}(A)$. The converse is also true. Namely, if A is a subset and H is a subgroup of G such that $\mathrm{Ann}(H) \subset \mathrm{Ann}(A)$, then there is an $X \subset G$ for which $A = XH$, where the product is direct. This is Theorem 1 of [3].

Suppose that a subset A of G is a disjoint union of two periodic subsets, that is, there are subsets X, Y and subgroups H, K of G such that $A = XH \cup YK$, where the products XH, YK are direct. Then clearly $\mathrm{Ann}(H) \cap \mathrm{Ann}(K) \subset \mathrm{Ann}(A)$. The converse is also true. Namely, if A is a subset and H, K are subgroups of G such that

$$\mathrm{Ann}(H) \cap \mathrm{Ann}(K) \subset \mathrm{Ann}(A),$$

then there are subsets X, Y of G for which

$$A = XH \cup YK,$$

where the union is disjoint and the products XH, YK are direct. This is Theorem 2 of [3]. We refer to this result as partition lemma.

The partition lemma plays a key part in proving various results about periodic factorizations. For examples see [1], [5], and [6]. The purpose of this note is to show that Hajós' theorem also can be proved by a standard application of the partition lemma.

2. Preliminaries

The structure of the annihilator of the cyclic subset (1) can be described easily. Namely, $\mathrm{Ann}(A)$ is the difference of two subgroups of the character group of G. If $\chi(a) = 1$, then $\chi(A) = p$ and so $\chi \notin \mathrm{Ann}(A)$. If $\chi(a) \neq 1$, then

$$\chi(A) = 1 + \chi(a) + \chi(a^2) + \cdots + \chi(a^{p-1}) = \frac{1 - \chi(a^p)}{1 - \chi(a)}.$$

From which it follows that $\mathrm{Ann}(A)$ consists of each character χ of G with $\chi(a) \neq 1$ and $\chi(a^p) = 1$.

To the cyclic subset (1) we assign the subgroup $K = \langle a^p \rangle$. It is clear that A is a subgroup of G if and only if $K = \{e\}$. Putting this in an other form A is periodic if and only if $K = \{e\}$.

Assume that $G = AB$ is a factorization and χ is a non-principal character of G. Then

$$0 = \chi(G) = \chi(AB) = \chi(A)\chi(B)$$

and it follows that $\chi(A) = 0$ or $\chi(B) = 0$. If $\chi \in \mathrm{Ann}(K)$, then $\chi(a^p) \neq 1$. Hence $\chi(A) \neq 0$ and so $\chi(B) = 0$. This means that $\mathrm{Ann}(K) \subset \mathrm{Ann}(B)$.

There is a neat elementary proof for Hajós' theorem for p-groups. See for instance [4] pages 157–161. From this reason we deal only with the non-p-group case of Hajós' theorem. Hajós' theorem can be stated in a sharper form and we need this sharper version. Let $G = A_1 \cdots A_n$ be a factorization of the finite abelian group G into the cyclic subsets $A_1, \ldots, A_n$. Suppose that $A_1 = H_1$ is a subgroup of G. Now, $G^{(1)} = A_2^{(1)} \cdots A_n^{(1)}$ is a factorization of the factor group $G^{(1)} = G/H_1$, where $A_i^{(1)} = (A_i H_1)/H_1$. If $A_2^{(1)}$ is a subgroup of $G^{(1)}$, then $A_1 A_2 = H_2$ is a subgroup of G and $G^{(2)} = A_3^{(2)} \cdots A_n^{(2)}$ is a factorization of the factor group $G^{(2)} = G/H_2$, where $A_i^{(2)} = (A_i H_2)/H_2$. Continuing in this way finally we get that there is a permutation $B_1, \ldots, B_n$ of the factors $A_1, \ldots, A_n$ such that the partial products

$$B_1, \quad B_1 B_2, \ldots, B_1 B_2 \cdots B_n$$

are subgroups of G.

We say that the subset A of G is *replaceable* by A' if $G = A'B$ is a factorization of G whenever $G = AB$ is a factorization of G. We need only the next two replacement results. If $G = AB$ is a factorization and g is an element of G, then multiplying the factorization by g we get the factorization $G = Gg = (Ag)B$. This means that A can be replaced by Ag for each $g \in G$. Note that A is periodic if and only if Ag is periodic. The other replacement result we need reads as follows. The cyclic subset (1) can be replaced by

$$A' = \{e, a^r, a^{2r}, \ldots, a^{(p-1)r}\}$$

for each integer r that is relatively prime to p. This is Lemma 1 of [4] page 158.

3. The result

After all these preparations we are now ready to prove Hajós' theorem for non-p-groups.

Theorem 1. *Let G be a finite abelian non-p-group and let*

$$G = A_1 \cdots A_n \tag{2}$$

be a factorization of G, where $A_1, \ldots, A_n$ are cyclic subsets of prime order. Then at least one of the factors must be a subgroup of G.

Proof. Let

$$A_i = \{e, a_i, a_i^2, \ldots, a_i^{p_i - 1}\}$$

be a typical factor in factorization (2) and let $K_i = \langle a_i^{p_i} \rangle$ be the subgroup assigned to A_i. We call the quantity

$$w(A_1, \ldots, A_n) = |a_1| \cdots |a_n|$$

the *weight* of the factorization (2). Assume the contrary that none of the factors in (2) is a subgroup of G, that is, $K_i \neq \{e\}$ for each i, $1 \leq i \leq n$. We choose counter examples with the smallest possible n. Among these counter examples we pick one with minimal weight.

Consider all the factors among $A_1,\dots,A_n$ whose order is p_1. Let they be $A_1,\dots,A_s$. If $|a_i|$ is a p_1 power for each i, $1\le i\le s$, then $A_1\cdots A_s$ forms a factorization of the p_1-component of G. From this by Lemma 3 of [4] page 160, it follows that one of $A_1,\dots,A_s$ is a subgroup. This is not the case so one of $a_1,\dots,a_s$, say a_1, is not a p_1-element. There is a prime r such that $r \bigm| |a_1|$ and $r\neq p_1$. In factorization (2) replace A_1 by

$$A_1' = \{e, a_1^r, a_1^{2r}, \dots, a_1^{(p_1-1)r}\}$$

to get the factorization $G = A_1'A_2\cdots A_n$. As $|a_1^r| < |a_1|$, it follows that

$$w(A_1', A_2, \dots, A_n) < w(A_1, \dots, A_n).$$

The weight of the factorization decreased so one of the factors $A_1', A_2, \dots, A_n$ is a subgroup of G. This can only be A_1'.

There is a permutation $B_1,\dots,B_n$ of the factors $A_1', A_2,\dots,A_n$ such that $B_1 = A_1'$ and the partial products

$$B_1, B_1B_2, \dots, B_1B_2\cdots B_n$$

are subgroups of G. We may assume that the permutation is the identity since this is only a matter of relabelling the factors. So the partial products

$$A_1',\quad A_1'A_2, \dots, A_1'A_2\cdots A_n$$

are subgroups of G.

Let $H_i = A_1'A_2\cdots A_i$ and $C_i = A_{i+1}\cdots A_n$. Note that as $a_i\in H_i$, it follows that $K_i\subset H_i$.

From the factorization $G = A_1C_1$ it follows that $0=\chi(C_1)$ for each $\chi\in \mathrm{Ann}(K_1)$, that is, $\mathrm{Ann}(K_1)\subset \mathrm{Ann}(C_1)$. As $C_1 = A_2C_2$ we get that $0 = \chi(A_2)\chi(C_2)$ for each $\chi\in\mathrm{Ann}(K_1)\subset\mathrm{Ann}(C_1)$. Hence

$$\mathrm{Ann}(K_1)\cap\mathrm{Ann}(K_2)\subset\mathrm{Ann}(C_2).$$

By the partition lemma there are subsets X_2, Y_2 of G such that

$$C_2 = X_2K_1\cup Y_2K_2,$$

where the union is disjoint and the products are direct. If there is an element y_2 of Y_2, then in the factorization $G = H_2C_2$ the factor C_2 can be replaced by $y_2^{-1}C_2$ to get the factorization $G = H_2(y_2^{-1}C_2)$. But here $K_2\subset H_2$ and $K_2\subset y_2^{-1}C_2$ violates the factorization as $K_2\neq\{e\}$. Thus $Y_2=\emptyset$ and so $C_2 = X_2K_1$. This is equivalent to $\mathrm{Ann}(K_1)\subset\mathrm{Ann}(C_2)$.

As $C_2 = A_3C_3$, it follows that $0=\chi(A_3)\chi(C_3)$ for each $\chi\in\mathrm{Ann}(K_1)\subset\mathrm{Ann}(C_2)$. Therefore

$$\mathrm{Ann}(K_1)\cap\mathrm{Ann}(K_3)\subset\mathrm{Ann}(C_3).$$

By the partition lemma there are subsets X_3, Y_3 of G such that

$$C_3 = X_3K_1\cup Y_3K_3,$$

where the union is disjoint and the products are direct. If there is an element $y_3 \in Y_3$, then in the factorization $G = H_3C_3$ the factor C_3 can be replaced by $y_3^{-1}C_3$ to get the factorization $G = H_2(y_3^{-1}C_3)$. But here $K_3 \subset H_3$ and $K_3 \subset y_3^{-1}C_3$ violates the factorization as $K_3 \neq \{e\}$. Thus $Y_3 = \emptyset$ and so $C_3 = X_3K_1$. This is equivalent to $\mathrm{Ann}(K_1) \subset \mathrm{Ann}(C_3)$.

Using $C_3 = A_4C_4$ it follows that $0 = \chi(A_4)\chi(C_4)$ for each $\chi \in \mathrm{Ann}(K_1) \subset \mathrm{Ann}(C_3)$. Therefore

$$\mathrm{Ann}(K_1) \cap \mathrm{Ann}(K_4) \subset \mathrm{Ann}(C_4).$$

By the partition lemma there are subsets X_4, Y_4 of G such that

$$C_4 = X_4K_1 \cup Y_4K_4,$$

where the union is disjoint and the products are direct. If there is an element $y_4 \in Y_4$, then the factorization $G = H_4(y_4^{-1}Y_4)$ leads to the contradiction $K_4 \subset H_4 \cap (y_4^{-1}Y_4) = \{e\}$. Thus $Y_4 = \emptyset$ and consequently $C_4 = X_4K_1$.

Continuing in this way finally we get that $C_{n-1} = X_{n-1}K_1$. But $C_{n-1} = A_n$ and it follows that $A_n = K_1$. This contradiction completes the proof.

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Received November 1, 2003