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About the Group Law for the Jacobi Variety of a Hyperelliptic Curve

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Abstract. We generalize the group law of curves of degree three by chords and tangents to the Jacobi variety of a hyperelliptic curve. In the case of genus 2 we accomplish the construction by a cubic parabola. We derive explicit rational formulas for the addition on a dense set in the Jacobian.

1. Introduction

The intention of this remark is an explicit description of the group law of hyperelliptic curves. It appears that it is possible to generalize the chord and tangent method for curves of degree three in a very naive way by replacing points by point groups of g points and by replacing lines by certain interpolation functions.

Explicit descriptions of the group law play a less important role in the history of the subject. They appear first in the new literature. Cassels remarked 1983: "I cannot even find in the literature an explicit set of equations for the Jacobian of a curve of genus 2 together with explicit expressions for the group operation in a form amenable to calculation ..." (cf. [2, 3]). Mazur remarked 1986: "... a naive attempt to generalize this group structure [of degree 3 curves] to curves of higher degree (even quartics) will not work." (cf. [8], p. 230). With the development of cryptography arose algorithms for the group law. In 1987 Cantor described the group law of a hyperelliptic curve in the context of cryptography (cf. [1, 6]). Later group laws of more general classes of curves were described in [4, 11]. These group laws work step by step and do not allow a visualization.

In this remark we derive explicit formulas for the group law for the Jacobi variety of a curve of genus 2 starting from an interpolating cubic parabola. As the above algorithms perform the reduction in several steps we execute the reduction in only one step. The case $g > 2$ can be performed by rational interpolation functions analogously. These interpolation functions were first considered by Jacobi in connection with Abel's theorem (cf. [5]). Our formulas are much simpler than analogous formulas derived by Theta functions in [3] p. 114–116, [7, 12]. A different geometric interpretation was given by Otto Staude in [10].

2. Preliminaries

Consider a hyperelliptic curve $C = \{\ (x, y) \in \mathbb{C}^2 \mid \ y^2 = p(x)\ \} \cup \{\infty\}$ of genus g where $p(x) = a_0x^{2g+1} + a_1x^{2g} + \cdots + a_{2g+1}$ is a complex polynomial with $a_0 \neq 0$, $g \geq 1$ without double zeros. C is endowed with the involution $\overline{(x, y)} := (x, -y)$, $\overline{\infty} := \infty$. The Jacobi variety of C is the Abelian group

$$\mathrm{Jac}(C) = \mathrm{Div}^0(C)/\mathrm{Div}^P(C),$$

where $\mathrm{Div}^0(C)$ denotes the group of divisors of degree 0 and $\mathrm{Div}^P(C)$ is the subgroup of principal divisors (i.e. the zeros and poles of analytic functions), cf. [9]. We find in every divisor class of Jac(C) an unique so called reduced divisor of the form

$$n_1P_1 + \cdots + n_mP_m - (n_1 + \cdots + n_m)\infty,$$

where $n_1 + \cdots + n_m \leq g$, $P_i \neq P_j, \overline{P_j}, \infty$ for $i \neq j$ and $n_i = 1$ if $P_i = \overline{P_i}$ (cf. [9]). We remark that

$$-(P - \infty) \sim \overline{P} - \infty \tag{$*$}$$

and

$$P_1 + \cdots + P_h \sim h\infty \tag{$**$}$$

if $P_1, \ldots, P_h$ are the finite intersections of C with an algebraic curve.

Now we consider the two reduced divisors

$$J_1 = P_1 + \cdots + P_{h_1} - h_1\infty, \qquad J_2 = Q_1 + \cdots + Q_{h_2} - h_2\infty$$

with $0 \leq h_1, h_2 \leq g$ (in this notation points P_i, Q_j can occur repeatedly). Without restriction of generality we have r $(0 \leq r \leq h_1, h_2)$ pairs $P_{h_1-k} = \overline{Q_{h_2-k}}$, $k = 0, \ldots, r-1$. Because of $P + \overline{P} \sim 2\infty$ it follows

$$J_1 + J_2 \sim P_1 + \cdots + P_{h_1-r} + Q_1 + \cdots + Q_{h_2-r} - (h_1 + h_2 - 2r)\infty.$$

In the case $h_1 + h_2 - 2r \leq g$ we have already a reduced divisor on the left side. Otherwise we consider the interpolation function

$$y = \frac{b_0x^p + \cdots + b_p}{c_0x^q + c_1x^{q-1} + \cdots + c_q} =: \frac{b(x)}{c(x)}$$

(cf. [5]) with $p = \frac{h_1+h_2+g-2r-\varepsilon}{2}$, $q = \frac{h_1+h_2-g-2r-2+\varepsilon}{2}$ where ε is the parity of $h_1 + h_2 + g$. We have $p + q + 1 = h_1 + h_2 - 2r$ degrees of freedom. We can determine the coefficients uniquely up to a constant factor so that we interpolate the points P_i, Q_j (in the case of a multiple point P we require a corresponding degree of contact with C). These $h_1 + h_2 - 2r$ points lie on the algebraic curve $yc(x) - b(x) = 0$. It follows $p(x)c^2(x) - b^2(x) = 0$. On the left side we have a polynomial of degree $\leq h_1 + h_2 - 2r + g$. Therefore we obtain $h_3 \leq g$ further finite intersections $R_1, \ldots, R_{h_3}$. With $(*), (**)$ it follows that

$$\overline{R_1} + \cdots + \overline{R_{h_3}} - h_3\infty$$

is the reduced divisor for $J_1 + J_2$. It appears that only for $g = 1, 2$ nonfractional interpolation functions are sufficient.

Consider the case $g = 2$. Let $J_1 = P_1 + P_2 - 2\infty$, $J_2 = Q_1 + Q_2 - 2\infty$ be two reduced divisors with $P_i \neq \overline{Q_j}$. The interpolation polynomial

$$y = b_0x^3 + b_1x^2 + b_2x + b_3$$

through the P_i, Q_i (possibly with multiplicities) intersects C for $b_0 \neq 0$ in two further finite points R_1 and R_2 with $R_1 \neq \overline{R_2}$. The result is

$$J_1 + J_2 = \overline{R_1} + \overline{R_2} - 2\infty.$$

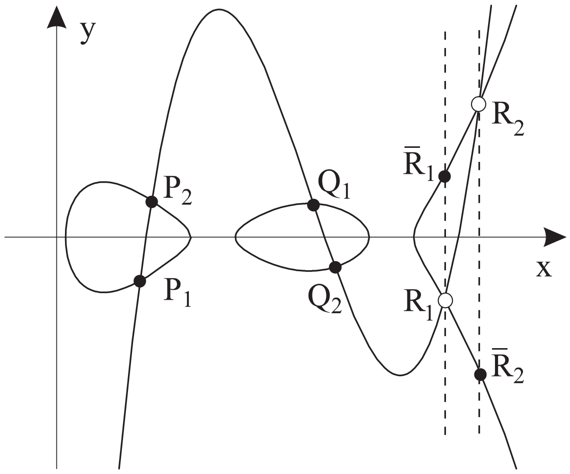

Figure 1. $(P_1 + P_2 - 2\infty) \ + \ (Q_1 + Q_2 - 2\infty) \ \sim \ \overline{R_1} + \overline{R_2} - 2\infty$

Remark. In the real case, contrarily to the case $g = 1$ for $g = 2$ the reduction of the sum of two divisors with real points can give a sum of two complex conjugated points.

3. Explicit formulas

We use the construction in order to derive explicit formulas in the case $g = 2$. We consider only the generic case where $b_0 \neq 0$ and all P_1, P_2, Q_1, Q_2 have different nonvanishing x-coordinates. In this case we have the interpolation polynomial

$$y = b(x) = b_0x^3 + b_1x^2 + b_2x + b_3 = \sum_{i=1}^{4} y_i \prod_{j \neq i} \frac{(x - x_j)}{(x_i - x_j)} .$$

For the x-coordinates of the intersections with the curve $y^2 = a_0x^5 + a_1x^4 + \cdots + a_5$ we obtain

$$(b_0x^3 + b_1x^2 + b_2x + b_3)^2 - a_0x^5 - a_1x^4 - \cdots - a_5 = 0.$$

For the six intersections it follows

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = \frac{a_0 - 2b_0 b_1}{b_0^2}, \quad x_1 x_2 x_3 x_4 x_5 x_6 = \frac{b_3^2 - a_5}{b_0^2}.$$

According to Vieta x_5 and x_6 are solutions of the quadratic equation

$$x^2 + \Big(x_1 + x_2 + x_3 + x_4 - \frac{a_0 - 2b_0 b_1}{b_0^2}\Big)x + \frac{b_3^2 - a_5}{b_0^2 x_1 x_2 x_3 x_4} = 0. \tag{1}$$

Therefore we obtain

$$\overline{R_1} = (x_5, -b_0 x_5^3 - b_1 x_5^2 - b_2 x_5 - b_3), \quad \overline{R_2} = (x_6, -b_0 x_6^3 - b_1 x_6^2 - b_2 x_6 - b_3).$$

4. Rational formulas

The group law of the previous section contains a root operation. It is possible to avoid roots by the representation of divisors by Mumford and Cantor (cf. [1, 9]). We present a reduced divisor $P_1 + P_2 = (x_1, y_1) + (x_2, y_2)$ by the pair of polynomials

$$\Big((x - x_1)(x - x_2), \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) + y_1\Big) =: \Big(A(x), B(x)\Big) = (x^2 + \alpha x + \beta, \gamma x + \delta)$$

if $x_1 \neq x_2$. A divisor $2P_1 = 2(x_1, y_1)$ has the representation $\Big((x - x_1)^2, \frac{p\,'(x_1)}{2y_1}(x - x_1) + y_1\Big)$. The divisors of the form $D = P_1 = (x_1, y_1)$ form the so called Theta divisor Θ. We can represent (x_1, y_1) by the pair $(x - x_1, y_1)$. Now we consider the sum

$$\Big(A_1(x), B_1(x)\Big) + \Big(A_2(x), B_2(x)\Big) = \Big(A_3(x), B_3(x)\Big).$$

The coordinates $\alpha, \beta, \gamma, \delta$ form a coordinate system on $\mathrm{Jac}(C) - \Theta$. We show that the group law has a rational form in the generic case $N b_0 \beta_1 \beta_2 \neq 0$ (cf. below for b_0, N). We can replace the x_i, y_i of the cubic interpolation polynomial through the $\alpha_i, \beta_i, \gamma_i, \delta_i$ by a Groebner basis calculation. We insert the expressions for y_i into $b(x)$ and we consider the ring $\mathbb{C}[x, y, a_1, a_2, b_1, b_2][x_1, x_2, x_3, x_4]$, the order $x_1 < x_2 < x_3 < x_4$ and the ideal

$$\Big((x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4)(x_3 - x_4)(y - b(x)),$$

$$\alpha_1 + x_1 + x_2, \alpha_2 + x_3 + x_4, \beta_1 - x_1 x_2, \beta_2 - x_3 x_4\Big).$$

By a computer calculation we find the first Groebner basis element

$$(a_1^2 - 4b_1)(a_2^2 - 4b_2)\ \Big(\ ((\beta_1 - \beta_2)^2 + (\alpha_1 - \alpha_2)(\alpha_1 \beta_2 - \alpha_2 \beta_1))y - \tilde{b}(x)\Big)$$

where $\tilde{b}(x)$ is independent from the x_i. We require that the discriminants of A_1, A_2 do not vanish. Furthermore we have

$$b_0 = \frac{1}{N}\Big((\beta_2 - \beta_1)(\gamma_1 - \gamma_2) + (\alpha_1 - \alpha_2)(\delta_1 - \delta_2)\Big),$$

$$b_1 = \frac{1}{N}\Big((\alpha_2\beta_2 - \alpha_1\beta_1)(\gamma_1 - \gamma_2) + (\alpha_1^2 - \alpha_2^2 - \beta_1 + \beta_2)(\delta_1 - \delta_2)\Big),$$

$$b_2 = \frac{1}{N}\Big(\alpha_2^2\beta_1\gamma_1 + \alpha_1^2\beta_2\gamma_2 - \alpha_1\alpha_2(\beta_1\gamma_1 + \beta_2\gamma_2) + (\beta_1 - \beta_2)(\beta_1\gamma_2 - \beta_2\gamma_1) +$$
$$+ (\alpha_1\alpha_2(\alpha_1 - \alpha_2) + (\alpha_1\beta_2 - \alpha_2\beta_1))(\delta_1 - \delta_2)\Big),$$

$$b_3 = \frac{1}{N}\Big((\alpha_2 - \alpha_1)\beta_1\beta_2(\gamma_1 - \gamma_2) + \alpha_1^2\beta_2\delta_1 + \alpha_2^2\beta_1\delta_2 - \alpha_1\alpha_2(\beta_2\delta_1 + \beta_1\delta_2) +$$
$$+ (\beta_1 - \beta_2)(-\beta_2\delta_1 + \beta_1\delta_2)\Big)$$

where N is the resultant $(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4)$ or

$$N = (\beta_1 - \beta_2)^2 + (\alpha_1 - \alpha_2)(\alpha_1\beta_2 - \alpha_2\beta_1).$$

Because of (1) we have

$$A_3(x) = x^2 + \Big(-\alpha_1 - \alpha_2 - \frac{a_0 - 2b_0b_1}{b_0^2}\Big)x + \frac{b_3^2 - a_5}{b_0^2\beta_1\beta_2} = 0$$

and

$$B_3(x) = -\Big(y_5\frac{x - x_6}{x_5 - x_6} + y_6\frac{x - x_5}{x_6 - x_5}\Big) = -\frac{b(x_5) - b(x_6)}{x_5 - x_6}x - \frac{b(x_6)x_5 - b(x_5)x_6}{x_5 - x_6}$$
$$= -(b_2 + b_1x_5 + b_1x_6 + b_0x_5^2 + b_0x_5x_6 + b_0x_6^2)x$$
$$+ b_3 + b_2x_5 + b_2x_6 + b_1x_5^2 + b_1x_6^2 + b_1x_5x_6 + b_0x_5^3 + b_0x_5^2x_6 + b_0x_5x_6^2 + b_0x_6^3\ .$$

Using $\alpha_3 = -x_5 - x_6$ and $\beta_3 = x_5x_6$ we obtain

$$B_3(x) = (-b_2 + b_1\alpha_3 - b_0\alpha_3^2 + b_0\beta_3)x - b_0\alpha_3\beta_3 + b_1\beta_3 - b_3.$$

Therefore we have the explicit rational group law

$$\begin{aligned}
\alpha_3 &= -\alpha_1 - \alpha_2 - \frac{a_0 - 2b_0b_1}{b_0^2},\\
\beta_3 &= \frac{b_3^2 - a_5}{b_0^2\beta_1\beta_2},\\
\gamma_3 &= -b_2 + b_1\alpha_3 - b_0\alpha_3^2 + b_0\beta_3,\\
\delta_3 &= -b_0\alpha_3\beta_3 + b_1\beta_3 - b_3
\end{aligned}$$

on the dense set of $\mathrm{Jac}(C) - \Theta$ with $(x_1 - x_2)(x_3 - x_4)Nb_0\beta_1\beta_2 \neq 0$.

Remark. The formulas are also true in the limit $x_1 = x_2$, $x_3 = x_4$. The remaining special cases can be treated similar.

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