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Harmonic Morphisms Between Degenerate Semi-Riemannian Manifolds

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Abstract. In this paper we generalize harmonic maps and morphisms to the degenerate semi-Riemannian category, in the case when the manifolds M and N are stationary and the map $\phi : M \to N$ is radical-preserving. We characterize geometrically the notion of (generalized) horizontal (weak) conformality and we obtain a characterization for (generalized) harmonic morphisms in terms of (generalized) harmonic maps.

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1. Introduction and preliminaries

Harmonic morphisms between (non-degenerate semi-)Riemannian manifolds are maps which preserve germs of harmonic functions. They are characterized in [8, 13, 9] as the subclass of harmonic maps which are horizontally weakly conformal. An up-to-date bibliography on this topic is given in [11]; see also [12] for a list of harmonic morphisms and construction techniques, and [1] for a comprehensive account of the topic.

However, when the manifold (M, g) is degenerate, then it fails, in general, to have a torsion-free, metric-compatible connection; moreover, in this case, the notion of 'trace', with respect to the metric g, does not make any sense, so that it is not possible to define the 'tension field' of a map, or, consequently, the notion of harmonic map, in the usual sense.

Degenerate manifolds arise naturally in the semi-Riemannian category: for example the restriction of a non-degenerate metric to a degenerate submanifold is a degenerate metric and the Killing-Cartan form on a non-semi-simple Lie group is a degenerate metric.

Such manifolds are playing an increasingly important role in quantum theory and string theory, as the action and field equations of particles and strings often do not depend on the inverse metric and are well-defined even when the metric becomes degenerate (cf. [4]). For example, an extension of Einstein's gravitational theory which contains degenerate metrics as possible solutions might lead to space-times with no causal structure (cf. [2]). As a (2-dimensional) degenerate manifold is not globally hyperbolic, it is of interest to study the influence of this degeneracy on the propagation of massless scalar fields (cf. [10]). A degenerate metric is used to build a 5-dimensional model of the universe, which is a degenerate extension to relativity, and allows us to incorporate electromagnetism in the geometry of space-time and unify it with gravitation (see [19] and its references).

In the mathematical literature, degenerate manifolds have been studied under several names: singular Riemannian spaces ([15, 28, 26]), degenerate (pseudo- or semi-Riemannian) manifolds ([5, 24, 14]), lightlike manifolds ([6]), isotropic spaces ([20, 21, 22, 23]), isotropic manifolds ([27]).

In this paper we define generalized harmonic maps and morphisms, characterize (generalized) horizontally weakly conformal maps with non-degenerate codomain into three types (Theorem 2.15), and give a Fuglede-Ishihara-type characterization for generalized harmonic morphisms (Theorem 3.5). We refer the reader to [18] for further details.

In this section, we aim to introduce the necessary background on semi-Riemannian geometry which will be used in the rest of the paper. We shall assume that all vector spaces, manifolds etc. have finite dimension.

1.1. Algebraic background

Let V be a vector space of dimension m.

Definition 1.1. *An inner product on V is a symmetric bilinear form $\langle\,,\rangle = \langle\,,\rangle_V$ on V. It is said to be non-degenerate (on V) if $\langle w, w'\rangle = 0$ for all $w' \in V$ implies $w = 0$, otherwise it is called degenerate.*

We shall refer to the pair $(V, \langle\,,\rangle)$ as an inner product space. Given two subspaces $W, W' \subseteq V$, we shall often write $W \perp_V W'$ to denote that W is orthogonal to W' (equivalently W' is orthogonal to W) with respect to the inner product $\langle\,,\rangle_V$, i.e. $\langle w, w'\rangle = 0$ for any $w \in W$ and $w' \in W'$.

Let $r, p, q \geq 0$ be integers and set $(\epsilon)_{ij} := (\epsilon_{r,p,q})_{ij}$ equal to the diagonal matrix

$$(\epsilon)_{ij} = \mathrm{diag}(\underbrace{0, \dots, 0}_{r\text{-times}}, \underbrace{-1, \dots, -1}_{p\text{-times}}, \underbrace{+1, \dots, +1}_{q\text{-times}}).$$

Given an inner product $\langle\,,\rangle$ on V, there exists a basis $\{e_i\}$, with $i = 1, \dots, m = \dim V$, of V such that $\langle e_i, e_i\rangle = (\epsilon_{r,p,q})_{ij}$. We call such a basis *orthonormal* and the triple (r, p, q) is called the *signature of the inner product* $\langle\,,\rangle$.

Example 1.2. The standard m-Euclidean space $\mathbb{R}^m_{r,p,q}$ of signature (r, p, q) is $\mathbb{R}^m$ endowed with the inner product $\langle\,,\rangle_{r,p,q}$ defined by $\langle E_i, E_j\rangle_{r,p,q} := (\epsilon_{r,p,q})_{ij}$; here $\{E_k\}_{k=1}^m$ is the canonical basis $E_1 = (1, 0, \dots, 0), \dots, E_m = (0, \dots, 0, 1)$.

Definition 1.3. *A subspace W of an inner product vector space $(V, \langle\, ,\rangle)$ is called degenerate (resp. null) if there exists a non-zero vector $X \in W$ such that $\langle X, Y\rangle = 0$ for all $Y \in W$ (resp. if, for all $X, Y \in W$, we have $\langle X, Y\rangle = 0$). Otherwise W is called non-degenerate (resp. non-null).*

Clearly if $W \neq \{\mathbf{0}\}$ is null then it is degenerate. Moreover W is degenerate if and only if $\langle , \rangle|_W$ is degenerate, but this does not necessarily mean that $\langle\, ,\rangle$ is degenerate on V.

Given a vector space V, we define the *radical of* V (cf. [6], p. 1, [14], p. 3 or [17], p. 53), denoted by $\mathcal{N}(V)$, to be the vector space:

$$\mathcal{N}(V) := V^{\perp} = \{X \in V : \langle X, Y\rangle = 0 \text{ for all } Y \in V\}.$$

We notice (cf. [17], p. 49) that $\mathcal{N}(V)$ is a null subspace of V. Moreover, V is non-degenerate if and only if $\mathcal{N}(V) = \{\mathbf{0}\}$, and V is null if and only if $\mathcal{N}(V) = V$. Note that, for any subspace W of V,

$$\mathcal{N}(V) \subseteq W^{\perp}. \tag{1}$$

The following proposition generalizes two well-known facts of linear algebra (cf. [17], Chapter 2, Lemma 22).

Proposition 1.4. *For any subspace $W \subseteq V$ of an inner product space $(V, \langle\, ,\rangle)$ we have:*

(i) $\dim W + \dim W^{\perp} = \dim V + \dim(\mathcal{N}(V) \cap W)$;

(ii) $(W^{\perp})^{\perp} = W + \mathcal{N}(V)$.

Proof. Let $t = \dim \mathcal{N}(V) - \dim\big(W \cap \mathcal{N}(V)\big)$. We can choose a basis $\{e_i\}_{i=1}^m$ on V, 'adapted' to $\mathcal{N}(V)$ and W, in the sense that $\mathcal{N}(V) = \mathrm{span}(e_1, \ldots, e_{\dim \mathcal{N}(V)})$ and $W = \mathrm{span}(e_{t+1}, \ldots, e_{t+\dim W})$; claim (i) follows immediately.

To prove (ii) we note that

$$W + \mathcal{N}(V) \subseteq (W^{\perp})^{\perp}.$$

From linear algebra (cf. [25], Theorem 1.9A) we have:

$$\dim(W + \mathcal{N}(V)) = \dim W + \dim \mathcal{N}(V) - \dim(W \cap \mathcal{N}(V));$$

on the other hand, (i) we get:

$$\dim W^{\perp} = \dim V + \dim(W \cap \mathcal{N}(V)) - \dim W;$$

on combining these and using (1) we obtain

$$\dim(W^{\perp})^{\perp} = \dim(W + \mathcal{N}(V));$$

claim (ii) follows. □

Let $W \subseteq V$ be a vector subspace of an inner product vector space $(V, \langle\, ,\rangle_V)$ and let $W^{\perp_V}$ be its orthogonal complement in V with respect to $\langle\, ,\rangle_V$. Denote by $\overline{V}$, $\overline{W}$ and $\overline{W^{\perp_V}}$ the spaces

$$\overline{V} := V/\mathcal{N}(V), \quad \overline{W} := W/(\mathcal{N}(V) \cap W), \text{ and } \overline{W^{\perp_V}} := W^{\perp_V}/\mathcal{N}(V), \tag{2}$$

having noted that, by (1), $\mathcal{N}(V) \subseteq W^{\perp_V}$. Let us also denote by $\langle\, ,\rangle_{\overline{V}}$ the inner product on $\overline{V}$ defined by

$$\langle \overline{v}, \overline{v}' \rangle_{\overline{V}} := \langle v, v' \rangle_V \qquad (v, v' \in V),$$

where $\overline{v} = \pi_V(v), \overline{v'} = \pi_V(v')$, $\pi_V : V \to \overline{V}$ being the natural projection. Note that this is well defined. For any subspace $E \subseteq \overline{V}$, let $E^{\perp_{\overline{V}}}$ denote its orthogonal complement in $(\overline{V}, \langle\, ,\rangle_{\overline{V}})$. Then we have the following

Proposition 1.5. *For any vector subspace $W \subseteq V$ we have the following canonical isomorphism:*

$$\overline{W} \cong (\overline{W^{\perp_V}})^{\perp_{\overline{V}}}. \tag{3}$$

Proof. Consider the composition

$$\theta : W \overset{i}{\hookrightarrow} V \overset{\pi_V}{\to} V/\mathcal{N}(V) =: \overline{V},$$

where $i : W \hookrightarrow V$ is the inclusion map and $\pi_V : V \to \overline{V}$ is the natural projection. We have

$$\theta(W) \subseteq \big(W^{\perp_V}/\mathcal{N}(V)\big)^{\perp_{\overline{V}}};$$

in fact, let $w \in W$ and $w' \in W^{\perp_V}$ and write $\theta(w) := \overline{w}$; then we have

$$0 = \langle w, w' \rangle_V = \langle \overline{w}, \overline{w}' \rangle_{\overline{V}}.$$

Next, note that $\ker \theta = \mathcal{N}(V) \cap W$. In fact for any $w \in W$, we have

$$\theta(w) = 0 \iff \overline{w} = 0 \iff w \in \mathcal{N}(V).$$

Hence θ factors to an injective map

$$\overline{\theta} : \overline{W} := W/\mathcal{N}(V) \cap W \longrightarrow \big(W^{\perp_V}/\mathcal{N}(V)\big)^{\perp_{\overline{V}}} =: (\overline{W^{\perp_V}})^{\perp_{\overline{V}}}.$$

We show that this is an isomorphism, by calculating the dimension of the spaces on either side of the equation (3). On the left-hand side we have

$$\dim \overline{W} = \dim W - \dim(\mathcal{N}(V) \cap W);$$

on the right-hand side, applying Proposition 1.4, we get

$$\dim W^{\perp_V} = \dim V + \dim(\mathcal{N}(V) \cap W) - \dim W,$$

so that

$$\dim \overline{W^{\perp_V}} = \dim V + \dim(\mathcal{N}(V) \cap W) - \dim W - \dim \mathcal{N}(V)$$

and, applying once more Proposition 1.4,

$$\begin{aligned}\dim(\overline{W^{\perp_V}})^{\perp_{\overline{V}}} &= \dim \overline{V} - \big(\dim V + \dim(\mathcal{N}(V) \cap W) - \dim W - \dim \mathcal{N}(V)\big)\\ &= \dim W - \dim(\mathcal{N}(V) \cap W)\\ &= \dim \overline{W},\end{aligned}$$

so that the map $\overline{\theta}$ is an isomorphism, and the claim follows. □

We shall use the proposition above to identify $\overline{W}$ and $(\overline{W^{\perp_V}})^{\perp_{\overline{V}}}$. Thus, any subspace $K \subseteq \overline{W}$ will sometimes be considered as a subspace of $(\overline{W^{\perp_V}})^{\perp_{\overline{V}}}$ and *vice versa.*

1.2. Background on semi-Riemannian geometry

Definition 1.6. *Let r, p, q be three non-negative integers such that $r + p + q = m$. A semi-Riemannian metric g of signature (r, p, q) on an m-dimensional smooth manifold M is a smooth section of the symmetric square $\odot^2 T^*M$ which defines an inner product $\langle\, , \rangle$ on each tangent space of constant signature (r, p, q). A semi-Riemannian manifold is a pair (M, g) where M is a smooth manifold and g is a semi-Riemannian metric on M. When $r > 0$ (resp. $r = 0$, $r < m$, or $r = m$) (M, g) is called degenerate (resp. non-degenerate, non-null, or null).*

Let $\mathcal{L}$ denote the Lie derivative and let $\mathcal{N} = \mathcal{N}(TM) := \cup_{x\in M}\mathcal{N}(T_xM)$; $\mathcal{N}$ is called the *radical distribution on M.*

Definition 1.7. ([14], Definition 3.1.3) *A semi-Riemannian manifold (M, g) is said to be stationary if $\mathcal{L}_A g = 0$ for any locally defined smooth section $A \in \Gamma(\mathcal{N})$.*

Such a manifold is also called a *Reinhart manifold* (cf. [6], p. 49, for alternative definition). The condition that M be stationary is equivalent to $\mathcal{N}$ being a Killing distribution (i.e. all vector fields in $\mathcal{N}$ are Killing). Trivially a non-degenerate manifold is stationary.

We introduce the following operator ([14], Definition 3.1.1):

Definition 1.8. (Koszul derivative) *Let (M, g) be a semi-Riemannian manifold. An operator $D : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ is called a Koszul derivative on (M, g) if, for any $X, Y, Z \in \Gamma(TM)$, it satisfies the Koszul formula*

$$\begin{aligned}2g(D_XY, Z) = {}& Xg(Y, Z) + Yg(Z, Y) - Zg(X, Y)\\ &-g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]).\end{aligned} \tag{4}$$

Remark 1.9. We note that, when g is non-degenerate, D is nothing but the Levi-Civita connection, and it is uniquely determined by (4) (cf. [17], Theorem 11, p. 61). However, when g is degenerate, the Koszul derivative is only determined up to a smooth section of the radical of M, in the sense that, given any two Koszul derivatives D, D' on M and any two vector fields $X, Y \in \Gamma(TM)$, we have $D_XY - D'_XY \in \Gamma(\mathcal{N})$.

We have the following fundamental lemma of degenerate semi-Riemannian geometry:

Lemma 1.10. ([14], Lemma 3.1.2) *Let (M, g) be a semi-Riemannian manifold. Then (M, g) admits a Koszul derivative if and only if it is stationary.*

□

For a later use, given an endomorphism $\sigma \in \Gamma(\mathrm{End}(TM))$ of the tangent bundle TM, we define its Koszul derivative by the Leibniz rule:

$$(D\sigma)(Y) := D(\sigma(Y)) - \sigma(DY), \qquad (Y \in \Gamma(TM)). \tag{5}$$

It is easy to see that given a Koszul derivative D on M, then

$$D_X A \in \Gamma(\mathcal{N}) \qquad (X \in \Gamma(TM),\ A \in \Gamma(\mathcal{N})) \tag{6}$$

In fact, for any $Z \in \Gamma(TM)$ we have

$$g(D_X A, Z) = X(g(A, Z)) - g(A, D_X Z) = 0, \qquad (X \in \Gamma(TM),\ A \in \Gamma(\mathcal{N})).$$

We have that:

Lemma 1.11. ([14], Lemma 3.1.4) *If the manifold (M, g) is stationary then $\mathcal{N}$ is integrable.*

Proof. Let $A, B \in \Gamma(\mathcal{N})$ and let D be a Koszul derivative on M. Then, for any $V \in \Gamma(TM)$:

$$\begin{aligned} g([A, B], V) = &\ g(D_A B, V) - g(D_B A, V) \\ = &\ A(g(B, V)) - g(B, D_A V) - B(g(A, V)) + g(A, D_B V) = 0, \end{aligned}$$

so that $[A, B] \in \Gamma(\mathcal{N})$. □

By the Frobenius Theorem, we obtain a foliation associated to $\mathcal{N}$; we shall call this the *radical foliation of M*.

Let (M, g) be a stationary semi-Riemannian manifold of (constant) signature (r, p, q), with $r \geq 0$. Let $E \to M$ be a *semi-Riemannian bundle* (i.e. a bundle whose fibres are semi-Euclidean spaces of (constant) signature (r, p, q)); by $\overline{E}$ (cf. (2)) we shall denote the quotient bundle

$$\overline{E} := E/\mathcal{N}(E) \equiv \cup_{x \in M} E_x/\mathcal{N}(E_x),$$

E_x being the fibre of E over $x \in M$. In particular, we define the *quotient tangent bundle of M* by $\overline{TM} := TM/\mathcal{N}(TM)$; this is endowed with the non-degenerate metric $\overline{g}(\overline{X}, \overline{Y}) := g(X, Y)$ of signature $(0, p, q)$, where $X, Y \in \Gamma(TM)$ and $\overline{X} = \pi_E(X)$, $\overline{Y} = \pi_E(Y)$, $\pi_E : E \to \overline{E}$ being the natural projection. Let $\overline{TM}^{\,*}$ $(= \overline{T^*M})$ be its dual bundle.

Definition 1.12. *We shall call an E-valued 1-form $\sigma \in \Gamma(T^*M \otimes E)$ radical-preserving (resp. radical-annihilating) if, for each $x \in M$,*

$$\sigma_x(\mathcal{N}(T_x M)) \subseteq \mathcal{N}(E_x) \qquad (\textit{resp. } \sigma_x(\mathcal{N}(T_x M)) = 0).$$

Denote by $\pi_{TM} : TM \to \overline{TM}$ *and* $\pi_E : E \to \overline{E}$ *the natural projections. Then there exists a linear bundle map* $\overline{\sigma} \in \Gamma(\overline{T^*M} \otimes \overline{E})$ *such that the following diagram*

$$\begin{array}{ccc} TM & \xrightarrow{\sigma} & E \\ \big\downarrow{\scriptstyle \pi_{TM}} & & \big\downarrow{\scriptstyle \pi_E} \\ \overline{TM} & \xrightarrow{\overline{\sigma}} & \overline{E} \end{array}$$

commutes if and only if σ *is radical-preserving.*

We shall say that a map $\phi : M \to N$ *is radical-preserving (resp. radical-annihilating) if its differential* $d\phi \in \Gamma(T^*M \otimes \phi^{-1}TN)$ *is radical-preserving (resp. radical-annihilating), i.e.* $d\phi_x\big(\mathcal{N}(T_xM)\big) \subseteq \mathcal{N}(T_{\phi(x)}N)$ *(resp.* $d\phi_x\big(\mathcal{N}(T_xM)\big) = 0$*, i.e.* $\mathcal{N}(T_xM) \subseteq \ker d\phi_x$*).*

Remark 1.13. Clearly a radical-annihilating map is radical-preserving. Furthermore, if N is a non-degenerate manifold, the reverse holds as, for any $x \in M$, we have $d\phi_x\big(\mathcal{N}(T_xM)\big) \subseteq \mathcal{N}(T_{\phi(x)}N) = \{\mathbf{0}\}$.

We note that a radical-annihilating map need not to have a non-degenerate codomain. In fact, let $(M, g) := \mathbb{R}^2_{1,0,1} = \big(\mathbb{R}^2, (dx^2)^2\big)$ and let $(N, h) := \mathbb{R}_0$ be the real line with the null metric; the map $\phi : \mathbb{R}^2_{1,0,1} \to \mathbb{R}_0$ given by $\phi(x^1, x^2) = x^2$, is radical-annihilating, but N is degenerate.

If $\phi : M \to N$ is radical-preserving, then we can define the *(generalized) differential of* ϕ, $\overline{d\phi} : \overline{TM} \to \overline{TN}$, to be the map (clearly well-defined)

$$\overline{d\phi}(\overline{X}) := \overline{d\phi(X)}, \qquad \text{for any } X \in \Gamma(TM). \tag{7}$$

We note that, if M and N are both non-degenerate, then the map $\phi : M \to N$ is automatically radical-preserving (in fact, it is radical-annihilating), and the notion of generalized differential $\overline{d\phi}$ agrees with that of (standard) differential $d\phi$.

We now state the fundamental theorem of degenerate semi-Riemannian geometry.

Theorem 1.14. ([14], Theorem 3.2.3) *Let* (M, g) *be a semi-Riemannian manifold. If* (M, g) *is stationary, then there exists a unique connection* $\overline{\nabla}$ *on* $(\overline{TM}, \overline{g})$ *which is torsion-free in the sense that* $\overline{T}^{\overline{\nabla}}(X, Y) := \overline{\nabla}_X\overline{Y} - \overline{\nabla}_Y\overline{X} - \overline{[X, Y]} = 0$, $\big(X, Y \in \Gamma(TM)\big)$*, and compatible with the metric* $\overline{g}$ *in the sense that* $\overline{\nabla}\,\overline{g} = 0$*; in fact* $\overline{\nabla}$ *is given by:*

$$\overline{\nabla}_X\overline{Y} := \overline{D_XY} \qquad \big(X, Y \in \Gamma(TM),$$

where D *is any Koszul derivative on* (M, g)*.*

Conversely, if there exists such a connection $\overline{\nabla}$*, then* (M, g) *is stationary.* □

The connection $\overline{\nabla}$ is called the *Koszul connection on* (M, g). If (M, g) is non-degenerate, then $\overline{\nabla}$ coincides with the usual Levi-Civita connection. Let us set $E \equiv TM$ and let $\sigma \in \Gamma(T^*M \otimes TM)$ be radical-preserving. We define the Koszul connection on $\overline{T^*M} \otimes \overline{TM}$ by the Leibniz rule

$$(\overline{\nabla}_X\overline{\sigma})\overline{Y} := \overline{\nabla}_X(\overline{\sigma}(\overline{Y})) - \overline{\sigma}(\overline{\nabla}_X\overline{Y}), \quad \big(X, Y \in \Gamma(TM)\big)$$

where $\overline{\nabla}$ is defined as in Theorem 1.14.

We note that the connection $\overline{\nabla}$ is defined for $(X,\overline{Y}) \in TM \otimes \overline{TM}$, as is the operator $\overline{\nabla}\overline{\sigma}$ defined above. It does not, in general, factor to an operator on $\overline{TM} \otimes \overline{TM}$. However, if $\sigma = d\phi$, i.e. if σ is the differential of a map $\phi : M \to N$, with ϕ radical-preserving, we have the following fact. Let $\phi^{-1}(\overline{TN}) \to M$ denote the pull-back of the bundle $\overline{TN} \to N$, equivalently,

$$\phi^{-1}(\overline{TN}) := \phi^{-1}(TN)/\phi^{-1}(\mathcal{N}(TN)).$$

Lemma 1.15. *The operator* $\overline{B}^{\phi} \in \Gamma\big(\otimes^2 \overline{TM}^* \otimes \phi^{-1}(\overline{TN})\big)$ *defined by*

$$\overline{B}^{\phi}(\overline{X},\overline{Y}) \equiv (\overline{\nabla}\,\overline{d\phi})(\overline{X},\overline{Y}) := (\overline{\nabla}_X\overline{d\phi})(\overline{Y}), \quad \big(\overline{X},\overline{Y} \in \Gamma(\overline{TM})\big),$$

is well-defined, tensorial and symmetric.

Proof. The operator $\overline{B}^{\phi}$ is clearly well-defined with respect to the second argument. In order to prove that is well-defined with respect to the first entry, it will be enough to show that

$$(\overline{\nabla}^{\phi}_A\overline{d\phi})(\overline{Y}) = 0,$$

for any $A \in \Gamma(\mathcal{N}(TM))$ and $Y \in \Gamma(TM)$. (Here for brevity we shall denote by $\overline{\nabla}^{\phi}$ both the induced connections on the pull-back bundles $\overline{TM}^* \otimes \phi^{-1}(\overline{TN})$ and $\phi^{-1}(\overline{TN})$; the context should make clear which of the two we are using). So we have

$$\begin{aligned}
(\overline{\nabla}^{\phi}_A\overline{d\phi})(\overline{Y}) &= \overline{\nabla}^{\phi}_A\overline{d\phi}(\overline{Y}) - \overline{d\phi}(\overline{\nabla}^M_A\overline{Y}) \\
&= \overline{\nabla}^N_{d\phi(A)}\overline{d\phi}(\overline{Y}) - \overline{d\phi}(\overline{\nabla}^M_A\overline{Y}) \\
&= \overline{D^N_{d\phi(A)}d\phi(Y)} - \overline{d\phi(D^M_A Y)} \\
&= \overline{D^N_{d\phi(Y)}d\phi(A) - [d\phi(A), d\phi(Y)]^N} - \Big(\overline{d\phi(D^M_Y A - [A,Y]^M)}\Big) \\
&= \overline{d\phi([A,Y]^M) - [d\phi(A), d\phi(Y)]^N},
\end{aligned}$$

the last step because of equation (6), and because ϕ is radical-preserving. Now, by the 'naturality' of the Lie brackets with respect to the map ϕ (cf. [3] , Theorem 7.9, p. 155), the last expression is zero, and so we have the claim. The symmetry of $\overline{B}^{\phi}$ also follows from the naturality of Lie brackets, as $d\phi([X,Y]^M) - [d\phi(X), d\phi(Y)]^N = 0$. The tensoriality is easy to prove. □

We shall call the operator $\overline{B}^{\phi}$ the *(generalized) second fundamental form of the map* ϕ.

2. Generalized harmonic maps and morphisms

Let $\phi : M \to N$ be a (C^1) radical-preserving map between stationary manifolds. We shall define the *(generalized) divergence* $\overline{\operatorname{div}}\,(\overline{d\phi})$ of $\overline{d\phi}$. Let $\{e_i\}_{i=1}^m$ be any basis of TM such that $\mathcal{N}(TM) = \operatorname{span}(e_1, \ldots, e_r)$ and let $\mathcal{V}_1 := \operatorname{span}(e_{r+1}, \ldots, e_m)$ be a *screen space*, i.e. a subbundle of TM such that $TM = \mathcal{N}(TM) \oplus \mathcal{V}_1$; we shall call such a basis a *(local) radical basis for TM*. Then

$$\overline{\mathrm{div}}\,(\overline{d\phi}) := \mathrm{tr}_{\overline{g}}(\overline{B}^{\phi}) := \sum_{a,b=r+1}^{m} \overline{g}^{ab}\left(\overline{\nabla}_{e_a}\overline{d\phi}\right)\overline{e_b}\,,$$

where $\overline{g}_{ab} := \overline{g}(\overline{e_a}, \overline{e_b})$. This is well defined and does not depend on the choice of the local radical basis $\{e_i\}_{i=1}^{m}$ on M.

We can now define the *(generalized) tension field* $\overline{\tau}(\phi)$ of a (C^2) radical-preserving map $\phi : M \to N$ between stationary manifolds by:

$$\overline{\tau}(\phi) := \overline{\mathrm{div}}\,(\overline{d\phi}).$$

Definition 2.1. *We shall say that a radical-preserving map $\phi : M \to N$ between stationary semi-Riemannian manifolds is (generalized) harmonic if its (generalized) tension field $\overline{\tau}(\phi)$ is identically zero.*

Note that this notion agrees with the usual notion of harmonicity when the manifolds M and N are both non-degenerate.

If $(x^1, \dots, x^m)$ and $(y^1, \dots, y^n)$ are *radical coordinates* (i.e. coordinates whose tangent vector fields form a radical basis) on M and N, respectively (with $\mathrm{rank}\,\mathcal{N}(TM) = r$ and $\mathrm{rank}\,\mathcal{N}(TN) = \rho$), then, analogously to the non-degenerate case, the (generalized) tension field of ϕ can be locally expressed by (cf. [7])

$$\overline{\tau}^{\gamma}(\phi) = \sum_{\alpha,\beta,\gamma=\rho+1}^{n}\ \sum_{i,j,k=r+1}^{m} \overline{g}^{ij}\left(\phi^{\gamma}_{ij} - {}^{M}\overline{\Gamma}^{k}_{ij}\phi^{\gamma}_{k} + {}^{N}\overline{\Gamma}^{\gamma}_{\alpha\beta}\phi^{\alpha}_{i}\phi^{\beta}_{j}\right), \tag{8}$$

where $\phi^{\gamma}_{k} := \partial\phi^{\gamma}/\partial x^{k}$, and ${}^{M}\overline{\Gamma}^{k}_{ij}\,\overline{\partial/\partial x^{k}} := \overline{\nabla}^{M}_{\partial/\partial x^{i}}\,\overline{\partial/\partial x^{j}}$, ${}^{N}\overline{\Gamma}^{\gamma}_{\alpha\beta}\,\overline{\partial/\partial y^{\gamma}} := \overline{\nabla}^{N}_{\partial/\partial y^{\alpha}}\,\overline{\partial/\partial y^{\beta}}$. In particular, if $N \equiv \mathbb{R}$, then $\overline{\tau}$ reduces to what we shall call the *(generalized) Laplace-Beltrami operator* $\overline{\Delta^{M}}$ and the radical-preserving functions $f \in C^{\infty}(M)$ satisfying $\overline{\Delta^{M}}f = 0$ will be called *(generalized) harmonic functions.*

Remark 2.2. We note that, as in the non-degenerate case, it is possible to define the notion of harmonicity in the degenerate context from a variational principle. Choosing radical coordinates $(x^1, \dots, x^m)$ and $(y^1, \dots, y^n)$ on M and N respectively, as above, we define the *(generalized) energy density* $\overline{e}$ of a radical-preserving map $\phi : (M, g) \to (N, h)$ between stationary manifolds by

$$\overline{e}(\phi) := \frac{1}{2}\sum_{\alpha,\beta=\rho+1}^{n}\ \sum_{i,j=r+1}^{m} \overline{h}_{\alpha\beta}\phi^{\alpha}_{i}\phi^{\beta}_{j}\overline{g}^{ij};$$

moreover, we define the *(generalized) volume form* $\overline{v}_g$ on (M, g) by

$$\overline{v}_g := \sqrt{\det(\overline{g})}\,dx^{r+1} \wedge \cdots \wedge dx^{m} \equiv v_{\overline{g}}\,;$$

it is not difficult to prove that the above definitions of $\overline{e}(\phi)$ and $\overline{v}_g$ do not depend on the choice of radical coordinates. Then, it is possible to prove that a map $\phi : M \to N$ as in

Definition 2.1 is (generalized) harmonic if and only if it is a critical point of the (generalized) energy functional $\overline{E}(\phi)$, defined by

$$\overline{E}(\phi) := \int_D \overline{e}(\phi)\, \overline{v}_g,$$

where D is a small enough compact domain on the leaf space (see next section).

Example 2.3. If $N = \mathbb{R}^n_{\rho,\pi,\sigma}$ then a map $\phi : (M,g) \to \mathbb{R}^n_{\rho,\pi,\sigma}$ is (generalized) harmonic if and only if each component $\phi^\alpha : (M,g) \to \mathbb{R},\ \alpha = \rho+1,\dots,n$, is a (generalized) harmonic function.

Now we can state the following

Definition 2.4. *We shall call a (C^2) radical-preserving map $\phi : (M,g) \to (N,h)$ between semi-Riemannian manifolds a (generalized) harmonic morphism if, for any (generalized) harmonic function $f : V \subseteq N \to \mathbb{R}$ on an open subset $V \subseteq N$, with $\phi^{-1}(V)$ non-empty, its pull-back $\phi^* f := f \circ \phi$ is a (generalized) harmonic function on M.*

Note that the usual definition of harmonic morphism does not make sense for degenerate manifolds since the trace, divergence and Laplacian are not defined when the metric is degenerate.

Let $\phi : M \to N$ be a radical-preserving map between two semi-Riemannian manifolds and $\overline{d\phi}_x : \overline{T_x M} \to \overline{T_{\phi(x)} N}$ its (generalized) differential at $x \in M$ (cf. (7)); then we define the *(generalized) adjoint* $\overline{d\phi}^*_{\phi(x)} : \overline{T_{\phi(x)}N} \to \overline{T_x M}$ of $d\phi$ as the adjoint of $\overline{d\phi}_x$, i.e. the linear map characterized by

$$\overline{g}_x(\overline{d\phi}^*_x(\overline{V}), \overline{X}) = \overline{h}_{\phi(x)}(\overline{V}, \overline{d\phi}_x(\overline{X})) = h_{\phi(x)}(V, d\phi_x(X)), \quad (V \in T_{\phi(x)}N,\ X \in T_x M). \tag{9}$$

We now generalize the notion of horizontal weak conformality.

Definition 2.5. *We shall call a radical-preserving map $\phi : (M,g) \to (N,h)$ between two non-null semi-Riemannian manifolds M and N (generalized) horizontally (weakly) conformal (or, for brevity, (generalized)* HWC*) at $x \in M$ with square dilation $\Lambda(x)$ if*

$$\overline{g}_x(\overline{d\phi}^*_x(\overline{V}), \overline{d\phi}^*_x(\overline{W})) = \Lambda(x)\, \overline{h}_{\phi(x)}(\overline{V}, \overline{W}), \qquad \left(\overline{V}, \overline{W} \in \overline{T_{\phi(x)}N}\right). \tag{10}$$

In particular, if Λ is identically equal to 1, we shall say that ϕ is a (generalized) Riemannian submersion.

Remark 2.6. If both M and N are non-degenerate, then the above notion of (generalized) horizontal weak conformality coincides with the better-known one of horizontal weak conformality. In the Riemannian case, if $\phi : M \to N$ is non-constant HWC, them $\dim M \geq \dim N$. However, this is no longer true in our case (or even in the non-degenerate semi-Riemannian case, see [9]).

Let (M, g) and (N, h) be stationary manifolds of signatures $\operatorname{sign} g = (r, p, q)$ and $\operatorname{sign} h = (0, \pi, \eta)$ (N non-degenerate), respectively, and let $\phi : M \to N$ be a radical-preserving map (therefore, by Remark 1.13, radical-annihilating, i.e. a map such that $\mathcal{N}(T_xM) \subseteq \ker d\phi_x$ for each $x \in M$). As usual, for any $x \in M$, set $\mathcal{V}_x := \ker d\phi_x$ and $\mathcal{H}_x := \mathcal{V}_x^{\perp}$. We shall also set:

$$\overline{\mathcal{V}}_x := \mathcal{V}_x/(\mathcal{N}(T_xM) \cap \mathcal{V}_x) = \mathcal{V}_x/(\mathcal{N}(T_xM), \quad \overline{\mathcal{H}}_x := \mathcal{H}_x/\mathcal{N}(T_xM),$$

having noticed that, by equation (1), $\mathcal{N}(T_xM) \subseteq \mathcal{H}_x$. We have the following

Lemma 2.7. *Let $\phi : M \to N$ be a radical-preserving map with N non-degenerate. Then, at any $x \in M$, the following identity holds:*

$$\operatorname{image} \overline{d\phi}_x^* = \overline{\mathcal{H}}_x \,. \tag{11}$$

Proof. First, we note that the following identity holds:

$$\overline{\ker d\phi_x} = \ker \overline{d\phi}_x. \tag{12}$$

In fact, let $\overline{X} \in \ker \overline{d\phi}_x$, and let Y be a representative of $\overline{X}$, i.e. $Y \in T_xM$ is such that $\overline{Y} = \overline{X}$. Then we have:

$$\begin{aligned} \overline{d\phi}_x(\overline{X}) = \overline{\mathbf{0}} &\iff \overline{d\phi_x(Y)} = \overline{\mathbf{0}} \\ &\iff d\phi_x(Y) \in \mathcal{N}(T_{\phi(x)}N) = \{\mathbf{0}\} \\ &\iff Y \in \ker d\phi_x \\ &\iff \overline{X} \in \overline{\ker d\phi_x} \,. \end{aligned}$$

Finally we have

$$\operatorname{image} \overline{d\phi}_x^* = (\ker \overline{d\phi}_x)^{\perp_{\overline{g}}} = (\overline{\mathcal{V}}_x)^{\perp_{\overline{g}}} = \overline{\mathcal{H}}_x \,,$$

the last equality following by Proposition 1.5. □

Remark 2.8. We note that the lemma above cannot be improved by letting N be (possibly) degenerate. In fact, in this case, equation (12) would no longer be true, as shown in the following example.

Example 2.9. Consider the manifolds $\mathbb{R}^3_{1,1,1} = \big(\mathbb{R}^3, -(dx^2)^2 + (dx^3)^2\big) =: (M, g)$ and $\mathbb{R}^3_{2,0,1} = \big(\mathbb{R}^3, (dy^3)^2\big) =: (N, h)$, and let $\phi : M \to N$ be the identity map. This map is easily seen to be radical-preserving. However, we have

$$\overline{\ker d\phi} = \{\overline{\mathbf{0}}\} \subsetneqq \operatorname{span}(\overline{\partial/\partial x^2}) = \ker \overline{d\phi}.$$

We have the following special sort of generalized HWC maps:

Lemma 2.10. *Let $\phi : M \to N$ be a radical-preserving map with N non-degenerate. Then ϕ is (generalized)* HWC *at $x \in M$ with square dilation $\Lambda(x) = 0$ if and only if*

$$\overline{\mathcal{H}}_x \subseteq \overline{\mathcal{V}}_x \,, \tag{13}$$

i.e. if and only if $\overline{\mathcal{H}}_x$ is null.

Proof. By Definition 2.5, ϕ is (generalized) HWC with square dilation $\Lambda(x) = 0$ if and only if

$$\overline{g}_x(\overline{d\phi}_x^*(V), \overline{d\phi}_x^*(W)) = 0 \qquad (V, W \in T_{\phi(x)}N).$$

By equation (11), this holds if and only if $\overline{\mathcal{H}}_x$ is null. □

Example 2.11. We note that the condition (13) does not imply $\mathcal{H}_x \subseteq \mathcal{V}_x$. In fact, taking the map $\phi : (\mathbb{R}^4, -(dx^3)^2) \to (\mathbb{R}^3, -(dx^2)^2)$, defined by $\phi((x^1, x^2, x^3, x^4)) = (y^1 = (x^1)^2, y^2 = 0, y^3 = x^4)$, it is not difficult to show that $\mathcal{V} = \mathrm{span}(\partial/\partial x^2, \partial/\partial x^3)$ and $\mathcal{H} = \mathrm{span}(\partial/\partial x^1, \partial/\partial x^2, \partial/\partial x^4)$. Hence, $\{\mathbf{0}\} = \overline{\mathcal{H}} \subseteq \mathrm{span}(\overline{\partial/\partial x^3}) = \overline{\mathcal{V}}$, but $\mathcal{H} \not\subseteq \mathcal{V}$.

We have the following characterization which generalizes a better-known characterization of HWC maps (cf. [1]).

Proposition 2.12. *A radical-preserving map $\phi : M \to N$, where M and N are non-null semi-Riemannian manifolds, is (generalized)* HWC *at $x \in M$ with square dilation $\Lambda(x)$ if and only if*

$$\overline{d\phi}_x \circ \overline{d\phi}_x^* = \Lambda(x)\, \mathbf{1}_{\overline{T_{\phi(x)}N}}. \tag{14}$$

Proof. From the characterization (9) of the adjoint map $\overline{d\phi}_x^*$, we have

$$\overline{g}_x(\overline{d\phi}_x^*(\overline{V}), \overline{d\phi}_x^*(\overline{W})) = h_{\phi(x)}(\overline{V}, \overline{d\phi}_x \circ \overline{d\phi}_x^*(\overline{W})), \qquad (V, W \in T_{\phi(x)}N). \tag{15}$$

Comparing with equation (10), gives the statement. □

Proposition 2.13. *Let ϕ be a (generalized)* HWC *between a non-null semi-Riemannian manifold M and a non-degenerate manifold N, and let $x \in M$. Then $\overline{\mathcal{H}}_x \subseteq \overline{\mathcal{V}}_x$ if and only if one of the following holds:*

(i) $\ker \overline{d\phi}_x \equiv \overline{T_xM}$ *(i.e.* $\ker d\phi_x = T_xM$*),*

(ii) $\ker \overline{d\phi}_x \subsetneqq \overline{T_xM}$ *is degenerate.*

Proof. If $\overline{\mathcal{H}}_x \subseteq \overline{\mathcal{V}}_x$ and (i) does not hold, then $\overline{\mathcal{H}}_x \neq \{\mathbf{0}\}$, so that there exists a vector $\overline{0} \neq \overline{X} \in \overline{\mathcal{H}}_x$ and, for such a vector, $\overline{g}(\overline{X}, \overline{V}) = 0$ for any $\overline{V} \in \overline{\mathcal{V}}_x$, so that (ii) holds. Conversely if $\overline{\mathcal{V}}_x := \ker \overline{d\phi}_x \equiv \overline{T_xM}$ then clearly $\overline{\mathcal{H}}_x \subseteq \overline{\mathcal{V}}_x$. If, on the other hand, $\ker \overline{d\phi}_x$ is degenerate, then since ϕ is (generalized) HWC, we get $\Lambda(x) = 0$; in fact, $\ker \overline{d\phi}_x$ is degenerate if and only if $\overline{\mathcal{H}}_x$ is degenerate if and only if $\overline{\mathcal{V}}_x \cap \overline{\mathcal{H}}_x \neq \{\mathbf{0}\}$, so that there exists a non-zero vector $V \in T_{\phi(X)}N$ such that

$$\overline{0} \neq \overline{d\phi}_x^*(V) \in \ker \overline{d\phi}_x \cap \mathrm{image}\, \overline{d\phi}_x^*.$$

Combining this with the (generalized) HWC condition gives

$$0 = \overline{g}(\overline{d\phi}_x^*(V), \overline{d\phi}_x^*(W)) = \Lambda(x)\, h(V, W) \quad \text{for any } W \in T_{\phi(X)}N,$$

and, as h is non-degenerate, we must have $\Lambda(x) = 0$. Then, from Lemma 2.10, $\overline{\mathcal{H}}_x \subseteq \overline{\mathcal{V}}_x$, and this gives the claim. □

In the case when the square dilation is non-zero, we have the following characterization:

Proposition 2.14. *A map $\phi : (M,g) \to (N,h)$ between a non-null semi-Riemannian manifold M and a non-degenerate manifold N is (generalized)* HWC *at a point $x \in M$ with square dilation $\Lambda(x) \neq 0$ if and only if*

$$h_{\phi(x)}(d\phi_x(X), d\phi_x(Y)) = \Lambda(x)\, g_x(X,Y), \qquad (X,Y \in \mathcal{H}_x). \tag{16}$$

Proof. Suppose that ϕ is (generalized) HWC; then by Lemma 2.7 we have image$(\overline{d\phi}_x^*) = \overline{\mathcal{H}}_x$, so that, for any $\overline{X}, \overline{Y} \in \overline{\mathcal{H}}_x$ there exist vectors V and $W \in T_{\phi(x)}N$ such that

$$\overline{d\phi}_x^*(V) = \overline{X} \quad \text{and} \quad \overline{d\phi}_x^*(W) = \overline{Y}. \tag{17}$$

Applying the operator $\overline{d\phi}_x$ to both sides of the identities (17), and using equation (14), since $\Lambda(x) \neq 0$ we obtain

$$V = (\Lambda(x))^{-1}\overline{d\phi(X)} \quad \text{and} \quad W = (\Lambda(x))^{-1}\overline{d\phi(Y)};$$

on substituting these into the definition of (generalized) HWC, we obtain the statement. The converse is similar. □

We thus obtain the following characterization for a (generalized) HWC map:

Theorem 2.15. *Let $\phi : M \to N$ be a radical-preserving map between a non-null semi-Riemannian manifold (M,g) and a non-degenerate manifold (N,h). Then ϕ is (generalized) HWC at $x \in M$, with square dilation $\Lambda(x)$, if and only if precisely one of the following possibilities holds:*

(a) *$d\phi_x = 0$ (so $\Lambda(x) = 0$);*

(b) *$\overline{\mathcal{V}}_x \subsetneqq \overline{T_xM}$ is degenerate and $\overline{\mathcal{H}}_x \subseteq \overline{\mathcal{V}}_x$ (equivalently $\overline{\mathcal{H}}_x$ is non-zero and null): then $\Lambda(x) = 0$ but $d\phi_x \neq 0$;*

(c) *$\Lambda(x) \neq 0$ and $h_{\phi(x)}(d\phi_x(X), d\phi_x(Y)) = \Lambda(x)\, g_x(X,Y), \qquad (X,Y \in \mathcal{H}_x)$.*

Proof. Let $x \in M$ and suppose that ϕ is (generalized) HWC at x. If $\Lambda(x) = 0$ then by Lemma 2.10 we have $\overline{\mathcal{H}}_x \subseteq \overline{\mathcal{V}}_x$, so by Proposition 2.13, either (i) $\ker \overline{d\phi_x} \equiv \overline{T_xM}$ (i.e. $d\phi_x = 0$, which is case (a)), or (ii) $\ker \overline{d\phi_x} \subsetneqq \overline{T_xM}$ is degenerate, so that case (b) holds. Otherwise $\Lambda(x) \neq 0$, so that by Proposition 2.14 we obtain case (c).

Conversely, if (a) or (b) holds, then clearly ϕ is (generalized) HWC at x with $\Lambda(x) = 0$. If (c) holds then, by Proposition 2.14, ϕ is (generalized) HWC at x with square dilation $\Lambda(x) \neq 0$. □

This result is analogous to the case when both manifolds M and N are non-degenerate (see [1], Proposition 14.5.4). We note that Theorem 2.15 cannot be improved by letting N be degenerate, as shown in Remark 2.8 and Example 2.9.

We have the following characterization of (generalized) horizontal weak conformality whose proof is similar to its (non-degenerate semi-)Riemannian analogue (cf. [1], Lemma 14.5.2):

Lemma 2.16. *A radical-preserving map $\phi : (M, g) \to (N, h)$ between stationary manifolds is (generalized) horizontally weakly conformal at a point $x \in M$ with square dilation $\Lambda(x)$ if and only if, in radical coordinates $\{x^j\}_{j=1}^m$ in a neighbourhood of $x \in M$ and $\{y^\alpha\}_{\alpha=1}^n$ around $\phi(x) \in N$, we have*

$$\sum_{i,j=r+1}^{m} \phi_i^\alpha \phi_j^\beta \overline{g}^{ij} = \Lambda(x)\, \overline{h}^{\alpha\beta}, \tag{18}$$

where $\rho + 1 \leq \alpha, \beta \leq n$ and $\phi_k^\gamma := \partial\phi^\gamma/\partial x^k$. Moreover, setting $\overline{\mathrm{grad}}\phi^\alpha := \overline{g}^{ij}\phi_i^\alpha \overline{\partial/\partial x^j}$, equation (18) *above reads:*

$$\overline{g}(\overline{\mathrm{grad}}\phi^\alpha, \overline{\mathrm{grad}}\phi^\beta) = \Lambda(x)\, \overline{h}^{\alpha\beta}. \tag{19}$$

3. A Fuglede-Ishihara-type characterization of (generalized) harmonic morphisms

3.1. Preliminaries

Recall (see [16]) that

(i) a foliation $\mathcal{F}$ on a manifold M is said to be *simple* if its leaves are the (connected) fibres of a smooth submersion defined on M;

(ii) the *leaf space* of a foliation $\mathcal{F}$ is the topological space $M/\mathcal{F}$ (whose points are the leaves), equipped with the quotient topology.

We note that this space, in general, is not Hausdorff. However, the following holds.

Proposition 3.1. [16] *A foliation $\mathcal{F}$ on M is simple if and only if its leaf space $M/\mathcal{F}$ can be given the structure of a Hausdorff (smooth) manifold such that the natural projection $M \to M/\mathcal{F}$ is a smooth submersion. Furthermore, if such a smooth structure exists, then it is unique.* □

Since each point $x \in M$ has a neighbourhood $W \subseteq M$ with $\mathcal{F}|_W$ simple, $\mathcal{F}$ is always simple locally. Hence, as all the considerations in this section will be local, by replacing the manifold M by a suitable open subset W if necessary, we shall assume that any foliation $\mathcal{F}$ on M is simple. We make the same assumption for N.

We recall (cf. Lemma 1.11) that, if a manifold M is stationary, then its radical distribution $\mathcal{N}(TM)$ is integrable. Let $\mathcal{F}^M$ be the radical foliation of M (i.e., the foliation whose leaves are tangent to $\mathcal{N}(TM)$); set $\overline{M} := M/\mathcal{F}^M$, the leaf space of $\mathcal{N}(TM)$, and denote by $\pi_M : M \to \overline{M}$ the natural projection; by Proposition 3.1, $\overline{M}$ is a smooth manifold. Elements of $\overline{M}$ will be denoted by $[x]_{\mathcal{F}^M} := \pi_M(x)$, where $x \in M$. Then, any radical-preserving map $\phi : M \to N$ between stationary manifolds factors to a map $\overline{\phi} : \overline{M} \to \overline{N}$ in the sense that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\phi} & N \\ \downarrow{\scriptstyle \pi_M} & & \downarrow{\scriptstyle \pi_N} \\ \overline{M} & \xrightarrow{\overline{\phi}} & \overline{N} \end{array} \quad .$$

Thus $\overline{\phi}([x]_{\mathcal{F}^M}) := [\phi(x)]_{\mathcal{F}^N}$. For any $[x] \in \overline{M}$, the map $\overline{\phi}$ naturally induces a linear operator $(d\overline{\phi})_{[x]} : T_{[x]}\overline{M} \to T_{\overline{\phi}([x])}\overline{N}$.

For each $x \in M$ define a following map

$$\Psi_x^M : \overline{T_x M} \to T_{\pi_M(x)}\overline{M}, \qquad \overline{X} \mapsto (d\pi_M)_x(X),$$

where $X \in T_x M$ is such that $\pi_{TM}(X) = \overline{X}$. It is easy to see that Ψ_x^M is a well-defined isomorphism, and that the following holds:

Lemma 3.2. *Let $\phi : M \to N$ be a radical-preserving map between stationary manifolds; then, for any $x \in M$,*

$$\Psi^N_{\phi(x)} \circ \overline{d\phi}_x = (d\overline{\phi})_{[x]} \circ \Psi_x^M \,, \tag{20}$$

equivalently, the following diagram commutes:

$$\begin{array}{ccc} \overline{T_x M} & \xrightarrow{\overline{d\phi}_x} & \overline{T_{\phi(x)} N} \\ \Big\downarrow {\scriptstyle \Psi_x^M} & & \Big\downarrow {\scriptstyle \Psi^N_{\phi(x)}} \\ T_{[x]}\overline{M} & \xrightarrow{(d\overline{\phi})_{[x]}} & T_{\overline{\phi}([x])}\overline{N} \end{array} \quad .$$

In particular, as the maps Ψ_x^M and $\Psi^N_{\phi(x)}$ are isomorphisms, we can identify $(\overline{d\phi})_x$ with $(d\overline{\phi})_{[x]}$. □

3.2. Horizontal weak conformality of $\overline{\phi}$

Let (M, g) be a non-null stationary semi-Riemannian manifold. Then we can endow $\overline{M}$ with the induced metric $g^{\overline{M}}$ defined by:

$$g^{\overline{M}} := ((\Psi^M)^{-1})^* \overline{g},$$

where $\overline{g}$ is defined by

$$\overline{g}(\overline{X}, \overline{Y}) := g(X, Y) \qquad (X, Y \in \Gamma(TM)).$$

Note that the metric $g^{\overline{M}}$ is non-degenerate.

The adjoint of $d\overline{\phi}_{[x]} : T_{[x]}\overline{M} \to T_{\overline{\phi}([x])}\overline{N}$ is the (unique) linear map $(d\overline{\phi})^*_{[x]} : T_{\overline{\phi}([x])}\overline{N} \to T_{[x]}\overline{M}$ characterized as usual by

$$g^{\overline{M}}_{[x]}((d\overline{\phi})^*_{[x]}(\widetilde{V}), \widetilde{X}) = h^{\overline{N}}_{\overline{\phi}([x])}(\widetilde{V}, d\overline{\phi}_{[x]}(\widetilde{X})), \qquad (\widetilde{X} \in \Gamma(T_{[x]}\overline{M}), \widetilde{V} \in \Gamma(T_{\overline{\phi}([x])}\overline{N})). \tag{21}$$

Setting $\widetilde{X} = \Psi^M(\overline{X})$ and $\widetilde{V} = \Psi^M(\overline{V})$ for some $\overline{X} \in \Gamma(\overline{TM})$, $\overline{V} \in \Gamma(\overline{TN})$ and using equation (20) we obtain:

$$(d\overline{\phi})^* \circ \Psi^N = \Psi^M \circ \overline{d\phi}^*. \tag{22}$$

Now we can state the

Proposition 3.3. *Let $\phi : (M,g) \to (N,h)$ be a radical-preserving map between stationary manifolds. Then ϕ is (generalized) HWC if and only if $\overline{\phi}$ is* HWC.

Proof. The map $\overline{\phi}$ is HWC with square dilation Λ if and only if:

$$g^{\overline{M}}\big((d\overline{\phi})^*(\widetilde{V}),(d\overline{\phi})^*(\widetilde{W})\big) = \Lambda\, h^{\overline{N}}(\widetilde{V},\widetilde{W}), \qquad (\widetilde{V},\widetilde{W} \in \Gamma(T\,\overline{N})). \tag{23}$$

Let $\overline{V},\overline{W} \in \Gamma(\overline{TN})$ be such that:

$$\widetilde{V} = \Psi^N(\overline{V}), \quad \widetilde{W} = \Psi^N(\overline{W}); \tag{24}$$

then, on using substitutions (24), equation (22) and the definition of $g^{\overline{M}}$, we see that (23) is equivalent to ϕ being (generalized) HWC. □

3.3. On harmonicity of $\overline{\phi}$

Let (M,g) and (N,h) be two stationary manifolds of dimension m and n respectively, whose radical distributions $\mathcal{N}(TM)$ and $\mathcal{N}(TN)$ have ranks r and ρ respectively. Then the quotient manifolds $(\overline{M},g^{\overline{M}})$ and $(\overline{N},h^{\overline{N}})$ are $(m-r)$- and $(n-\rho)$-dimensional non-degenerate semi-Riemannian manifolds, thus they admit uniquely determined Levi-Civita connections $\nabla^{\overline{M}}$ and $\nabla^{\overline{N}}$, respectively.

As $\overline{M}$ and $\overline{N}$ are non-degenerate, we have the usual notion of tension field τ, for a map $\overline{\phi} : \overline{M} \to \overline{N}$:

$$\tau(\overline{\phi}) := \mathrm{tr}_{g^{\overline{M}}}(\nabla d\,\overline{\phi}), \tag{25}$$

where ∇ is the connection on the bundle $(T\,\overline{M})^* \otimes (\overline{\phi})^{-1}(T\,\overline{N})$ induced from $\nabla^{\overline{M}}$ and $\nabla^{\overline{N}}$. Then $\overline{\phi}$ is harmonic if and only if $\tau(\overline{\phi}) = 0$. Endow (M,g) (resp. (N,h)) with (local) radical coordinates $(x^1,\dots,x^r,x^{r+1},\dots,x^m)$ (resp. $(y^1,\dots,y^\rho,y^{\rho+1},\dots,y^n)$); then $\overline{M}$ (resp. $\overline{N}$) has the same coordinates as M with the first r (resp. ρ) coordinates omitted. In these coordinates, (25) reads:

$$\tau^\gamma(\overline{\phi}) = \sum_{\alpha,\beta,\gamma=\rho+1}^{n} \sum_{i,j,k=r+1}^{m} (g^{\overline{M}})^{ij}\big(\overline{\phi}^\gamma_{ij} - {}^{\overline{M}}\Gamma^k_{ij}\overline{\phi}^\gamma_k + {}^{\overline{N}}\Gamma^\gamma_{\alpha\beta}\overline{\phi}^\alpha_i\overline{\phi}^\beta_j\big),$$

where ${}^{\overline{M}}\Gamma^k_{ij}\partial/\partial x^k := \nabla^{\overline{M}}_{\partial/\partial x^i}\partial/\partial x^j$ and ${}^{\overline{N}}\Gamma^\gamma_{\alpha\beta}\partial/\partial y^\gamma := \nabla^{\overline{N}}_{\partial/\partial y^\alpha}\partial/\partial y^\beta$. Since the coordinates are radical, we have $\overline{\phi}^\gamma = \phi^\gamma$ (for $\gamma = \rho+1,\dots,n$), and the Christoffel symbols ${}^{\overline{M}}\Gamma^k_{ij}$ and ${}^{\overline{N}}\Gamma^\gamma_{\alpha\beta}$ agree with the symbols ${}^{M}\overline{\Gamma}^k_{ij}$ and ${}^{N}\overline{\Gamma}^\gamma_{\alpha\beta}$ of formula (8) (for $r+1 \le i,j,k \le m$ and $\rho+1 \le \alpha,\beta,\gamma \le n$); hence, we have:

Proposition 3.4. *Let $\phi : M \to N$ be a radical-preserving map between stationary manifolds. Then, on identifying $\overline{T_yN}$ with $T_{\overline{y}}\overline{N}$ ($\overline{y} := \pi_N(y)$), $\overline{\tau}(\phi)_x \in \overline{T_{\phi(x)}N}$ can be identified with $\tau(\overline{\phi})_{\overline{x}} \in T_{\overline{\phi}(\overline{x})}\overline{N}$; in particular, $\overline{\phi}$ is harmonic if and only if ϕ is (generalized) harmonic.* □

3.4. Main characterization of (generalized) harmonic morphisms and examples

Now we state the Fuglede-Ishihara-type characterization for (generalized) harmonic morphisms.

Theorem 3.5. *Let $\phi : M \to N$ be a radical-preserving map between stationary manifolds. Then ϕ is a (generalized) harmonic morphism if and only if it is (generalized) harmonic and (generalized)* HWC.

Proof. Any (generalized) harmonic function $f : U \subseteq N \to \mathbb{R}$ is, by definition, radical-preserving, and so factors to a smooth function $\overline{f} : \pi_N(U) \subseteq \overline{N} \to \mathbb{R}$, with $f = \overline{f} \circ \pi_N$; this function $\overline{f}$ is harmonic, by Proposition 3.4. Conversely, if $\overline{f} : V \subseteq \overline{N} \to \mathbb{R}$ is harmonic, then $f := \overline{f} \circ \pi_N$ is (generalized) harmonic. Hence, the map ϕ is a (generalized) harmonic morphism if and only if $\overline{\phi} : \overline{M} \to \overline{N}$ is a harmonic morphism. By Fuglede's Theorem (cf. [9], Theorem 3) this is equivalent to $\overline{\phi}$ being harmonic and HWC, then the claim follows from Propositions 3.4 and 3.3. $\square$

Now we give few examples of (generalized) harmonic morphisms.

Example 3.6. Let ϕ be a (C^2) map

$$\phi : \mathbb{R}^3_{1,1,1} \to \mathbb{R}, \quad (x^1, x^2, x^3) \mapsto \phi(x^1, x^2, x^3).$$

Clearly $\mathcal{N}(\mathbb{R}^3_{1,1,1}) = \mathrm{span}(\partial/\partial x^1)$ and $\mathcal{N}(\mathbb{R}) = \{\mathbf{0}\}$. Moreover we have $d\phi\,(\partial/\partial x^1) = \partial\phi/\partial x^1$, so ϕ is radical-preserving if and only if $\partial\phi/\partial x^1 = 0$. We notice that the coordinates (x^1, x^2, x^3) are radical. Identifying the vector fields $\partial/\partial x^2$ and $\partial/\partial x^3 \in \Gamma(T\mathbb{R}^3_{1,1,1})$ and $\partial/\partial t \in \Gamma(T\mathbb{R})$ with their natural projections in $\overline{T\mathbb{R}^3_{1,1,1}}$ and $\overline{T\mathbb{R}}$ respectively, a simple calculation gives the following expression for $\overline{d\phi}^*$:

$$\overline{d\phi}^*\left(\frac{\partial}{\partial t}\right) = -\frac{\partial\phi}{\partial x^2}\frac{\partial}{\partial x^2} + \frac{\partial\phi}{\partial x^3}\frac{\partial}{\partial x^3},$$

from which we get:

$$\left\langle \overline{d\phi}^*\left(\frac{\partial}{\partial t}\right), \overline{d\phi}^*\left(\frac{\partial}{\partial t}\right) \right\rangle_{\overline{T\mathbb{R}^3_{1,1,1}}} = -\left(\frac{\partial\phi}{\partial x^2}\right)^2 + \left(\frac{\partial\phi}{\partial x^3}\right)^2 =: \Lambda. \tag{26}$$

As ϕ is a function, it is automatically (generalized) HWC, and its square dilation is Λ. Moreover ϕ is (generalized) harmonic if and only if

$$\frac{\partial^2\phi}{(\partial x^2)^2} - \frac{\partial^2\phi}{(\partial x^3)^2} = 0, \tag{27}$$

i.e. if and only if ϕ is of the form $\phi(x^1, x^2, x^3) = \mu(x^2 + x^3) + \nu(x^2 - x^3)$, where $\mu, \nu \in C^2(\mathbb{R})$. By Theorem 3.5, ϕ is a (generalized) harmonic morphism.

We note that along $x^2 = x^3$, equations (26) and (27) are trivial and $\Lambda = 0$.

Example 3.7. (An anti-orthogonal multiplication) Identify $\mathbb{R}^3_{1,1,1}$ with the (associative) algebra

$$\{x = \epsilon x^1 + \eta x^2 + j x^3, \quad (x^1, x^2, x^3) \in \mathbb{R}^3\},$$

where ϵ, η and j satisfy the following relations:

$$\epsilon^2 = \epsilon\eta = \eta\epsilon = \epsilon j = j\epsilon = 0, \quad j^2 = \eta^2 = \eta, \quad \eta j = j\eta = j.$$

Given two elements $x, y \in \mathbb{R}^3_{1,1,1}$ we can define their product

$$\theta : \mathbb{R}^3_{1,1,1} \times \mathbb{R}^3_{1,1,1} \to \mathbb{R}^2_{0,1,1} \subseteq \mathbb{R}^3_{1,1,1}, \qquad \theta(x,y) = x \cdot y,$$

as follows:

$$\begin{aligned} \theta(x,y) &= x \cdot y \\ &= (\epsilon x^1 + \eta x^2 + j x^3)(\epsilon y^1 + \eta y^2 + j y^3) \\ &= \epsilon \cdot 0 + \eta(x^2 y^2 + x^3 y^3) + j(x^2 y^3 + x^3 y^2), \end{aligned}$$

where $\mathbb{R}^2_{0,1,1} \equiv \mathbb{R}^2_{1,1}$ is the non-degenerate 2-dimensional Minkowski space, naturally embedded in $\mathbb{R}^3_{1,1,1}$. For any $x \in \mathbb{R}^3_{1,1,1}$ we define the *square norm* $\|x\|^2_{1,1,1}$ (induced from the metric on $\mathbb{R}^3_{1,1,1}$) by:

$$\|x\|^2_{1,1,1} := -(x^2)^2 + (x^3)^2.$$

Then $\|\theta(x,y)\|^2_{1,1,1} = -\|x\|^2_{1,1,1} \cdot \|y\|^2_{1,1,1}$, so θ is an anti-orthogonal multiplication.

Take standard coordinates $(x^1, x^2, x^3, y^1, y^2, y^3)$ in $\mathbb{R}^3_{1,1,1} \times \mathbb{R}^3_{1,1,1}$, and (z^1, z^2, z^3) in $\mathbb{R}^3_{1,1,1}$. They are radical coordinates. It is easy to see that:

$$\mathcal{N}(\mathbb{R}^3_{1,1,1} \times \mathbb{R}^3_{1,1,1}) := \operatorname{span}\left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial y^1}\right)$$

and

$$\mathcal{N}(\mathbb{R}^3_{1,1,1}) := \operatorname{span}\left(\frac{\partial}{\partial z^1}\right).$$

Moreover

$$d\theta = (0, x^2 dy^2 + y^2 dx^2 + x^3 dy^3 + y^3 dx^3, x^2 dy^3 + y^3 dx^2 + x^3 dy^2 + y^2 dx^3),$$

so that θ is radical-preserving.

The components $\theta^\alpha, \quad \alpha = 2, 3$ of θ are easily seen to be (generalized) harmonic, so that θ is (generalized) harmonic.

In order to check the (generalized) horizontal weak conformality, we make use of Lemma 2.16. So, in this case, θ is (generalized) HWC since

$$\overline{g}^{ij}(\theta^\alpha_i)(\theta^\beta_j) = \Lambda \overline{h}^{\alpha\beta},$$

where $(\overline{g}^{ij}) = \operatorname{diag}(-1, 1, -1, 1)$ and $(\overline{h}^{\alpha\beta}) = \operatorname{diag}(-1, 1)$ and $\Lambda = -\big(-(y^2)^2 + (y^3)^2 - (x^2)^2 + (x^3)^2\big) = -(\|x\|^2_{1,1,1} + \|y\|^2_{1,1,1})$. Finally, applying Theorem 3.5, we see that θ is a (generalized) harmonic morphism.

Example 3.8. (Radial projection) Let $\mathbb{R}^3_{1,1,1}$ be $\mathbb{R}^3$ endowed with the degenerate metric $g = -(dx^2)^2 + (dx^3)^2$, where (x^1, x^2, x^3) are the canonical (and so radical) coordinates on $\mathbb{R}^3$. We set

$$(\mathbb{R}^3_{1,1,1})^+ := (\mathbb{R}^3 \backslash \{-(x^2)^2 + (x^3)^2 \leq 0\}, g).$$

We define the *degenerate* 2-*pseudo-sphere* $S^2_{1,1,1}$ as the manifold:

$$S^2_{1,1,1} := \{x \in \mathbb{R}^3 : -(x^2)^2 + (x^3)^2 = 1\},$$

endowed with the induced metric $h := i^*g$, where $i : S^2_{1,1,1} \hookrightarrow \mathbb{R}^3_{1,1,1}$ is the natural inclusion. We can then define the following map:

$$\phi : (\mathbb{R}^3_{1,1,1})^+ \to S^2_{1,1,1} \subseteq \mathbb{R}^3_{1,1,1}, \quad x \mapsto x/\|x\|,$$

where $\|x\| := \sqrt{-(x^2)^2 + (x^3)^2}$ is the norm with respect to the metric of $\mathbb{R}^3_{1,1,1}$.
As $\dim \overline{T_x S^2_{1,1,1}} = 1$, ϕ is automatically (generalized) HWC. Set $\phi^\alpha_i := \partial\phi^\alpha/\partial x^i$ ($\alpha = 1, 2, 3$ and $i = 1, 2$). From Lemma 2.16, by parametrizing the upper half of $S^2_{1,1,1}$ by $X = X(t, u) := (t, \sinh u, \cosh u) \subseteq \mathbb{R}^3_{1,1,1}$, we find that, for $x^2 \neq 0$,

$$\Lambda(x) = (\phi^2_2)^2 - (\phi^2_3)^2 = \frac{1}{(x^3)^2}\left(1 - \left(\frac{x^2}{\|x\|}\right)^2\right)^2 - \frac{1}{(x^2)^2}\left(1 - \left(\frac{x^3}{\|x\|}\right)^2\right)^2,$$

and

$$\ker d\phi_x = \operatorname{span}\left(\frac{x^1(x^3 - \gamma x^2)}{\|x\|^2}\frac{\partial}{\partial x^1} - \gamma\frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^3}\right),$$

where

$$\gamma := \left(1 - \left(\frac{x^3}{\|x\|}\right)^2\right)x^3\left\{\left(1 - \left(\frac{x^2}{\|x\|}\right)^2\right)x^2\right\}^{-1}.$$

For $x^2 = 0$ we have $\Lambda(x) = 0$ and

$$\ker d\phi_x = \operatorname{span}\left(\frac{x^1}{x^3}\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^3}\right).$$

As we have

$$\frac{\partial^2 u}{(\partial x^2)^2} = \frac{\partial^2 u}{(\partial x^3)^2} = \frac{2x^2x^3}{\|x\|^4},$$

then

$$\overline{\tau}(\phi) = -\frac{\partial^2 u}{(\partial x^2)^2} + \frac{\partial^2 u}{(\partial x^3)^2} = 0$$

so that ϕ is (generalized) harmonic. By Theorem 3.5, the map ϕ is a (generalized) harmonic morphism.

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