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Conjugacy for Closed Convex Sets

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Abstract. Even though the polarity is a well defined operation for arbitrary subsets in the Euclidean n-dimensional space, the related operation of conjugacy of faces appears defined in the literature exclusively for either convex bodies containing the origin as interior point and their polar sets, or for closed convex cones. This paper extends the geometry of closed convex cones and convex bodies to unbounded convex sets (and, in a dual way, to those closed convex sets containing the origin at the boundary), not only for the sake of theoretical completeness, but also for the potential applications of this theory in the fields of Convex Programming and Semi-infinite Programming. Introducing the recession cone into the analysis we develop a general theory of conjugacy which, together with the new concept of curvature index of a convex set on a face, allows us to establish a strong result on complementary dimensions of conjugate faces which extends a well-known result on polytopes.

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1. Introduction

In contrast with polarity, which is defined for any subset in $\mathbb{R}^n$, conjugacy of faces has been only defined for convex bodies (i.e., for full-dimensional convex and compact sets) which are neighborhoods of the origin (see, e.g., [4] and [13]), and for pointed full-dimensional closed

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convex cones (see, e.g., [3]). This paper extends the geometry of convex cones and polyhedra to unbounded closed convex sets and, in a dual way, to those convex sets for which the origin is a boundary point. The geometrical properties of the (possibly unbounded) closed convex sets have been used for algorithm constructions in Semi-infinite Programming ([2], [9], [5], and others), and for the characterization of the solution set of linear inequality systems (in [1], [10], [7], and [8]).

On the other hand, the new concept of "curvature index" of a convex set on a face, allows us to establish a strong result about complementary dimensions of conjugate faces, first on cones and then, on general closed convex sets by using the extended conjugation. These results complete the co-dimension theorem of conjugate faces ([4], Theorem 6.10, reproduced here as Theorem 1.1) known for polytopes as an equality.

The paper is organized as follows. In the rest of this introduction the notation is fixed and some necessary results are recalled. In Section 2 we define the conjugacy of faces, the so-called $\triangle$-operation, to general closed convex sets, extending and unifying to this general frame classic results on conjugacy. Theorem 2.2 is the main result in this section and Theorem 2.5 summarizes the resulting structure of considering the extended $\triangle$-operation. In Section 3 we introduce the new concept of curvature index of a convex set on a face. Then, for closed convex cones provided with the classical $\triangle$-operation, results on complementary dimensions are established and the invariance of the curvature index through conjugacy is proved. In particular, we give a strong result about complementary dimensions of conjugate faces, first on cones and then, on general closed convex sets. Finally, in Section 4, results about the curvature index and co-dimensions of faces are extended to general closed convex sets provided with the $\triangle$-operation introduced in Section 2. Theorem 4.2 is the main result in this section which includes graphical examples visualizing the construction of polar sets and the techniques developed in the paper.

For notation and general concepts (e.g., those of proper face and exposed face of a convex set) the main references are [4], [12], and [13]. We denote by θ the origin of coordinates in $\mathbb{R}^n$. Given a set $\emptyset \neq M \subset \mathbb{R}^n$, the convex, affine, conical convex (containing the origin) and linear hulls of M are denoted by $\operatorname{conv} M$, $\operatorname{aff} M$, $\operatorname{cone} M$ and $\operatorname{span} M$, respectively. Also, $\operatorname{cl} M$, $\operatorname{int} M$, $\operatorname{bd} M$ denote the closure, the interior and the boundary of M; by $\operatorname{ri} M$ and $\operatorname{rbd} M$, the relative interior and the relative boundary of M (with regard to $\operatorname{aff} M$). For $C \neq \emptyset$ convex and closed, the dimension of C is $\dim C = \dim \operatorname{aff} C$, 0^+C denotes its recession cone, and the linearity subspace of C is $\operatorname{lin} C = 0^+C \cap (-0^+C)$. The feasible directions cone of C at x is denoted by $D(C, x)$ and the corresponding tangent cone by $T_C(x) = \operatorname{cl} D(C, x)$ (see [6], p. 64).

We define the *tangent space* of C at x as $TS_C(x) = \operatorname{lin} T_C(x)$. We associate with $\theta \neq x \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ the hyperplane

$$H(x, \alpha) = \{y \in \mathbb{R}^n \mid \langle x, y \rangle = \alpha\}$$

and the corresponding half-space

$$H^-(x, \alpha) = \{y \in \mathbb{R}^n \mid \langle x, y \rangle \leq \alpha\}.$$

The (positive) polar set of $M \subset \mathbb{R}^n$, $M \neq \emptyset$, is defined as

$$M^o = \{y \in \mathbb{R}^n \mid \langle x, y\rangle \geq -1, \ \forall x \in M\}.$$

Obviously, M^o is always convex and closed, and $\theta \in M^o$. Moreover, polarity inverts the inclusion, $(\rho M)^o = \rho^{-1} M^o$ for all $\rho > 0$, and $M^o = [\operatorname{cl} \operatorname{conv}(M \cup \{\theta\})]^o$.

If C is convex, closed, and $\theta \in C$, then $C^{oo} := (C^o)^o = C$, and we have

$$\theta \in \operatorname{int} C \Longleftrightarrow C^o \text{ is bounded},$$

statement where C and C^o can be interchanged. Hence, C is a convex body and $\theta \in \operatorname{int} C$ if and only if C^o satisfies the same properties.

Now assume that C is a convex body such that $\theta \in \operatorname{int} C$. Then, given a face F of C, its conjugate face is

$$F^{\triangle} = \{y \in C^o \mid \langle x, y\rangle = -1, \ \forall x \in F\} \tag{1}$$

(we respect the usual notation even though it may be sometimes ambiguous because the set C which appears in the definition is not explicit). In fact, $F^{\triangle}$ is an exposed face of C^o ($C^{\triangle} = \emptyset$ and $\emptyset^{\triangle} = C^o$). The $\triangle$-operation defined by (1) inverts the inclusion, i.e., if $F \subset F' \subset C$ then $(F')^{\triangle} \subset F^{\triangle} \subset C^o$, and F is an exposed face of C if and only if $F^{\triangle\triangle} := \left(F^{\triangle}\right)^{\triangle} = F$. If C is a convex body such that $\theta \in \operatorname{int} C$, then the $\triangle$-operation establishes a one-to-one correspondence between the exposed faces of C and C^o and the dimensions of each pair of proper faces are related as follows ([4], Theorem 6.10).

Theorem 1.1. *Let $C \subset \mathbb{R}^n$ be a convex body such that $\theta \in \operatorname{int} C$, and let F be an exposed proper face of C. Then $F^{\triangle}$ is also an exposed proper face of C^o and*

$$\dim F + \dim F^{\triangle} \leq n - 1.$$

It is known that the equality holds if C is a polytope, i.e., conjugate faces have complementary dimensions.

In the case of a cone K (a non-empty set which is closed under non-negative scalar multiplications), its (positive) polar cone turns out to be

$$K^o = \{y \in \mathbb{R}^n \mid \langle x, y\rangle \geq 0, \ \forall x \in K\}.$$

There exists a parallel conjugacy theory for pointed full-dimensional closed convex cones (see, e.g., [3]). The conjugate of a face $G \neq \emptyset$ of K is defined as

$$G^{\triangle} = \{y \in K^o \mid \langle x, y\rangle = 0, \ \forall x \in G\} = K^o \cap G^{\perp}, \tag{2}$$

which is an exposed face of K^o ($K^{\triangle} = \{\theta\}$ and $\{\theta\}^{\triangle} = K^o$). Moreover, if $G \subset G' \subset K$ then $(G')^{\triangle} \subset G^{\triangle} \subset K^o$, and G is an exposed face of K if and only if $G^{\triangle\triangle} = G$.

The pointedness assumption on K (or the dual condition of the full-dimensionality) can be removed as follows ([7]). Denoting $L := \operatorname{lin} K$, and $\overline{K} := K \cap L^{\perp}$, where $L^{\perp}$ is the orthogonal complement of L. The cone K turns out to be the direct sum $K = L \oplus \overline{K}$, where

$\overline{K}$ is a pointed cone. Similarly, if $G \neq \emptyset$ is a face (an exposed face) of K and $\overline{G} := G \cap L^{\perp}$, then $G = L \oplus \overline{G}$, where $\overline{G}$ is a face (an exposed face) of $\overline{K}$. The operations

$$K^o = \overline{K}^o \cap L^{\perp} \qquad \text{and} \qquad G^{\triangle} = \overline{G}^{\triangle} \cap L^{\perp}$$

coincide with the corresponding operations carried out on the linear subspace $L^{\perp}$. Observe that $K^{\triangle} = \operatorname{lin} K^o$, $(\operatorname{lin} K)^{\triangle} = K^o$, and that G is an exposed face of K if and only if $G^{\triangle\triangle} = G$.

Working with unbounded convex sets, it is suitable to use the one-to-one correspondence between convex cones on $\mathbb{R}^{n+1}$ contained in the lower half-space and convex sets on $\mathbb{R}^n$ (see [12], p. 20). We will use the notation $(x;\chi) \in \mathbb{R}^{n+1}$, where $x \in \mathbb{R}^n$ and $\chi \in \mathbb{R}$. The lower closed half-space on $\mathbb{R}^{n+1}$ will be denoted by $H_0^- = H^-((\theta;1),0)$, and we shall identify convex sets in $\mathbb{R}^n$ with convex subsets in the hyperplane $H_{-1} = \mathbb{R}^n \times \{-1\} \subset \mathbb{R}^{n+1}$. Given a convex set $C \subset \mathbb{R}^n$, the closed cone

$$\widehat{C} := \operatorname{cl}\operatorname{cone}(C \times \{-1\}) \subset \mathbb{R}^{n+1}$$

will be called *associated cone* to C. Similarly, given a cone $K \subset \mathbb{R}^n$ the *horizon cone* of K will be

$$\widetilde{K} := K \times \{0\} \subset \mathbb{R}^{n+1}.$$

Obviously, $\operatorname{lin}\widetilde{K} = \widetilde{\operatorname{lin} K}$, $\dim K = \dim \widetilde{K}$ and $\widetilde{K}$ is closed if K is closed.

For $x \in \mathbb{R}^n$ we shall denote $\widehat{x} := (x;-1)$ and $\widetilde{x} := (x;0)$. If C is closed and convex, by Theorem 8.2 in [12] we have

$$\widehat{C} = \operatorname{cone}(C \times \{-1\}) \cup \widetilde{0^+C}. \tag{3}$$

Clearly, $\widetilde{0^+C}$ is an exposed face of $\widehat{C}$, and $\widetilde{x} \in \widehat{C}$ if and only if $x \in 0^+C$. Moreover, it can be shown ([7]) that, if K is a closed convex cone such that $K \subset H_0^- \subset \mathbb{R}^{n+1}$ and C satisfies $C \times \{-1\} = K \cap H_{-1}$ then $K = \widehat{C}$.

Proposition 1.2. *If $C \subset \mathbb{R}^n$ is a closed convex set containing the origin then $\widehat{C^o} = \widehat{C}^o$.*

Proof. For $u \in \mathbb{R}^n$,

$$\begin{aligned}
&\widehat{u} \in C^o \times \{-1\} \Leftrightarrow u \in C^o\\
&\Leftrightarrow \lambda\langle x,u\rangle + \lambda \geq 0, \quad \forall x \in C,\ \forall\lambda \geq 0\\
&\Leftrightarrow \langle \lambda\widehat{x}, \widehat{u}\rangle \geq 0, \quad \forall x \in C,\ \forall \lambda \geq 0\\
&\Leftrightarrow \widehat{u} \in (\operatorname{cone}(C \times \{-1\}))^o,
\end{aligned}$$

so that

$$C^o \times \{-1\} = (\operatorname{cone}(C \times \{-1\}))^o \cap H_{-1} = \widehat{C}^o \cap H_{-1}$$

and we obtain

$$\widehat{C^o} \cap H_{-1} = C^o \times \{-1\} = \widehat{C}^o \cap H_{-1} = \left(\widehat{C}^o \cap H_0^-\right) \cap H_{-1}.$$

This means that both sets, $\widehat{C^o}$ and $\widehat{C}^o \cap H_0^-$, are closed convex cones contained in the lower half-space having the same section on H_{-1}, and so $\widehat{C^o} = \widehat{C}^o \cap H_0^-$. Since

$$\left(H_0^-\right)^o = \operatorname{cone}\{(\theta; -1)\} \subset \operatorname{cone}(C \times \{-1\}) \subset \widehat{C},$$

$\widehat{C}^o \subset H_0^-$ and we get $\widehat{C^o} = \widehat{C}^o$. $\square$

Lemma 1.3. *Let $C \subset \mathbb{R}^n$ be a closed convex set and let F and G be nonempty sets such that $F \subset C$ and $G \subset 0^+C$. Then the following statements hold:*

1. *F is a (exposed) face of C if and only if $\widehat{F}$ is a (exposed) face of $\widehat{C}$ which is not contained in $\widetilde{0^+C}$.*
2. *G is a face of 0^+C if and only if $\widetilde{G}$ is a face of $\widehat{C}$.*
3. *If $\widetilde{G}$ is an exposed face of $\widehat{C}$, then G is an exposed face of 0^+C.*

Proof. We will prove 2. first.

2. Since $\widetilde{0^+C}$ is a face of $\widehat{C}$ and $\widetilde{G} \subset \widetilde{0^+C}$, we get that $\widetilde{G}$ is face of $\widehat{C}$ if and only if $\widetilde{G}$ is face of $\widetilde{0^+C}$, and $\widetilde{G}$ is face of $\widetilde{0^+C}$ if and only if G is face of 0^+C.

3. If $\widetilde{G}$ is an exposed face of $\widehat{C}$, $\widetilde{G} = \widehat{C} \cap H$ for some hyperplane $H \subset \mathbb{R}^{n+1}$. Moreover $\widetilde{G} \subset H_0$ then

$$\widetilde{G} = \left(\widehat{C} \cap H\right) \cap H_0 = \left(\widehat{C} \cap H_0\right) \cap H = \widetilde{0^+C} \cap H,$$

thus $\widetilde{G}$ is an exposed face of $\widetilde{0^+C}$ in H_0 and so, G is an exposed face of 0^+C. (If $H = H_0$ then G is the improper exposed face 0^+C.)

1. Let F be a face of C. Let $(y;\gamma), (z;\zeta) \in \widehat{C}$ and $\lambda \in]0,1[$ be such that

$$\begin{pmatrix} x \\ \chi \end{pmatrix} := \lambda \begin{pmatrix} y \\ \gamma \end{pmatrix} + (1-\lambda) \begin{pmatrix} z \\ \zeta \end{pmatrix} \in \widehat{F}.$$

We will prove $(y;\gamma), (z;\zeta) \in \widehat{F}$. Obviously $\gamma \leq 0$, $\zeta \leq 0$, and $\chi \leq 0$.

Case I: $\gamma < 0$ and $\zeta < 0$. Then $y/|\gamma|$ and $z/|\zeta|$ belong to C, moreover $\chi < 0$, $x/|\chi| \in F$, and

$$\frac{x}{|\chi|} = \frac{\lambda y + (1-\lambda) z}{|\lambda\gamma + (1-\lambda)\zeta|} = \frac{-\lambda\gamma}{|\lambda\gamma + (1-\lambda)\zeta|} \frac{y}{|\gamma|} + \frac{-(1-\lambda)\zeta}{|\lambda\gamma + (1-\lambda)\zeta|} \frac{z}{|\zeta|},$$

with

$$\frac{-\lambda\gamma}{|\lambda\gamma + (1-\lambda)\zeta|} + \frac{-(1-\lambda)\zeta}{|\lambda\gamma + (1-\lambda)\zeta|} = 1.$$

Since F is a face of C, we get $y/|\gamma|, z/|\zeta| \in F$, therefore $(y;\gamma), (z;\zeta) \in \widehat{F}$.

Case II: $\gamma < 0$ and $\zeta = 0$. Now $y/|\gamma| \in C$, $z \in 0^+C$, and $\chi = \lambda\gamma < 0$. We will prove that $y/|\gamma| \in F$ and $z \in 0^+F$. Since $(x;\chi) \in \widehat{F}$ then $\frac{1}{\lambda|\gamma|}(x;\chi) \in \widehat{F}$ so $\frac{1}{\lambda|\gamma|}x \in F$. Moreover, for all $\mu \neq 1$,

$$\frac{x}{\lambda|\gamma|} = \frac{y}{|\gamma|} + \frac{1-\lambda}{\lambda|\gamma|}\,z = \mu\frac{y}{|\gamma|} + (1-\mu)\left(\frac{y}{|\gamma|} + \frac{1-\lambda}{(1-\mu)\lambda|\gamma|}\,z\right).$$

Since F is a face of C, $y/|\gamma| \in F$, moreover

$$\frac{y}{|\gamma|} + \frac{1-\lambda}{(1-\mu)\lambda|\gamma|}\,z \in F$$

for all $\mu < 1$, which imply $z \in 0^+F$.
Case III: $\gamma = \zeta = 0$. Now $y, z \in 0^+C$, and $\chi = 0$. We will prove $y, z \in 0^+F$. For any $w \in F$, $\alpha \geq 0$, we get $w + \alpha y \in C$, $w + \alpha z \in C$, and $w + \alpha x \in F$. Then

$$\lambda(w + \alpha y) + (1-\lambda)(w + \alpha z) = w + \alpha x \in F,$$

which implies $w + \alpha y \in F$ and $w + \alpha z \in F$, because F is a face of C. Therefore $y, z \in 0^+F$.
At this point we have proved that if F is a face of C then $\widehat{F}$ is a face of $\widehat{C}$ (obviously $\widehat{F}$ is not contained in $\widetilde{0^+C}$ if $F \neq \emptyset$). Conversely, suppose $\widehat{F}$ is a face of $\widehat{C}$ which is not contained in $\widetilde{0^+C}$. Denoting $F \times \{-1\} := \widehat{F} \cap H_{-1}$, we must prove F is a face of C. Let $y, z \in C$, and $\lambda \in]0,1[$ be such that $x := \lambda y + (1-\lambda) z \in F$. Then $(y;-1) \in \widehat{C}$, $(z;-1) \in \widehat{C}$, and

$$(x;-1) = \lambda(y;-1) + (1-\lambda)(z;-1) \in \widehat{F} \cap H_{-1}.$$

Since $\widehat{F}$ is a face of $\widehat{C}$ we conclude that $(y;-1)$ and $(z;-1)$ belong to $\widehat{F}$, thus $y, z \in F$. Therefore F is a face of C.

Now we will prove that if F is an exposed face of C then $\widehat{F}$ is an exposed face of $\widehat{C}$. Since $F = H \cap C$ for some hyperplane $H \subset \mathbb{R}^n$, the hyperplane span $\widehat{H} \subset \mathbb{R}^{n+1}$ satisfies

$$\operatorname{span}\widehat{H} \cap \widehat{C} = \widehat{H} \cap \widehat{C} = \widehat{H \cap C} = \widehat{F}.$$

Conversely, let $\widehat{F}$ be an exposed face of (the cone) $\widehat{C}$ which is not contained in $\widetilde{0^+C}$, i.e., $\widehat{F} = H \cap \widehat{C}$ for some hyperplane (through the origin) $H \subset \mathbb{R}^{n+1}$, $H \neq H_0$. Denoting $F \times \{-1\} := \widehat{F} \cap H_{-1}$ we have

$$\begin{aligned} F \times \{-1\} &= \left(H \cap \widehat{C}\right) \cap H_{-1} \\ &= (H \cap H_{-1}) \cap \left(\widehat{C} \cap H_{-1}\right) \\ &= (H \cap H_{-1}) \cap (C \times \{-1\}). \end{aligned}$$

Since $H \cap H_{-1}$ is a hyperplane in the affine space $\mathbb{R}^n \times \{-1\}$ we conclude that $F \times \{-1\}$ is an exposed face of $C \times \{-1\}$ in the affine space $\mathbb{R}^n \times \{-1\}$. Therefore F is an exposed face of C. □

Later, helped with Figure 4, we shall prove that the converse of statement 3 in Lemma 1.3 is not true.

Corollary 1.4. *If F is a nonempty face of the closed convex set C and S_F is the minimal exposed face of C containing F, then $\widehat{S_F}$ is the minimal exposed face of $\widehat{C}$ containing $\widehat{F}$ and, consequently, $\widehat{F}^{\triangle} = \widehat{S_F}^{\triangle}$.*

We conclude this section recalling Lemma 2.2 in [11], rewritten here on our notation.

Lemma 1.5. *For $x \in C$, $\mu > 0$, the feasible directions cone of $\widehat{C}$ at $\mu\widehat{x}$ satisfies:*

$$D\left(\widehat{C}, \mu\widehat{x}\right) = \widetilde{D(C,x)} + \operatorname{span}\{\widehat{x}\}.$$

2. Conjugacy of faces

Two difficulties can arise in the application of the $\triangle$-operation (1) to a closed convex set C such that either it is unbounded or $\theta \in \operatorname{bd} C$. If C is unbounded, C^o may have exposed faces containing the origin which do not correspond to any face of C. In Figure 1 the exposed faces of C^o (labelled α, β and γ) are not conjugate of any face of C. On the other hand, the face α of C^o seems to be the conjugate of face 1 of C, but actually $1^{\triangle} = \emptyset$.

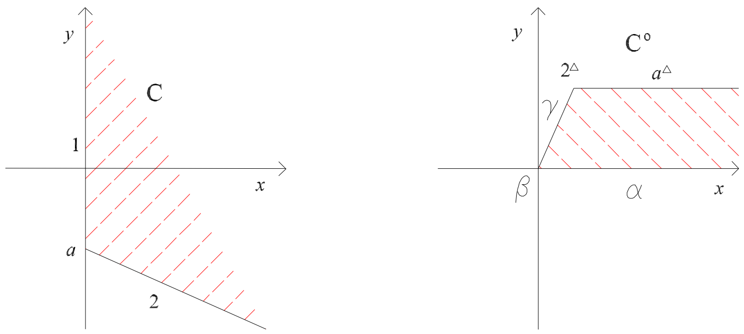

Figure 1

Similarly, if $\theta \in \operatorname{bd} C$, exposed faces of C can exist for which it is not possible to assign any conjugate face in C^o. This is visible in Figure 2, where the exposed face 2 of C does not correspond with any face of C^o (and the same happens in Figure 1 with the face 1).

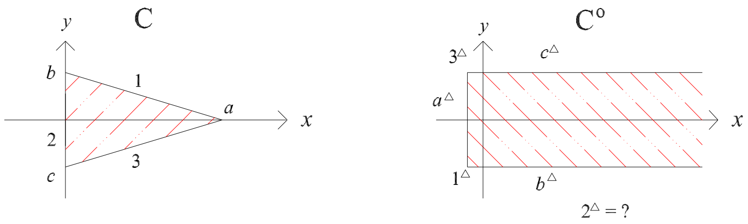

Figure 2

In order to overcome these difficulties, we introduce the recession cones 0^+C and $0^+(C^o)$ into the analysis. In addition, it is not necessary to consider the empty face as an exposed face, because its role will be played by the minimal faces of these cones. So from now on, "face" will mean "nonempty face". Since the next definition extends the classical $\triangle$-operation, we will use the same notation. However, observe that the conjugate face of any (nonempty) face will always be nonempty.

Definition 2.1. *Let $C \subset \mathbb{R}^n$ be a closed convex set containing the origin and let F be a face of C. We define its conjugate face as*

$$F^{\triangle} := \{y \in C^o \mid \langle x, y\rangle = -1,\ \forall x \in F\}, \tag{4}$$

if (4) *is not empty. Otherwise,*

$$F^{\triangle} := \left\{y \in 0^+(C^o) \mid \langle x, y\rangle = 0,\ \forall x \in F\right\}. \tag{5}$$

If G is a face of 0^+C, we define its conjugate face as

$$G^{\triangle} = \{y \in C^o \mid \langle x, y\rangle = 0,\ \forall x \in G\}. \tag{6}$$

If $\theta \notin F$ and F is an exposed face of C, then there $F^{\triangle}$ is the set (4), i.e. $\emptyset \neq F^{\triangle} \subset C^o$ and $\theta \notin F^{\triangle}$. Which is true because exists $c \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ such that $\langle c, x\rangle \geq \alpha$ for all $x \in C$ and $\langle c, x\rangle = \alpha$ for all $x \in F$. Since $\theta \in C \backslash F$, $\alpha < 0$ and it can be easily shown that $-\alpha^{-1}c$ belongs to the set in (4).

Clearly, given two faces $F \subset F' \subset C$, if both sets $F^{\triangle}$ and $F'^{\triangle}$ are defined by (4) (by (5), e.g., if $\theta \in F$), then $F'^{\triangle} \subset F^{\triangle}$, these sets are subsets of C^o (of $0^+(C^o)$ and so cones, respectively). For faces $G \subset G' \subset 0^+C$ we have $\theta \in G'^{\triangle} \subset G^{\triangle} \subset C^o$.

Let us compare Definition 2.1 with the classical one for a convex body C such that $\theta \in \operatorname{int} C$ (recall (1)). Given a face $F \neq C$, the minimal exposed face of C containing F verifies $F \subset S_F \subset \operatorname{bd} C$, so that $\theta \notin S_F$, and we have

$$\emptyset \neq \{y \in C^o \mid \langle x, y\rangle = -1,\ \forall x \in S_F\} \subset \{y \in C^o \mid \langle x, y\rangle = -1,\ \forall x \in F\}.$$

Observe that $C^{\triangle} = \emptyset$ for the classical definition whereas $C^{\triangle} = \{\theta\} = 0^+(C^o)$ for the previous one.

Now we consider given a closed convex cone K and a face G of K. According to Definition 2.1, since $\theta \in G$, the set in (4) is empty and so

$$G^{\triangle} := \left\{y \in 0^+(K^o) \mid \langle x, y\rangle = 0,\ \forall x \in G\right\} = \{y \in K^o \mid \langle x, y\rangle = 0,\ \forall x \in G\},$$

and this coincides with the classical definition (recall (2)).

Recalling again Figure 1, observe that the face α of C^o is associated by conjugacy with a face of 0^+C.

The next theorem is based upon the identity $\widehat{C^o} = \widehat{C}^o$ of Proposition 1.2.

Theorem 2.2. *Let $C \subset \mathbb{R}^n$ be a closed convex set containing the origin. Then the following statements hold:*

1. *If F is a face of C and $F^{\triangle}$ is the nonempty set* (4)*, then:*

   1.A $F^{\triangle}$ *is an exposed face of C^o and $\theta \notin F^{\triangle}$, and*

   1.B $\widehat{F^{\triangle}} = \widehat{F}^{\triangle}$.

2. *If F is a face of C and $F^{\triangle}$ is the cone* (5), *then*

   2.A $\widetilde{F^{\triangle}}$ *is an exposed face of* $\widetilde{C^o}$, *therefore* $F^{\triangle}$ *is an exposed face of* $0^+(C^o)$, *and*

   2.B $\widetilde{F^{\triangle}} = \widehat{F}^{\triangle}$.

3. *If G is a face of 0^+C, then*

   3.A $G^{\triangle}$ *is an exposed face of* C^o *and* $\theta \in G^{\triangle}$, *and*

   3.B $\widehat{G^{\triangle}} = \widetilde{G}^{\triangle}$.

Proof. Since $\theta \in C$, by Proposition 1.2 we have

$$\widehat{C}^o = \widehat{C^o}. \tag{7}$$

The first part of 1.A is consequence of 1.B, (7), and Lemma 1.3, part 1. In fact, since $\widehat{F}^{\triangle}$ is an exposed face of $\widehat{C}^o$ by conjugacy on cones, then $\widehat{F^{\triangle}}$ is an exposed face of $\widehat{C^o}$ and thus $F^{\triangle}$ turns out to be an exposed face of C^o. Moreover, if $\theta \in F^{\triangle}$ is true then $(\theta; -1) \in \widehat{F^{\triangle}} = \widehat{F}^{\triangle}$ and, by conjugacy on cones, $\widehat{F} \subset H_0$ contradicting $F \neq \emptyset$.

The equality 1.B, $\widehat{F^{\triangle}} = \widehat{F}^{\triangle}$, is equivalent to

$$\widehat{F}^{\triangle} \cap H_{-1} = F^{\triangle} \times \{-1\}, \tag{8}$$

because $\widehat{F}^{\triangle} \subset \widehat{C}^o$ is a closed convex cone contained in the lower half-space. In order to prove (8), taking into account that $F^{\triangle}$ is the set in (4) together with (7), we have

$$\begin{aligned}
&\widehat{u} \in F^{\triangle} \times \{-1\} \\
&\Leftrightarrow u \in F^{\triangle} \\
&\Leftrightarrow \langle x, u\rangle = -1,\ \forall x \in F, \text{ and } u \in C^o \\
&\Leftrightarrow \lambda \langle x, u\rangle + \lambda = 0,\ \forall x \in F,\ \forall \lambda \geq 0, \text{ and } u \in C^o \\
&\Leftrightarrow \langle \lambda\widehat{x}, \widehat{u}\rangle = 0,\ \forall x \in F,\ \forall \lambda \geq 0, \text{ and } \widehat{u} \in \widehat{C^o} \\
&\Leftrightarrow \widehat{u} \in \left\{\widehat{y} \in \widehat{C}^o \mid \langle \lambda\widehat{x}, \widehat{y}\rangle = 0,\ \forall \widehat{x} \in F \times \{-1\},\ \forall \lambda \geq 0\right\} \\
&\Leftrightarrow \widehat{u} \in \left\{\widehat{y} \in \widehat{C}^o \mid \langle \widehat{x}, \widehat{y}\rangle = 0,\ \forall \widehat{x} \in \widehat{F}\right\},
\end{aligned}$$

and, by (2), the last set is $\widehat{F}^{\triangle} \cap H_{-1}$.

Analogously, the first assertion in 2.A is consequence of 2.B and (7), because if $\widehat{F}^{\triangle}$ is an exposed face of $\widehat{C}^o$ then $\widetilde{F^{\triangle}}$ is an exposed face of $\widehat{C^o}$. On the other hand, we know that $\widetilde{F^{\triangle}} \subset \widetilde{0^+(C^o)}$ and, by Lemma 1.3, part 3, we conclude that $F^{\triangle}$ is an exposed face of $0^+(C^o)$.

Concerning 2.B we first prove that

$$\widehat{F}^{\triangle} \subset \widetilde{0^+(C^o)}. \tag{9}$$

Note that if $\widehat{F}^{\triangle}$ is the conjugate face of $\widehat{F}$ with regard to the cone $\widehat{C}$, then $\widehat{F}^{\triangle} \subset \widehat{C}^o$. Moreover, using (7) and (3), we have

$$\widehat{F}^{\triangle} \subset \widehat{C}^o = \widehat{C^o} = \operatorname{cone}(C^o \times \{-1\}) \cup \widetilde{0^+(C^o)}.$$

If (9) fails, there exists $\widehat{z} = (z; -1) \in \widehat{F}^{\triangle}$ such that $z \in C^o$. Recalling Definition 2.1 and the hypothesis of 2, we get

$$\{y \in C^o \mid \langle x, y\rangle = -1, \ \forall x \in F\} = \emptyset,$$

so that $\langle x, z\rangle \neq -1$ for some $x \in F$. But this is not possible because $\widehat{x} \in \widehat{F}$ and $\widehat{z} \in \widehat{F}^{\triangle}$ imply

$$0 = \langle \widehat{z}, \widehat{x}\rangle = \langle z, x\rangle + 1.$$

Therefore, $\widehat{F}^{\triangle} \subset \widetilde{0^+ (C^o)}$. This allows us to write:

$$\begin{aligned}\widehat{F}^{\triangle} &= \left\{(y;0) \in \mathbb{R}^{n+1} \mid y \in 0^+ (C^o), \ \langle u, \widetilde{y}\rangle = 0 \quad \forall u \in \operatorname{cl}\operatorname{cone}(F \times \{-1\})\right\} \\ &= \left\{(y;0) \in \mathbb{R}^{n+1} \mid y \in 0^+ (C^o), \ \langle x, y\rangle = 0 \quad \forall x \in F\right\} = \widetilde{F^{\triangle}},\end{aligned}$$

where the 2nd identity is consequence of the continuity on the inner product. This proves 2.B.

Now we prove 3.A using (7) and 3.B. Since $\widetilde{G}^{\triangle}$ is an exposed face of $\widehat{C}^o$, one has that $\widehat{G^{\triangle}}$ is an exposed face of $\widehat{C^o}$. By Lemma 1.3, part 1, $G^{\triangle}$ is an exposed face of C^o, with $\theta \in G^{\triangle}$ according to Definition 2.1.

Finally, we shall prove 3.B: $\widehat{G^{\triangle}} = \widetilde{G}^{\triangle}$. Since $\widetilde{G}^{\triangle}$ is a closed convex cone and $\widetilde{G}^{\triangle} \subset \widehat{C}^o \subset H_0^-$, we have just to prove that

$$\widetilde{G}^{\triangle} \cap H_{-1} = G^{\triangle} \times \{-1\}. \tag{10}$$

In fact, for $u \in C^o$, $\widehat{u} \in \widehat{C^o} = \widehat{C}^o$, we have

$$\begin{aligned}&\widehat{u} \in G^{\triangle} \times \{-1\} \\ &\Leftrightarrow u \in G^{\triangle} \\ &\Leftrightarrow \langle x, u\rangle = 0, \quad \forall x \in G \\ &\Leftrightarrow \langle \widetilde{x}, \widehat{u}\rangle = 0, \quad \forall \widetilde{x} \in \widetilde{G} \\ &\Leftrightarrow \widehat{u} \in \widetilde{G}^{\triangle},\end{aligned}$$

which proves (10). □

Corollary 2.3. *Let $C \subset \mathbb{R}^n$ be a closed convex set containing the origin, let F be a face of C and let S_F be the minimal exposed face of C containing F. Then $S_F^{\triangle} = F^{\triangle}$, and $\theta \notin S_F$ if and only if $\theta \in F^{\triangle}$.*

Proof. By Corollary 1.4,

$$\widehat{S_F}^{\triangle} = \widehat{F}^{\triangle}. \tag{11}$$

We shall discuss four possible cases.

Case I: $F^\triangle$ is the set (4) and $\theta \notin S_F$. By Theorem 2.2, part 1.B, $\widehat{F}^\triangle = \widehat{F^\triangle}$, and taking into account that S_F is exposed and $\theta \notin S_F$, the same result applies again, so that $\widehat{S_F}^\triangle = \widehat{S_F^\triangle}$. Recalling the identity (11) we obtain $\widehat{F^\triangle} = \widehat{S_F^\triangle}$. Therefore $F^\triangle = S_F^\triangle$.

Case II: If $F^\triangle$ is the set (4) and $\theta \in S_F$, again $\widehat{F}^\triangle = \widehat{F^\triangle}$ and, using now Theorem 2.2, part 2.B, we get $\widehat{S_F}^\triangle = \widetilde{S_F^\triangle}$. Adding the equality (11) we would obtain $\widehat{F^\triangle} = \widetilde{S_F^\triangle}$ which implies $\widehat{F^\triangle} \subset H_0$, in contradiction with $F^\triangle \subset C^o$. Therefore Case II is impossible.

Case III: If $F^\triangle$ is the cone (5) and $\theta \notin S_F$, by Theorem 2.2, part 1.B, $\widehat{S_F}^\triangle = \widehat{S_F^\triangle}$ and by Theorem 2.2, part 2.B, $\widehat{F}^\triangle = \widetilde{F^\triangle}$. Appealing again to (11), $\widehat{S_F^\triangle} = \widetilde{F^\triangle}$ so that $\widehat{S_F^\triangle} \subset H_0$, which is absurd. Therefore Case III is also impossible.

Case IV: $F^\triangle$ is the cone (5) and $\theta \in S_F$. By Theorem 2.2, part 2.B, $\widehat{S_F}^\triangle = \widetilde{S_F^\triangle}$ and $\widehat{F}^\triangle = \widetilde{F^\triangle}$, which together with (11) imply $\widetilde{S_F^\triangle} = \widetilde{F^\triangle}$. Hence $S_F^\triangle = F^\triangle$. □

Theorem 2.4. *Let $C \subset \mathbb{R}^n$ be a closed convex set containing the origin, $F \subset C$, and $G \subset 0^+C$. The following statements hold:*

1. *F is an exposed face of C if and only if $F = F^{\triangle\triangle}$.*
2. *$\widetilde{G}$ is an exposed face of $\widehat{C}$ if and only if $G = G^{\triangle\triangle}$.*

Proof. The reciprocal statements are consequences of Theorem 2.2.
1. Let F be an exposed face of C. By Lemma 1.3 we have

$$\widehat{F} = \widehat{F}^{\triangle\triangle}. \tag{12}$$

If $\theta \notin F$ then $\theta \notin F^\triangle$, and by using Theorem 2.2, part 1.B, twice, we get

$$\widehat{F}^{\triangle\triangle} = \widehat{F^\triangle}^{\triangle} = \widehat{F^{\triangle\triangle}}.$$

If $\theta \in F$, by Theorem 2.2 (statements 2.B and 3.B), we obtain

$$\widehat{F}^{\triangle\triangle} = \widetilde{F^\triangle}^{\triangle} = \widehat{F^{\triangle\triangle}}.$$

In both cases $\widehat{F}^{\triangle\triangle} = \widehat{F^{\triangle\triangle}}$. Combining this identity with (12) we have $\widehat{F} = \widehat{F^{\triangle\triangle}}$, from which we conclude $F = F^{\triangle\triangle}$.

2. Since G is a face of 0^+C and $\widetilde{G}$ is an exposed face of (the cone) $\widehat{C}$, Theorem 2.2 (parts 3.B and 2.B) yields

$$\widetilde{G} = \widetilde{G}^{\triangle\triangle} = \widehat{G^\triangle}^{\triangle} = \widetilde{G^{\triangle\triangle}},$$

and this entails $G = G^{\triangle\triangle}$. □

As a consequence of Theorems 2.2 and 2.4 it is easy to prove the next result on exposed faces corresponding to a given closed convex set containing θ and its polar.

Theorem 2.5. *Let $C \subset \mathbb{R}^n$ be a closed convex set containing the origin. Let $\mathcal{E}(C)$ be the set of all nonempty exposed faces of C not containing the origin, let $\mathcal{E}_0(C)$ be the set of all exposed faces of C containing the origin (including the improper face C) and let $\mathcal{E}_1(C)$ be the set of nonempty faces of 0^+C (including 0^+C) for which its horizon cone is an exposed face of $\widehat{C}$. Analogously for C^o, the three sets of exposed faces are $\mathcal{E}(C^o)$, $\mathcal{E}_0(C^o)$ and $\mathcal{E}_1(C^o)$. Then the conjugacy of nonempty faces produces three isomorphisms,*

$$\triangle : \mathcal{E}(C) \longrightarrow \mathcal{E}(C^o),$$
$$\triangle : \mathcal{E}_0(C) \longrightarrow \mathcal{E}_1(C^o),$$
$$\triangle : \mathcal{E}_1(C) \longrightarrow \mathcal{E}_0(C^o),$$

all of them reverting the order given by the inclusion of faces.

3. Curvature index

We shall introduce the new concept of curvature index of a closed convex set C on a face F, which will allow us to establish the complementary dimensions relation between conjugate faces on cones. Observe that $D(C,x)$ (and so $TS_C(x)$) is the same for all $x \in \operatorname{ri} F$. We denote this cone by $D(C,F)$.

Definition 3.1. *The curvature index of a nonempty closed convex set C at $x \in C$ is*

$$\operatorname{cin}(C,x) := \dim TS_C(x) - \dim \operatorname{lin} D(C,x).$$

For F being a face of C, the curvature index of C on F is

$$\operatorname{cin}(C,F) := \operatorname{cin}(C,x),$$

where x is an arbitrary element of $\operatorname{ri} F$.
For G being a face of 0^+C, the curvature index of C on G is

$$\operatorname{cin}(C,G) := \operatorname{cin}(\widehat{C},\widetilde{G}).$$

Observe that, since $\dim \operatorname{lin} D(C,F) = \dim F$, one has

$$\operatorname{cin}(C,F) = \dim TS_C(F) - \dim F.$$

Figure 3 illustrates this definition. For example, $\operatorname{cin}(C_1,p_1) = 2$ whereas $\operatorname{cin}(C_2,p_2) = 1$.

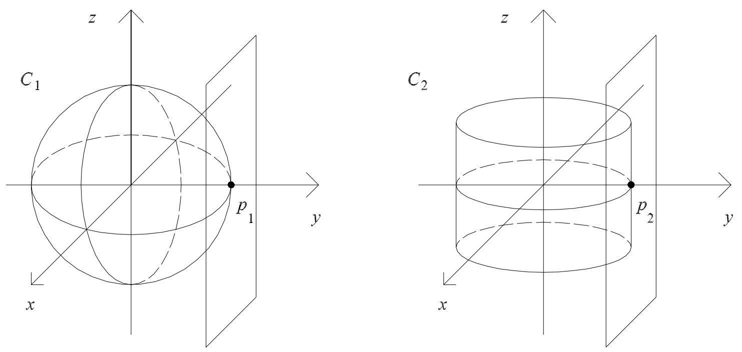

Figure 3

The curvature index of C on a face of the recession cone can only be seen on the cone $\widehat{C}$. In fact, in Figure 4, $\operatorname{cin}(C,3)=1$ and not 0, as one might suppose from just looking at 0^+C.

In the rest of this section K denotes a given closed convex cone, for which the $\triangle$-operation can be thought of in the classical way (recall 2).

Theorem 3.2. *If G is a face of $K \subset \mathbb{R}^n$, then*

$$\dim G + \dim G^{\triangle} + \operatorname{cin}(K,G) = n.$$

Proof. Observing that $D(K,G) = K + \operatorname{span} G$ we get $G^{\triangle} = [D(K,G)]^o$, and applying Corollary 14.6.1 in [12], one has

$$\begin{aligned}\dim G^{\triangle} &= \dim [\operatorname{cl} D(K,G)]^o \\ &= n - \dim \operatorname{lin} \operatorname{cl} D(K,G) \\ &= n - \dim TS_K(G),\end{aligned}$$

and because $\dim G = \dim \operatorname{lin} D(K,G)$,

$$\begin{aligned}\dim G + \dim G^{\triangle} &= \dim \operatorname{lin} D(K,G) + n - \dim TS_K(G) \\ &= n - \operatorname{cin}(K,G),\end{aligned}$$

which is the aimed identity. □

For cones, the curvature index is invariant under conjugacy as the next corollary shows.

Corollary 3.3. *If G is an exposed face of $K \subset \mathbb{R}^n$ then*

$$\operatorname{cin}(K,G) = \operatorname{cin}\left(K^o, G^{\triangle}\right).$$

Proof. By Theorem 3.2, and noting that $G = G^{\triangle\triangle}$ because G is exposed, we have

$$\begin{aligned}\operatorname{cin}(K,G) &= n - \dim G - \dim G^{\triangle} \\ &= n - \dim G^{\triangle\triangle} - \dim G^{\triangle} \\ &= \operatorname{cin}\left(K^o, G^{\triangle}\right),\end{aligned}$$

which ends the proof. □

Corollary 3.4. *If G is a face of $K \subset \mathbb{R}^n$, then*

$$\dim G + \operatorname{cin}(K,G) = \dim S_G + \operatorname{cin}(K,S_G),$$

where S_G is the minimal exposed face of K containing G.

Proof. By Theorem 3.2, and recalling that $G^{\triangle} = S_G^{\triangle}$, one has

$$\begin{aligned}\dim G + \operatorname{cin}(K,G) &= n - \dim G^{\triangle} = n - \dim S_G^{\triangle} \\ &= \dim S_G + \operatorname{cin}(K,S_G),\end{aligned}$$

which proves the corollary. □

4. Co-dimensions of conjugate faces

Now we extend the results on complementary dimensions and curvature index from cones to general closed convex sets.

Lemma 4.1. *Let F be a face of a closed convex set $C \subset \mathbb{R}^n$. Then*

$$\operatorname{cin}(C, F) = \operatorname{cin}\left(\widehat{C}, \widehat{F}\right).$$

Proof. Since $\dim F = \dim \operatorname{lin} D(C, F)$ we have

$$\operatorname{cin}(C, F) = \dim TS_C(F) - \dim F,$$

and

$$\operatorname{cin}\left(\widehat{C}, \widehat{F}\right) = \dim TS_{\widehat{C}}\left(\widehat{F}\right) - \dim \widehat{F}.$$

If $x \in \operatorname{ri} F$, one has $\widehat{x} \in \operatorname{ri} \widehat{F}$ ([12], Corollary 6.8.1), so that we only need to prove that

$$\dim TS_{\widehat{C}}(\widehat{x}) = \dim TS_C(x) + 1.$$

We will use now Lemma 1.5, taking into account that

$$\operatorname{cl}\left(\widetilde{D(C, x)} + \operatorname{span}\{\widehat{x}\}\right) = \operatorname{cl}\widetilde{D(C, x)} + \operatorname{span}\{\widehat{x}\},$$

because $\operatorname{cl}\widetilde{D(C, x)} \cap \operatorname{span}\{\widehat{x}\}$ is a linear subspace on $\mathbb{R}^{n+1}$ ([12], Corollary 9.1.3). Then

$$\begin{aligned} TS_{\widehat{C}}(\widehat{x}) &= \operatorname{lin}\operatorname{cl} D\left(\widehat{C}, \widehat{F}\right) \\ &= \operatorname{lin}\operatorname{cl}\left(\widetilde{D(C, x)} + \operatorname{span}\{\widehat{x}\}\right) \\ &= \operatorname{lin}\widetilde{T_C(x)} + \operatorname{span}\{\widehat{x}\}. \end{aligned}$$

Therefore

$$\begin{aligned} \dim TS_{\widehat{C}}(\widehat{x}) &= \dim\left(\operatorname{lin}\widetilde{T_C(x)} + \operatorname{span}\{\widehat{x}\}\right) \\ &= \dim \operatorname{lin} T_C(x) + 1, \end{aligned}$$

which is what we want to prove. □

The next result extends Theorems 1.1 and 3.2 on complementary dimensions of conjugate exposed faces.

Theorem 4.2. *Let $C \subset \mathbb{R}^n$ be a closed convex set containing the origin. Let F be a face of C and let S_F be the minimal exposed face of C containing F.*

1. *If $\theta \notin S_F$ then:*

$$\dim F + \dim F^{\triangle} + \operatorname{cin}(C, F) = n - 1. \tag{13}$$

2. *If $\theta \in S_F$ then:*

$$\dim F + \dim F^{\triangle} + \operatorname{cin}(C, F) = n. \tag{14}$$

3. *If G is a face of 0^+C then*

$$\dim G + \dim G^{\triangle} + \operatorname{cin}(C, G) = n. \tag{15}$$

Proof. First note that $\dim \widehat{F} = \dim F + 1$.

1. By using successively Lemma 4.1, Theorem 2.2 (part 1) and Theorem 3.2, we get

$$\begin{aligned}\dim F + \dim F^{\triangle} + \operatorname{cin}(C, F) &= (\dim \widehat{F} - 1) + (\dim \widehat{F^{\triangle}} - 1) + \operatorname{cin}(\widehat{C}, \widehat{F}) \\ &= (\dim \widehat{F} + \dim \widehat{F}^{\triangle} + \operatorname{cin}(\widehat{C}, \widehat{F})) - 2 \\ &= (n+1) - 2.\end{aligned}$$

2. By using successively Theorem 2.2 (part 2), Lemma 4.1 and Theorem 3.2, we obtain

$$\begin{aligned}\dim F + \dim F^{\triangle} + \operatorname{cin}(C, F) &= (\dim \widehat{F} - 1) + \dim \widetilde{F^{\triangle}} + \operatorname{cin}(\widehat{C}, \widehat{F}) \\ &= (\dim \widehat{F} + \dim \widehat{F}^{\triangle} + \operatorname{cin}(\widehat{C}, \widehat{F})) - 1 \\ &= (n+1) - 1.\end{aligned}$$

3. By using successively Theorem 2.2 (part 3), Theorem 3.2 and Lemma 4.1, we have

$$\begin{aligned}\dim G + \dim G^{\triangle} &= \dim \widetilde{G} + (\dim \widehat{G^{\triangle}} - 1) \\ &= \dim \widetilde{G} + \dim \widetilde{G}^{\triangle} - 1 \\ &= (n+1) - \operatorname{cin}(\widehat{C}, \widetilde{G}) - 1 \\ &= n - \operatorname{cin}(C, G),\end{aligned}$$

and the proof ends. □

Following [4], for a convex body C containing the origin as an interior point, we call a face F such that

$$\dim F + \dim F^{\triangle} = n - 1$$

a "perfect" face. Then we can conclude that a face F is perfect if and only if $\operatorname{cin}(C, F) = 0$. We can use $\operatorname{cin}(C, F) = 0$ as the definition of a perfect face, extending this notion to general closed convex sets because the curvature index on a face does not depend neither on the position of C with regard to the origin nor on the boundlessness of C. Recalling that $\operatorname{cin}(C, F) = \dim TS_C(F) - \dim F$, we would say that a face F of C is perfect when the dimensions of F and of its tangent space coincide.

The polytopes and, in a more general way, the quasi-polyhedral sets (convex sets for which its nonempty intersections with polytopes are polytopes), have all faces perfect. However, there exist convex sets with all their faces being perfect which are not quasi-polyhedral. A non quasi-polyhedral set satisfying this condition is: $\operatorname{conv}\left(\left\{(t,t^2)\in\mathbb{R}^2 \mid t=0,1,\frac{1}{2},\frac{1}{3},\dots\right\}\cup\{(0,1)\}\right)$.

The next result shows that the curvature index of exposed faces is invariant under conjugacy.

Corollary 4.3. *Let $C\subset\mathbb{R}^n$ be a closed convex set containing the origin. If F is an exposed face of C then*

$$\operatorname{cin}(C,F)=\operatorname{cin}\left(C^o,F^\triangle\right). \tag{16}$$

If G is a face of 0^+C such that $\widetilde{G}$ is an exposed face of $\widehat{C}$, then

$$\operatorname{cin}(C,G)=\operatorname{cin}\left(C^o,G^\triangle\right). \tag{17}$$

Proof. First note that $F=F^{\triangle\triangle}$ (Theorem 2.4, part 1). We discuss two cases:

If $\theta\notin F$, then $F^\triangle$ is an exposed face of C^o and $\theta\notin F^\triangle$. From (13) we get

$$\begin{aligned}\operatorname{cin}(C,F)&=(n-1)-\dim F-\dim F^\triangle\\&=(n-1)-\dim F^{\triangle\triangle}-\dim F^\triangle\\&=\operatorname{cin}\left(C^o,F^\triangle\right).\end{aligned}$$

If $\theta\in F$, then $F^\triangle$ is an exposed face of $0^+(C^o)$. Successive applications of (14) and (15) yield

$$\begin{aligned}\operatorname{cin}(C,F)&=n-\dim F-\dim F^\triangle\\&=n-\dim F^{\triangle\triangle}-\dim F^\triangle\\&=\operatorname{cin}\left(C^o,F^\triangle\right).\end{aligned}$$

In both cases we conclude that (16) holds.

In order to prove (17), first note that $G=G^{\triangle\triangle}$ (Theorem 2.4, part 2). Applying successively (15) and (14):

$$\begin{aligned}\operatorname{cin}(C,G)&=n-\dim G-\dim G^\triangle\\&=n-\dim G^{\triangle\triangle}-\dim G^\triangle\\&=\operatorname{cin}\left(C^o,G^\triangle\right),\end{aligned}$$

which proves (17). □

Corollary 4.4. *Let $C\subset\mathbb{R}^n$ be a closed convex set containing the origin. Then the following statements hold:*

1. *If F is a face of C and S_F is the minimal exposed face of C containing F, then*

$$\dim F+\operatorname{cin}(C,F)=\dim S_F+\operatorname{cin}(C,S_F).$$

2. *If G is a face of 0^+C and S_G is the minimal exposed face of $\widehat{C}$ containing $\widetilde{G}$, then*

$$\dim G+\operatorname{cin}(C,G)=\dim S_G+\operatorname{cin}(C,S_G).$$

Proof. It is direct consequence of Theorem 4.2, repeating the same argument as in the proof of Corollary 4.3, but using now $F^{\triangle} = S_F^{\triangle}$ (Corollary 2.3) and $G^{\triangle} = S_G^{\triangle}$ (by conjugacy on cones). □

Finally we illustrate the usefulness of the tools developed along the paper by means of two representative examples.

First, let us consider the closed convex set $C \subset \mathbb{R}^2$ in Figure 4. Since C has recession directions and $\theta \in \operatorname{bd} C$, we must include the recession cones into the analysis. At exposed points, as b and c, the curvature index is 1 (recall Definition 3.1) and, according to Theorem 4.2, these points are associated by conjugacy with exposed points of C^o which have a curvature index of 1 (Corollary 4.3). The face a is not exposed and, by Corollary 2.3, its behavior under conjugacy depends on the minimal exposed face containing it, in this case the face 1. Since the origin belongs to face 1 we have that $1^{\triangle}\left(= a^{\triangle}\right)$ is an exposed face of $0^+(C^o)$ with dimension 1 and curvature index 0 (because $\operatorname{cin}(C,1) = 0$). By a straightforward application of Definition 2.1 we infer that $C^{\triangle} \subset 0^+(C^o)$ and $C^{\triangle} = \{\theta\}$, i.e., $\{\theta\}$ is an exposed face of $0^+(C^o)$. According to Theorem 2.5, $(0^+(C^o))^{\triangle}$ is an exposed face of C containing the origin; in fact, it is the minimal exposed face containing the origin, so that $(0^+(C^o))^{\triangle}$ is the face 1. This fact allows us to establish that $0^+(C^o)$ is the halfline of nonnegative x's. In order to analyze the behavior of the faces on 0^+C we need to look at the associated cone $\widehat{C}$, since the curvature index on these faces can only be appreciated on this cone. Looking at Figure 4, it is clear that $\operatorname{cin}(C,2) = 0$ and $\operatorname{cin}(C,3) = 1$ from where we conclude that $\dim\left(2^{\triangle}\right) = 1$ and $\dim\left(3^{\triangle}\right) = 0$ according to Theorem 4.2. Moreover, from Definition 2.1, $2^{\triangle}$ and $3^{\triangle}$ are exposed faces of C^o containing the origin. Trivially $d^{\triangle} = C^o$ and $(0^+C)^{\triangle} = 3^{\triangle} = \{\theta\}$. Concerning statement 3 in Lemma 1.3, observe from faces 3 and $\widetilde{3}$ that G being exposed face of 0^+C does not imply $\widetilde{G}$ being exposed face of $\widehat{C}$.

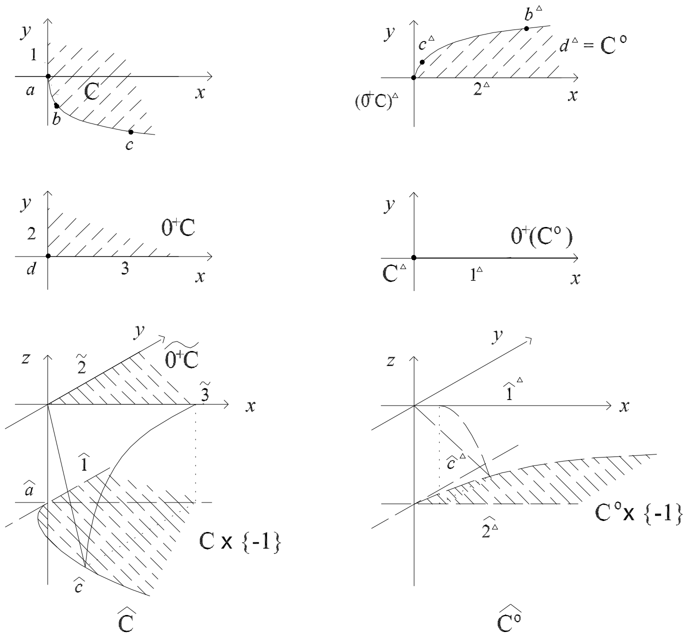

Figure 4

Now we calculate the polar body of the double cone C_1 in Figure 5. For the sake of simplicity $\theta \in \operatorname{int} C_1$. The exposed points α and β in C_1 correspond by conjugacy with 2-dimensional faces of C_1^o (Theorem 4.2). Exposed points, as 1 and 2 in C_1, are associated with 1-dimensional faces of C_1^o, because the curvature index is 1 on these faces. On edges of C_1, as a,b and c, the curvature index is 1. Therefore their corresponding conjugate faces have dimension zero.

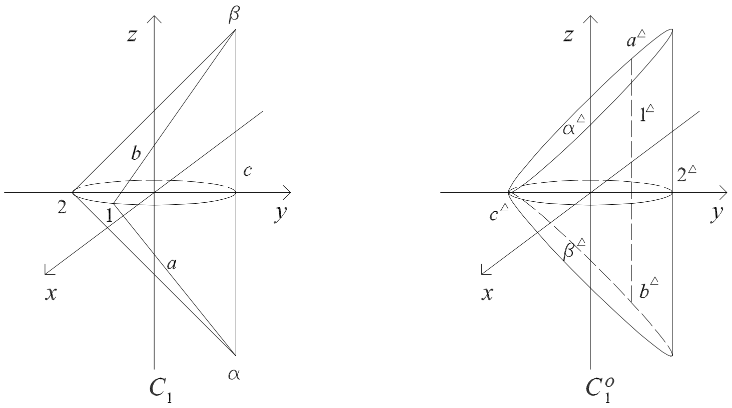

Figure 5

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