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Isotone Analogs of Results by Mal'tsev and Rosenberg

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Abstract. We prove an analog of a lemma by Mal'tsev and deduce the following analog of a result of Rosenberg [11]: let Q be a finite poset with n elements, let $\underline{k}$ denote the k-element chain, and let h be an integer such that $2 \leq h < n \leq k$. Consider the set of all order-preserving maps from Q to $\underline{k}$ whose image contains at most h elements, viewed as an n-ary relation $\mu_{Q,h}$ on $\underline{k}$. Then an l-ary order-preserving operation f on $\underline{k}$ preserves this relation if and only if it is either (i) essentially unary or (ii) the cardinality of $f(e(Q))$ is at most h for every isotone map $e : Q \to \underline{k}^l$. In other words, if an increasing k-colouring of the grid $\underline{k}^l$ assigns more than h colours to a homomorphic image of the poset Q, then there is such an image that lies in a subgrid $G_1 \times \ldots \times G_l$ where each G_i has size at most h, or otherwise the colouring depends only on one variable.

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1. Introduction and Results

Let $k \geq 2$ be a positive integer. We let $\underline{k}$ denote the k-element chain, i.e. the poset on $\underline{k} = \{1, 2, \ldots, k\}$ with the usual ordering of the integers. Let A be a finite set, $|A| \geq 2$ and let $l \geq 1$ be a positive integer. An *operation of arity l on A* is a function $f : A^l \to A$. An operation of arity l is also said to be *l-ary*. Let $G_1, \ldots, G_l, H$ be non-empty sets, and let $S \subseteq G_1 \times G_2 \times \cdots \times G_l$. A function

$$f : S \longrightarrow H$$

is said to *depend on the j-th variable* if there exist $a_i \in G_i$ $(i = 1, \dots, l)$ and $b_j \in G_j$ such that

$$f(a_1, \dots, a_{j-1}, a_j, a_{j+1}, \dots, a_l) \neq f(a_1, \dots, a_{j-1}, b_j, a_{j+1}, \dots, a_l).$$

The function f is *essentially unary* if it depends on at most one of its variables; otherwise it is *essentially at least binary.* In 1958, Jablonskii proved the following:

Lemma 1.1. [5] *Let f be an l-ary operation on $\underline{k}$, $k \geq 3$. If f is essentially at least binary and its image contains at least 3 elements then there exist l sets $G_i \subseteq \underline{k}$ each containing at most 2 elements, and l-tuples $\overline{x} = (x_1, \dots, x_l)$, $\overline{y} = (y_1, \dots, y_l)$ and $\overline{z} = (z_1, \dots, z_l)$ such that $x_i, y_i, z_i \in G_i$ for all $i = 1, \dots, l$ and $f(\overline{x})$, $f(\overline{y})$ and $f(\overline{z})$ are distinct.*

This was later improved slightly by Mal'tsev:

Lemma 1.2. [9] *Let f be an l-ary operation on $\underline{k}$, $k \geq 3$. If f is essentially at least binary and its image contains at least 3 elements then there exist l-tuples $\overline{x} = (x_1, \dots, x_l)$, $\overline{y} = (y_1, \dots, y_l)$ and $\overline{z} = (z_1, \dots, z_l)$ and an index $1 \leq i \leq l$ such that $x_j = y_j$ for all $j \neq i$, $y_i = z_i$, and $f(\overline{x})$, $f(\overline{y})$ and $f(\overline{z})$ are distinct.*

If f is an s-ary operation on A and $g_1, \dots, g_s$ are operations on A of arity l, then the *composition* of f and $g_1, \dots, g_s$ is the l-ary operation h defined by

$$h(x_1, \dots, x_l) = f(g_1(x_1, \dots, x_l), \dots, g_s(x_1, \dots, x_l)).$$

A *projection* is an operation π which satisfies

$$\pi(x_1, \dots, x_l) = x_i$$

for all $x_1, \dots, x_l \in A$. A *clone* on A is a set of operations of finite arity on A which contains all projections and is closed under composition.[1] It is well-known that clones admit a purely relational presentation, which we proceed to describe. Let $n \geq 1$ be a positive integer. A *relation of arity n on A* is a subset of A^n. We say that the l-ary operation f *preserves* the n-ary relation θ if the following holds: given any $n \times l$ matrix M whose columns are in θ, if we apply f to the rows of M then the resulting column is in θ. If R is a set of relations of finite arity on A then $Pol\, R$ denotes the set of all operations on A that preserve each relation in R. It is easy to verify that $Pol\, R$ is a clone, and in fact, it can be shown that every clone on A is of this form.

Let h be an integer such that $2 \leq h < k$. It is well-known and easy to verify that the following sets of operations are clones: B_h consists of all operations whose image contains at most h elements or that are essentially unary. These so-called *Burle clones* (see [1]) admit a simple relational description, first stated by I. G. Rosenberg in 1977. Let n be a positive integer such that $2 \leq h < n \leq k$. Define $\mu_{n,h}$ as the n-ary relation on $\underline{k}$ that consists of all n-tuples $\overline{a} = (a_1, \dots, a_n)$ such that $|\{a_1, \dots, a_n\}| \leq h$. Rosenberg's result is easy to deduce from Mal'tsev's lemma:

[1]We refer the reader to [13] for basic results and terminology concerning clones.

Theorem 1.3. [11] *Let* $2 \le h < n \le k$. *An* l*-ary operation* f *on* $\underline{k}$ *is in* $Pol\, \mu_{n,h}$ *if and only if either (i)* f *is unary or (ii) the image of* f *contains at most* h *elements.*

Proof. If f is unary or the image of f contains at most h elements then it clearly preserves the relation $\mu_{n,h}$. Conversely suppose that f depends on at least two variables and that its image contains at least $h+1$ elements. Let $a_1, \dots, a_{h+1}$ be distinct values in the image of f, and let u_i be elements of $\underline{k}^l$ such that $f(u_i) = a_i$ for all i. We may assume that $\overline{x} = u_1$, $\overline{y} = u_2$ and $\overline{z} = u_3$ are l-tuples with the properties guaranteed by the last lemma. Let M be any $n \times l$ matrix whose $h+1$ first rows are the u_i. Then M has all its columns in $\mu_{n,h}$, but f maps it to an n-tuple with at least $h+1$ distinct entries, so f does not preserve $\mu_{n,h}$. □

A class of clones that has attracted a great deal of attention in the last few years is that of so-called *isotone clones*, i.e. clones of the form $Pol\ \le$ where $\le$ is a partial order on A (see for example [2], [3], [4], [6], [7], [8], [10], [12].) While studying clones of the form $Pol \le$ where $\le$ denotes a total order, we discovered with A. Krokhin that many of their subclones admit a description not unlike that of the Burle clones. Our characterisation required a result concerning the relational description of these clones which is the main result of the present paper. More precisely, let $Pol\ \le$ denote the clone of all order-preserving operations on the k-element chain (as usual, a map f from a poset P to a poset Q is *order-preserving* if $f(x) \le f(y)$ in Q whenever $x \le y$ in P). We determined in [4] that for $k \le 5$ the number of subclones of $Pol\ \le$ that contain all unary order-preserving maps (the so-called monoidal interval, see [13]) is finite. This question remains open for $k \ge 6$, but our investigations showed that the interval of clones, although possibly finite, has a very intricate structure. We now describe some of the clones in this interval.

As above, let k, h, n be integers such that $2 \le h < n \le k$. Let Q be a poset on n elements. We say that the poset Q' is an *extension of* Q if Q and Q' have the same base set and every comparability of Q is also a comparability of Q'. We shall assume that the base set of Q is $\{1, 2, \dots, n\}$ and we'll use the symbol $\sqsubseteq$ to denote the ordering of Q. An n-tuple $\overline{a} = (a_1, \dots, a_n)$ of elements of $\underline{k}$ *respects the ordering* Q if $a_i \le a_j$ whenever $i \sqsubseteq j$. Define $\mu_{Q,h}$ as the n-ary relation on $\underline{k}$ that consists of all n-tuples $\overline{a}$ that respect the ordering Q and such that $|\{a_1, \dots, a_n\}| \le h$. Let $C_{Q,h}$ denote the set of all order-preserving operations on $\underline{k}$ that are either (i) essentially unary or that satisfy the following: (ii) f is l-ary and for every order-preserving map $e : Q \to (\underline{k})^l$ the set $f(e(Q))$ contains at most h elements. For convenience, let $C_{n,h}$ denote the clone $C_{Q,h}$ when Q is the n-element antichain; obviously we have that $C_{n,h} = B_h \cap Pol\ \le$. Notice also that if Q is the n-element antichain, then $\mu_{Q,h} = \mu_{n,h}$.

To illustrate, consider the 4-element poset Q (the 4-crown) on $\{1, 2, 3, 4\}$ where 3 and 4 cover 1 and 2 and these are the only coverings. Then for any $2 \le h \le 3$ the relation $\mu_{Q,h}$ consists of all tuples (a, b, c, d) with at most h distinct entries and such that $a \le c$, $a \le d$, $b \le c$ and $b \le d$.

For another example, consider an ordering Q on $\{1, 2, \dots, n\}$ where only 1 and 2 are comparable. Then it is easy to verify that the clone $C_{Q,h}$ consists of all order-preserving operations f on $\underline{k}$ that are either essentially unary, or have the following property: if the image under f of an $(h+1)$-element set $S \subseteq \underline{k}^l$ contains $h+1$ distinct elements, then S must be an antichain. However, suppose that the image of f contains more than h elements; certainly there are two comparable elements in $\underline{k}^l$ with different values under f, but this means we can find a set S as above which is not an antichain; hence f is essentially unary. This shows that for this particular choice of Q we have $C_{Q,h} = C_{n,h}$.

Clearly the clones $Pol\,\mu_{Q,h}$ and $C_{Q,h}$ are subclones of the clone $Pol\,\leq$ of all order-preserving operations on the k-element chain, unless Q is the n-element antichain. Indeed, if Q has at least one comparability it guarantees that every operation that preserves $\mu_{Q,h}$ is order-preserving. The system of inclusions between these clones appears intricate and quite interesting. We have the following easy inclusion:

Lemma 1.4. *Let $2 \leq h < n \leq k$. Then $C_{Q,h} \subseteq Pol\,\mu_{Q,h}$ for all n-element posets Q.*

Proof. Let $f \in C_{Q,h}$ be l-ary and let M be an $n \times l$ matrix whose columns are in $\mu_{Q,h}$. If we apply f to the rows of M then the resulting column $f(M)$ respects the ordering of Q since f is order-preserving. If f is unary and depends only on the i-th variable, it follows that $f(M)$ contains at most h entries, since the i-th column contains at most h entries; thus f preserves $\mu_{Q,h}$. If f is not essentially unary, consider the map $e : Q \to (\underline{k})^l$ that sends i to the i-th row of M. By definition of $\mu_{Q,h}$ this map is order-preserving, and hence $f(M) = f(e(Q))$ contains at most h elements. Consequently, $f(M) \in \mu_{Q,h}$ and this completes the proof. □

Our main result is that in fact $C_{Q,h} = Pol\,\mu_{Q,h}$ for all h and for all non-trivial n-element posets Q. In other words, for any Q, the order-preserving operations that preserve $\mu_{Q,h}$ are exactly those operations in $C_{Q,h}$.

Theorem 1.5. *Let $2 \leq h < n \leq k$. Let Q be a finite poset on n elements. An l-ary order-preserving operation f on $\underline{k}$ is in $Pol\,\mu_{Q,h}$ if and only if either (i) f is unary or (ii) $|f(e(Q))| \leq h$ for any isotone map $e : Q \to (\underline{k})^l$.*

Notice that if Q in an antichain, then our result is Rosenberg's Theorem 1.3 restricted to order-preserving operations on $\underline{k}$.

We may state this result in a more combinatorial way.

Theorem 1.6. *Let $2 \leq h < n \leq k$, and let Q be a finite poset on n elements. Let f be a colouring of the grid $\underline{k}^l$ by k colours which is non-decreasing on every path from $(1, \ldots, 1)$ to $(k, \ldots, k)$. Suppose that the colouring assigns more than h colours to some homomorphic image of Q in $\underline{k}^l$; then either it assigns more than h colours to a homomorphic image of Q which lies in a subgrid $G_1 \times \ldots \times G_l$ where each G_i has size at most h, or the colouring depends only on one variable.*

Our proof will follow the pattern of that of Theorem 1.3, and for this we shall require an order-theoretic analog of Mal'tsev's lemma:

Lemma 1.7. *Let $k_1, \ldots, k_l$ be positive integers. Let f be an order-preserving map from $\underline{k_1} \times \underline{k_2} \times \ldots \times \underline{k_l}$ onto the 3-element chain that depends on at least two variables. Then there exist l-tuples $\overline{x} = (x_1, \ldots, x_l)$, $\overline{y} = (y_1, \ldots, y_l)$ and $\overline{z} = (z_1, \ldots, z_l)$ in $\underline{k_1} \times \underline{k_2} \times \ldots \times \underline{k_l}$ and an index $1 \leq i \leq l$ such that*

1. *$x_j = y_j$ for all $j \neq i$ and $y_i = z_i$,*
2. *$\{\overline{x}, \overline{y}, \overline{z}\}$ is a chain, and*
3. *$f(\overline{x})$, $f(\overline{y})$ and $f(\overline{z})$ are distinct.*

2. Proofs

To prove Theorem 1.5 we shall proceed as follows: first we prove the analog of Mal'tsev's lemma, Lemma 1.7. From this we will deduce a special case of the main result, namely for Q a chain. This will provide us with our induction base, as we'll prove our main result by induction on the number of incomparabilities in the poset Q. Notice that by the remarks following the statement of Theorem 1.5 we may assume throughout that Q contains at least one comparability.

Before we begin, we make one slight simplification: we claim that it will suffice to prove our result for $n = h + 1$. Let Q be a poset. We say that Q' is an *induced subposet* of Q if the base set of Q' is a subset of the base set of Q and its ordering is the restriction of the ordering of Q to this subset.

Lemma 2.1. [4] *$Pol\, \mu_{Q,h} = \cap_{Q' \in \mathcal{A}} Pol\, \mu_{Q',h}$ where $\mathcal{A}$ is the set of all $(h+1)$-element induced subposets Q' of Q.*

If $n = h + 1$ then we shall drop the subscript h and simply write μ_Q and C_Q respectively.

Lemma 2.2. *If the statement of Theorem 1.5 holds when $h = n + 1$ then it holds for all values of h.*

Proof. Fix a non-trivial poset Q on n elements. By Lemma 1.4 it suffices to show that $Pol\, \mu_{Q,h} \subseteq C_{Q,h}$. By the last lemma, we have that

$$Pol\, \mu_{Q,h} \subseteq Pol\, \mu_{Q'} \subseteq C_{Q'}$$

for every $(h+1)$-element induced subposet Q' of Q. Suppose there exists an isotone map $e : Q \to \underline{k}^l$ such that $|f(e(Q))| > h$. Then there certainly exists an $(h+1)$-element induced subposet Q' of Q such that the restriction e' of e to Q' satisfies $|f(e'(Q'))| > h$. It follows that f must be essentially unary, and we're done. □

Proof of Lemma 1.7. We use induction on $S = \sum k_i$. Certainly $S \geq 4$, and if $S = 4$ we have without loss of generality that $k_1 = k_2 = 2$ and $l = 2$. The claim is now obvious. So fix $k_1, \ldots, k_l$ such that the result holds for all $S' < S$. Let f be an order-preserving map of $\underline{k}_1 \times \underline{k}_2 \times \ldots \times \underline{k}_l$ onto $\{1, 2, 3\}$. Certainly we have that $f(1, \ldots, 1) = 1$ and $f(k_1, \ldots, k_l) = 3$. By induction hypothesis, may suppose that f depends on all its variables.

Choose some variable x_i, $1 \leq i \leq l$. For ease of presentation we'll assume that $i = 1$. Clearly if $f(1, k_2, \ldots, k_l) = 2$ or $f(k_1, 1, \ldots, 1) = 2$ we are done. Now suppose that $f(1, k_2, \ldots, k_l) = f(k_1, 1, \ldots, 1) = 1$. There exists some tuple $(t_1, \ldots, t_l)$ that f maps to 2; it follows that $f(k_1, t_2, \ldots, t_l) \in \{2, 3\}$, $f(1, t_2, \ldots, t_l) = 1$ and $f(t_1, 1, \ldots, 1) = 1$. Hence either we have $f(1, t_2, \ldots, t_l) = 1$, $f(k_1, t_2, \ldots, t_l) = 2$ and $f(k_1, \ldots, k_l) = 3$ and we're done, or else

$f(t_1, 1, \ldots, 1) = 1$, $f(t_1, \ldots, t_l) = 2$ and $f(k_1, t_2, \ldots, t_l) = 3$ which proves our claim. The case where $f(1, k_2, \ldots, k_l) = f(k_1, 1, \ldots, 1) = 3$ is dual. So we are left with the following cases:

Case A. Suppose that $f(1, k_2, \ldots, k_l) = 1$ and $f(k_1, 1, \ldots, 1) = 3$.

Since f is isotone we have that $f(1, x_2, \ldots, x_l) = 1$ and $f(k_1, x_2, \ldots, x_l) = 3$ for all x_i. Since f depends on all its variables, for each $j > 1$ there are tuples such that

$$f(a_1, \ldots, a_{j-1}, a_j, a_{j+1}, \ldots, a_l) \neq f(a_1, \ldots, a_{j-1}, b_j, a_{j+1}, \ldots, a_l).$$

Let $X = \{2, \ldots, k_1\}$ and let $Y = \{1, 2, \ldots, k_1 - 1\}$. Obviously $a_1 \in X \cap Y = \{2, \ldots, k_1 - 1\}$. Let f_X and f_Y denote the restrictions of f to $X \times \underline{k_2} \times \ldots \times \underline{k_l}$ and to $Y \times \underline{k_2} \times \ldots \times \underline{k_l}$ respectively.

Claim 1. f_X and f_Y depend on at least two variables.

Proof of Claim 1. Since f is onto there exists a tuple such that $f(t_1, \ldots, t_l) = 2$. Since $f(k_1, t_2, \ldots, t_l) = 3$, and obviously $t_1 \neq 1$, we have that the restriction of f to $X \times \underline{k_2} \times \ldots \times \underline{k_l}$ depends on its first variable. The tuples $(a_1, \ldots, a_{j-1}, a_j, a_{j+1}, \ldots, a_l)$ and $(a_1, \ldots, a_{j-1}, b_j, a_{j+1}, \ldots, a_l)$ defined above show that f_X depends also on its j-th variable. The proof for Y is similar.

As we remarked in the proof of Claim 1, the images of f_X and f_Y contain 2; and certainly the image of f_X contains 3 and the image of f_Y contains 1. Since we have

$$f(a_1, \ldots, a_{j-1}, a_j, a_{j+1}, \ldots, a_l) \neq f(a_1, \ldots, a_{j-1}, b_j, a_{j+1}, \ldots, a_l),$$

one of these values is not equal to 2. It follows that either f_X or f_Y is onto. By Claim 1 and by induction hypothesis we are done.

Case B. $f(k_1, 1, \ldots, 1) = 1$ and $f(1, k_2, \ldots, k_l) = 3$.

Since f is isotone we have that $f(x, k_2, \ldots, k_l) = 3$ and $f(x, 1, \ldots, 1) = 1$ for all x. Suppose for a contradiction that there is no triple $(\overline{x}, \overline{y}, \overline{z})$ for f. We claim that in this case, if $f(t_1, \ldots, t_l) = 2$ then $f(x, t_2, \ldots, t_l) = 2$ for all $x \in \underline{k_1}$. Indeed, we have that $f(t_1, 1, \ldots, 1) = 1$ and $f(t_1, \ldots, t_l) = 2$ so if f has no triple then $f(k_1, t_2, \ldots, t_l) = 2$. Similarly, since $f(t_1, t_2, \ldots, t_l) = 2$ and $f(t_1, k_2, \ldots, k_l) = 3$ we conclude that $f(1, t_2, \ldots, t_l) = 2$. Hence $2 \leq f(x, t_2, \ldots, t_l) \leq 2$ for all $x \in \underline{k_1}$.

Since we had chosen the variable x_i of f arbitrarily, we have obtained the following: if there is no triple for f, then for any $1 \leq i \leq l$, if

$$f(t_1, \ldots, t_{i-1}, t_i, t_{i+1}, \ldots, t_l) = 2$$

then $f(t_1, \ldots, t_{i-1}, x, t_{i+1}, \ldots, t_l) = 2$ for all $x \in \underline{k_i}$, from which it follows that f is constant, a contradiction. □

The next auxiliary result will be used in the proofs of Lemma 2.4 and Theorem 1.5.

Lemma 2.3. *Let $h \geq 2$. Let f be an l-ary order-preserving operation on $\underline{k}$, and let $\alpha_1 < \ldots < \alpha_{h+1}$ be an $(h+1)$-element chain in the image of f, where $\alpha_1 = 1$ and $\alpha_{h+1} = k$. Then there exists an order-preserving map $g : \underline{k} \to \underline{k}$ with the following properties:*

1. *the image of g is equal to $\{1, 2, \ldots, h+1\}$;*

2. *$g(\alpha_i) = i$ for all $1 \leq i \leq h+1$;*
3. *if $g \circ f$ is essentially unary then so is f.*

Proof. Assume that f depends on at least two variables, without loss of generality suppose they are the first and last. Hence there exist tuples u and v which share all the coordinates but the first such that $f(u) = a \neq b = f(v)$, and similarly there are tuples u' and v' which have the same coordinates but the last such that $f(u') = a' \neq b' = f(v')$. Hence it will suffice to find an order-preserving map $g : \underline{k} \to \underline{k}$ that satisfies the following:

(1) the image of g is equal to $\{1, 2, \ldots, h+1\}$;

(2) $g(\alpha_i) = i$ for all $1 \leq i \leq h+1$;

(3) $g(a) \neq g(b)$ and $g(a') \neq g(b')$.

Maps that satisfy conditions (1) and (2) are easily described: it suffices to map the elements between each α_i and α_{i+1} to $\{i, i+1\}$ in an order-preserving way. Let $\{a, b, a', b'\} = \{u_1, u_2, u_3, u_4\}$ where $u_1 \leq u_2 \leq u_3 \leq u_4$. There are 3 cases: (i) if $\{u_2, u_3\} = \{a, b\}$ or $\{u_2, u_3\} = \{a', b'\}$, choose g as follows: send u_j to the largest i such that $\alpha_i \leq u_j$ if $j = 1, 2$ and to the smallest i such that $\alpha_i \geq u_j$ if $j = 3, 4$. Otherwise, we have two possibilities: (ii) if $u_2 \neq u_3$, send u_j to the largest i such that $\alpha_i \leq u_j$ if $j = 1, 3$ and to the smallest i such that $\alpha_i \geq u_j$ if $j = 2, 4$. It is easy to see that condition (3) is satisfied in both cases. It remains to consider the case where $u_2 = u_3$; in fact, one sees easily that we may suppose that there exists an i such that $\alpha_i \leq u_1 < u_2 = u_3 < u_4 \leq \alpha_{i+1}$. We may suppose without loss of generality that $i < h$, that $\{u_1, u_2\} = \{a, b\}$ and that $\{u_3, u_4\} = \{a', b'\}$. Since the image of f contains elements above α_{i+1}, there exist tuples u'' and v'' which differ in one coordinate only such that $f(u'') = \alpha_{i+1} < f(v'')$. If these tuples differ in any coordinate but the first, then we're back in case (ii) by choosing $u_1, u_2, \alpha_{i+1}, f(w')$ instead of $\{u_1, u_2, u_3, u_4\}$. Otherwise, u'' and v'' differ in the first coordinate. We may then choose g as follows: send u_1, u_2 to i, send u_4 to $i+1$ and send $f(v'')$ to $i+2$. Then $g(a') \neq g(b')$ implies that $g \circ f$ depends on its last variable; and $g(f(u'')) = i+1 \neq i+2 = g(f(v''))$ implies that $g \circ f$ depends on its first variable. □

Lemma 2.4. *Let Q be a chain. Then $Pol\, \mu_Q = C_Q$.*

Proof. By Lemma 1.4 it suffices to prove that $Pol\, \mu_Q \subseteq C_Q$. We proceed as follows: Let f be an l-ary order-preserving operation on $\underline{k}$ that maps some chain onto $h+1$ distinct elements. Let this chain be denoted by $\{e_1, e_2, \ldots, e_{h+1}\}$ where $e_1 < e_2 < \ldots < e_{h+1}$, and let $f(e_i) = \alpha_i$ for all i. We may certainly assume without loss of generality that $\alpha_1 = 1$ and $\alpha_{h+1} = k$, and consequently we may also assume that $e_1 = (1, 1, \ldots, 1)$ and $e_{h+1} = (k, k, \ldots, k)$.

We show that if $f \in Pol\, \mu_Q$ then f is essentially unary. Let g be the map whose existence is guaranteed by Lemma 2.3. Since $f \in Pol\, \mu_Q$ then so is gf; and if we can show that gf is essentially unary it will follow that f is also essentially unary. Thus it will suffice to prove our claim for the map gf, whose image is precisely $\{1, 2, \ldots, h+1\}$ and such that $gf(1, 1, \ldots, 1) = 1$ and $gf(k, k, \ldots, k) = h+1$. For convenience, we simply assume that f (instead of gf) has these properties.

If $a < b$ in $\underline{k}^l$ let $[a, b]$ denote the set of all $x \in \underline{k}^l$ such that $a \leq x \leq b$.

Claim 1. Let $X = \{x_1, x_2, \ldots, x_{h+1}\}$ where $x_1 < x_2 < \ldots < x_{h+1}$ be any chain such that $|f(X)| = h + 1$. Then for every $1 \leq s \leq h - 1$, the restriction of f to the interval $[x_s, x_{s+2}]$ is essentially unary.

Proof of Claim 1. Fix $1 \leq s \leq h - 1$, and let f' denote the restriction of f to $[x_s, x_{s+2}]$. It is clear that f' is onto $\{s, s+1, s+2\}$. By Lemma 1.7, if f' were not essentially unary, we could find l-tuples $\overline{x} = (x_1, \ldots, x_l)$, $\overline{y} = (y_1, \ldots, y_l)$ and $\overline{z} = (z_1, \ldots, z_l)$ in $[x_s, x_{s+2}]$ and an index $1 \leq i \leq l$ such that

1. $x_j = y_j$ for all $j \neq i$ and $y_i = z_i$,
2. $\{\overline{x}, \overline{y}, \overline{z}\}$ is a chain, and
3. $f'(\overline{x})$, $f'(\overline{y})$ and $f'(\overline{z})$ are distinct.

We may assume without loss of generality that $\overline{x} < \overline{y} < \overline{z}$ (the case $\overline{z} < \overline{y} < \overline{x}$ is similar). Consider the $n \times l$ matrix M whose rows are

$$x_1, \ldots, x_{i-1}, \overline{x}, \overline{y}, \overline{z}, x_{i+3}, \ldots, x_{h+1}.$$

Clearly the columns are in μ_Q, but the column $f'(M)$ is not, since f maps $\overline{x}, \overline{y}, \overline{z}$ onto $\{i, i+1, i+2\}$. Since f preserves μ_Q we conclude that f' is essentially unary.

Recall that a subset X of a poset Q is *convex* if it satisfies the following condition: if $a \leq b \leq c$ and $a, c \in X$ then $b \in X$.

Claim 2. Let X and Y be sets such that the restrictions of f to X and to Y are essentially unary. If $X \cap Y$ is convex and there exist $u < v$ in $X \cap Y$ such that $f(u) \neq f(v)$ then the restrictions of f to X and to Y depend on the same variable.

Proof of Claim 2. It is clear that there exist u' and v' between u and v such that $f(u') \neq f(v')$ and v' covers u'. In particular, u' and v' differ in exactly one coordinate. Since $X \cap Y$ is convex it contains both u' and v'. Since the restrictions of f to X and to Y are essentially unary, the coordinate in which u' and v' differ must be the variable on which the restrictions depend.

By Claim 1 we may assume without loss of generality that f depends only on the first variable on the interval $[e_1, e_3]$. We prove by induction that f depends only on the first variable on the interval $[e_1, e_{h+1}] = \underline{k}^l$. Suppose this is true for $[e_1, e_r]$, and we now wish to prove it for $[e_1, e_{r+1}]$. Let e_i' denote the smallest element of $[e_1, e_r]$ that is mapped to i, say $e_i' = (a^{(i)}, 1, 1, \ldots, 1)$ for all $1 \leq i \leq r$. By Claims 1 and 2 f depends only on its first variable on the interval $[e_{r-1}', e_{r+1}]$. Let $e_r = (y_1, \ldots, y_l)$. Let $x \in [e_1, e_{r+1}]$ which is not in the subintervals $[e_1, e_r]$ nor $[e_{r-1}', e_{r+1}]$, i.e. $x = (t_1, t_2, \ldots, t_l)$ where $t_1 < a^{(r-1)}$ and $t_i > y_i$ for some $2 \leq i \leq l$. We must show that $f(x) = f(t_1, 1, \ldots, 1)$. We proceed by induction: choose x maximal with the property that $f(x) \neq f(t_1, 1, \ldots, 1)$. Suppose that $f(t_1, 1, \ldots, 1) = p$. This means that $a^{(i)} > t_1$ for all $i > p$, and in particular $f(x \vee e_i') = i$ for all $i > p$, by maximality of x (here $\vee$ denotes as usual the join operation on $\underline{k}^l$). Consider then the $n \times l$ matrix M whose rows are

$$e_1', \ldots, e_{p-1}', (t_1, 1, \ldots, 1), x, x \vee e_{p+2}', \ldots, x \vee e_r', e_{r+1}, \ldots, e_{h+1}.$$

Clearly its columns are in μ_Q. f maps the first p rows to $\{1, 2, \ldots, p\}$. Since $f(x) > p$ and $f(x) \leq f(x \vee e_{p+1}') = p + 1$ we must have that $f(x) = p + 1$. Thus f maps the rows of M onto $h + 1$ distinct elements, which contradicts our hypothesis that $f \in Pol\, \mu_Q$. $\square$

Proof of Theorem 1.5. It suffices by Lemma 2.2 to prove the result for $n = h + 1$, i.e. to prove that $Pol\,\mu_Q = C_Q$; and by Lemma 1.4 it will suffice to prove that $Pol\,\mu_Q \subseteq C_Q$. We use induction on the number of incomparabilities in Q: suppose the result does not hold, and choose Q with the largest number of comparabilities such that C_Q is a proper subset of $Pol\,\mu_Q$. Notice that Q has at least one comparability by Theorem 1.3. Let $f \in Pol\,\mu_Q$ such that $f \notin C_Q$; then f is essentially at least binary of arity l and there exists an isotone map $e : Q \to \underline{k}^l$ such that $|f(e(Q))| = n$. Let $P = \{x_1, \ldots, x_n\}$ be the image of Q under e, where $f(x_i) < f(x_j)$ if $i < j$. In fact, by using Lemma 2.3 we may assume without loss of generality that $f(x_i) = i$ for all $1 \leq i \leq n$. Since f is isotone and x_1 is minimal in P, we may assume that $x_1 \leq x_i$ for all $i = 1, \ldots, n$; indeed, simply replace x_1 by the meet of all the x_i and modify e accordingly. By Lemma 2.4 there exists at least one pair of incomparable elements in Q. It follows that we may assume that there exists an integer $1 \leq m \leq n - 2$ such that (a) $x_1 < x_2 < \ldots < x_{m-1} < x_m$, (b) $x_i \geq x_m$ for all $i \geq m$ and (c) there exist at least two upper covers of x_m in P. Define an element y of $\underline{k}^l$ as follows:

$$y = x_{m+1} \wedge x_{m+2} \wedge \cdots \wedge x_n.$$

Let M denote the $n \times l$ matrix whose rows are $x_1, \ldots, x_n$, and let M' be the matrix obtained from M by replacing the m-th row by y.

Claim 1. The columns of M' are in μ_Q.

Proof of Claim 1. Let z denote the j-th column of M, and z' the corresponding column of M'. Clearly z respects the ordering Q. Since we have that $x_m \leq y < x_i$ for all $i > m$, it follows that z' also respects the ordering Q. By definition the j-th coordinate of y is the j-th coordinate of some x_i with $i > m$ which implies that z' must contain this coordinate twice.

Claim 2. $f(y) = m + 1$.

Proof of Claim 2. Since $f \in Pol\,\mu_Q$, we must have $f(M') \in \mu_Q$; in particular, the set $\{f(x_1), \ldots, f(x_{m-1}), f(y), f(x_{m+1}), \ldots, f(x_n)\}$ contains at most $n - 1$ elements. But since $f(x_i) = i$ for all $1 \leq i \leq n$, this means that $f(y) \neq m$. But $x_m < y < x_{m+1}$ means that $m \leq f(y) \leq m + 1$ so the claim follows.

Consider the map $\gamma : P \to \underline{k}^l$ defined by

$$\gamma(x_i) = \begin{cases} y & \text{if } i = m + 1, \\ x_i & \text{otherwise.} \end{cases}$$

Since x_{m+1} covers x_m in P it is easy to see that γ is order-preserving. Let P' denote the image of γ; it certainly contains more comparabilities than P does. This means that there exists a proper extension Q' of Q and an order-preserving map $e' : Q' \to P'$. Indeed, if $e(a) = x_{m+1}$ and $e(b) = x_{m+2}$ add the comparability $a < b$ to $\sqsubseteq$ and take the transitive closure; the map $e' = \gamma \circ e$ is then an order-preserving map of Q' onto P'.

Claim 3. If Q contains at least one comparability and if Q' is an extension of Q then $Pol\,\mu_Q \subseteq Pol\,\mu_{Q'}$.

Proof of Claim 3. Consider the relation θ which consists of all tuples in μ_Q that respect the ordering Q'. Since Q has at least one comparability all the operations in $Pol\,\mu_Q$ are order-preserving, and it follows easily that $Pol\,\mu_Q \subseteq Pol\,\theta$. It is also easy to see that if Q' is an extension of Q then $\theta = \mu_{Q'}$.

It follows from Claim 3 that $f \in Pol\,\mu_{Q'}$ which is equal to $C_{Q'}$ by induction hypothesis. Since f is essentially at least binary, it follows that $|f(e'(Q'))|$ should be at most $n-1$. However, by Claim 2 we have that $f(e'(Q')) = f(P') = \{1, 2, \ldots, n\}$, a contradiction. $\square$

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