Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 46 (2005), No. 1, 71-102.

Linear Extensions and Nilpotence of Maltsev Theories

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Abstract. Relationship is clarified between the notions of linear extension of algebraic theories, and central extension, in the sense of commutator calculus, of their models. Varieties of algebras turn out to be nilpotent Maltsev precisely when their theories may be obtained as results of iterated linear extensions by bifunctors from the so called abelian theories. The latter theories are described; they are slightly more general than theories of modules over a ring.

Introduction

The notion of linear extension of categories introduced in [3] is basic in the study of cohomological properties of algebraic theories. Roughly, linear extensions play the same rôle for theories as extensions with abelian kernel for groups. It seems that many remarkable properties of theories are preserved under linear extensions. As an example, one can mention the fact (proved in [17]) that if a theory has the property that all of its projective models are free, then the same is true for any other theory obtained from it by linear extensions.

One of the goals of the present paper is to investigate behaviour of Maltsev theories (the ones possessing a ternary operation p which is Maltsev, i.e. obeys the identity $p(x, x, y) = p(y, x, x) = y$) under linear extensions. In particular, it turns out (Proposition 2.7) that any linear extension of a Maltsev theory is itself Maltsev.

Another aspect of the rôle that linear extensions can play in the study of algebraic theories is related to the notion of nilpotence for theories. One might ask whether there is an analog

[1]Research supported by the RTN Network HPRN-2002-00287 "Algebraic K-Theory, Linear Algebraic Groups and Related Structures"

for theories of the fact that nilpotence of an algebra (say, Lie algebra, or a group, or an associative algebra without unit) is equivalent to the existence of a finite tower of central extensions starting with an abelian algebra and ending with the given algebra. So one might call a theory nilpotent if there is a tower of linear extensions of theories starting with a theory which is "abelian" and ending with the given one. Such definition is inherent in [10], where it is proved that algebraic theories corresponding to the varieties of nilpotent groups (resp. algebras) of nilpotence class n fit into towers of linear extensions of length n by certain bifunctors, starting from abelian, or linear, theories, i.e. theories of modules over some ring. See also [17].

On the other hand, there is a well understood generalization of the commutator calculus from groups or Lie algebras to much more general varieties of universal algebras (the initial idea is contained in [19]; a maximally exhaustive treatment is probably [5]). In particular there is a notion of abelian (linear) and nilpotent varieties, generalizing the ones for groups and algebras. This approach yields most satisfactory results for Maltsev varieties. One can ask, what is the relationship between these two approaches.

It will be proved that these approaches are indeed equivalent for Maltsev theories. That is, a Maltsev variety is nilpotent of class n in the sense of commutator calculus if and only if the corresponding theory can be obtained by n-fold linear extensions of particular "untwisted" type, starting from a Maltsev theory which is abelian in the sense of commutator calculus. In the last section, a description of such abelian Maltsev theories and linear extensions between them is given.

The first author gratefully acknowledges discussions with G. Janelidze which helped him to realize the rôle of torsors under constant groups for characterizing centrality of an abelian extension, as in 1.5 below.

Concerning notation: throughout the paper, it is of set-theoretic style, as is by now usual in category theory. For example, for a morphism $f : G \to H$ between internal groups in a category with products, an expression like $f(x+y)$ might mean composite of f with $+(x,y)$, for variable morphisms $x, y : X \to G$, with $+ : G \times G \to G$ being the group operation. Or, for a congruence on an object A, i.e. a parallel pair $(a_1, a_2) : R \rightrightarrows A$ such that the resulting map $\langle a_1(-), a_2(-)\rangle : \hom(X, R) \to \hom(X, A) \times \hom(X, A)$ is an equivalence relation for any X, we might use notation xRy or $x \sim_R y$, or just $x \sim y$ for morphisms $x, y : X \to A$ such that (x, y) factors through R.

1. Linear extensions as torsors

Various kinds of extensions that appear in this paper are based on the notion of a *principal G-object* or *G-torsor*, or *torsor under G*, for an internal group G in a category $\mathbf{C}$ with finite products. It is a G-object T which satisfies two conditions; the first condition is that the morphism (action, projection): $G \times T \to T \times T$ is an isomorphism. The second condition says that T has global support and may have different meanings, depending on exactness properties of $\mathbf{C}$. In this paper this condition usually means that $T \times T \rightrightarrows T \to 1$ is a coequalizer.

The first condition is often expressed equationally, using a "subtraction" map $- : T \times T \to G$, namely the composite of the projection $G \times T \to G$ with the inverse of the above isomorphism. It is easy to see that the condition is equivalent to requiring two identities

$$(g+x)-x=g,$$
$$(x-y)+y=x$$

for $g:X\to G$, $x,y:X\to T$. In fact the whole torsor structure can be expressed by requiring that the morphism $m:T\times T\times T\to T$ given by $m(x,y,z)=(x-y)+z$ be an associative Maltsev operation, in the following sense:

Definition 1.1. *A morphism $m:T\times T\times T\to T$ is called a Maltsev operation on T if it satisfies*

$$m(x,y,y)=x=m(y,y,x).$$

It is called associative if one has

$$m(u,v,m(x,y,z))=m(m(u,v,x),y,z)$$

and commutative if $m(x,y,z)=m(z,y,x)$.

Note for future reference:

Lemma 1.2. *Any associative Maltsev operation m satisfies*

$$m(u,v,m(x,y,z))=m(u,m(y,x,v),z).$$

Proof. For readability, denote $m(x,y,z)=x-y+z$. We thus have

$$x-y+y=x=y-y+x$$

and

$$u-v+(x-y+z)=(u-v+x)-y+z.$$

We can thus save parentheses and denote the latter expression by $u-v+x-y+z$. One then has

$$\begin{aligned} u-v+x-y+z &= (u-(y-x+v)+(y-x+v))-v+x-y+z \\ &= u-(y-x+v)+(y-x+v-v+x-y+z) \\ &= u-(y-x+v)+z. \end{aligned}$$

□

One has the following (well known, in various guises) fact:

Proposition 1.3. *Let* **C** *be a category with coequalizers of congruences and finite products, which commute (that is, coequalizer of a product of diagrams is product of their coequalizers). Then there is a torsor structure on an object T iff it has global support and there is an associative Maltsev operation on T. In fact there is a one-to-one correspondence between such structures. Moreover commutative Maltsev operations correspond to structures of torsors under abelian groups.*

Proof. Given a torsor structure, the Maltsev conditions for $m(x, y, z) = (x - y) + z$ are just the two identities above. As for associativity, it means

$$(u - v) + ((x - y) + z) = (((u - v) + x) - y) + z,$$

which follows easily from

$$(g + x) - y = (g + ((x - y) + y)) - y = ((g + (x - y)) + y) - y = g + (x - y).$$

If moreover the group is commutative, one has

$$(x - y) + z = (x - y) + (z - y) + y = (z - y) + (x - y) + y = (z - y) + x.$$

Conversely, for an associative Maltsev operation m on T, the relation

$$(x, y) \sim (m(x, y, z), z)$$

is a congruence on $T \times T$. Indeed, it is reflexive since $(m(x, y, y), y) = (x, y)$, symmetric since $(m(x, y, z), z) \sim (m(m(x, y, z), z, y), y) = (m(x, y, m(z, z, y)), y) = (x, y)$, and transitive since $(m(m(x, y, z), z, t), t) = (m(x, y, m(z, z, t)), t) = (m(x, y, t), t)$. Let G be the coequalizer, and let $- : T \times T \twoheadrightarrow G$ be the quotient map. Since $(m(x, x, y), y) = (y, y)$, one has $(x, x) \sim (y, y)$, i.e. $x - x = y - y$. In particular, taking for x, y the projections $T \times T \to T$ we get a map from their coequalizer to G, i.e. a global element $0 : 1 \to G$, by the global support condition on T. Addition on G is defined by $(x - y) + (z - t) = m(x, y, z) - t$ which is legitimate since cartesian product of two coequalizers is a coequalizer in our category. Then additive inverse of $x - y$ is $y - x$, and the action of G on T is given by $(x - y) + z = m(x, y, z)$. It is straightforward to verify all the remaining identities. □

So torsors in such categories determine (at least one of) their own groups. In view of this, objects equipped with an associative Maltsev operation will be also referred to as torsors. In the literature they are also known as *herds*.

Definition 1.4. *A morphism $p : E \to B$ in a category $\mathbf{C}$ with products is called an abelian extension, or simply abelian, if it admits a structure of a torsor in $\mathbf{C}/B$, for some internal abelian group in $\mathbf{C}/B$. If furthermore the group has the form $B^*(A) = (B \times A \to B)$, for some internal abelian group A in $\mathbf{C}$, then the morphism is called central extension.*

So in view of 1.3, a morphism $p : E \to B$ in a sufficiently nice category is an abelian extension iff it is coequalizer of its own kernel pair $E \times_B E \rightrightarrows E$, and there is an associative Maltsev operation $E \times_B E \times_B E \to E$ over B. As for central extensions, one has another omnipresent fact:

Proposition 1.5. *A morphism $p : E \to B$ in $\mathbf{C}/B$ is a torsor under a constant group $B^*(G)$, for some group G in $\mathbf{C}$, if and only if the corresponding Maltsev operation $m : E \times_B E \times_B E \to E$ extends to an associative Maltsev operation $E \times_B E \times E \to E$, with $pm(x, y, z) = pz$ for any $(x, y, z) : X \to E \times_B E \times E$.*

Proof. If the group is $B^*(G)$, then the action can be written as $+ : G \times E \cong (B \times G) \times_B E \to E$, and the subtraction is given by $(p, -) : E \times_B E \to B \times G$ for some map $- : E \times_B E \to G$. Thus $(x - y) + z$ is defined for $px = py$ and any z. Moreover using $p(g + x) = px$, all the identities are proved in exactly the same way as in 1.3.

Conversely if m as above is given, we construct the group G as quotient of $E \times_B E$ by $(x, y) \sim (m(x, y, z), z)$ again, with $(x, y) : X \to E \times_B E$ and any $z : X \to E$. That is, we coequalize the maps $(x, y, z) \mapsto (x, y)$ and $(x, y, z) \mapsto (m(x, y, z), z)$ from $E \times_B E \times E$ to $E \times_B E$. This then gives the maps $+ : G \times E \to E$ and $- : E \times_B E \to G$ just as in 1.3, satisfying the required identities. □

Terminology in 1.4 above is motivated by one important case, when $\mathbf{C}$ is the slice $\mathbf{V}/B$ of some variety $\mathbf{V}$ of universal algebras over one of its objects B. Then the object $p : E \to B$ of $\mathbf{V}/B$ having global support simply means that p is surjective. It is well known that this notion of torsor gives various kinds of extensions of universal algebras. For example, when $\mathbf{V}$ is the variety of groups, then for any internal group G in $\mathbf{V}/B$ there is a B-module M such that G is isomorphic to the projection $B \rtimes M \to B$ of the semidirect product of B with M, the group structure given by homomorphisms $(+ : (B \rtimes M) \times_B (B \rtimes M) \to B \rtimes M$, $- : B \rtimes M \to B \rtimes M$, $0 : B \to B \rtimes M)$ with $((b, x_1), (b, x_2)) \mapsto (b, x_1 + x_2)$, $(b, x) \mapsto (b, -x)$, $b \mapsto (b, 0)$ respectively. Furthermore a G-torsor is the same as a short exact sequence

$$M \overset{i}{\rightarrowtail} E \overset{p}{\twoheadrightarrow} B$$

with $i(p(e)x) = e + i(x) - e$, the action $(B \rtimes M) \times_B E \to E$ being given by $((b, x), e) \mapsto i(x) + e$. Similarly when $\mathbf{V}$ is the variety of Lie rings, an internal group in $\mathbf{V}/B$ amounts to a B-module M, and a torsor under this group to an extension $0 \to M \to E \to B \to 0$; when $\mathbf{V}$ is the variety of associative algebras with unit, one gets bimodules and singular extensions, etc. Note that in all these cases torsors under *constant* groups, i.e. internal groups in $\mathbf{V}/B$ represented by projections $B \times G \to B$ for an internal group G in $\mathbf{V}$, correspond to central extensions of B.

Another situation where torsors are important for us arises from a small category $\mathbb{B}$, with $\mathbf{C}$ the full subcategory $\mathbf{Cat}/\!\!\underline{\,}\,\mathbb{B}$ of the slice $\mathbf{Cat}/\mathbb{B}$ of categories over $\mathbb{B}$, consisting of those functors $p : \mathbb{E} \to \mathbb{B}$ which are identity on objects. In this case the global support condition is that p is full, i.e. surjective on morphisms. The resulting notion turns then out to be equivalent to the notion of *linear extension* of categories, which we now recall.

For a small category $\mathbb{B}$, let $\mathbb{B}^{\#}$ denote the category called *twisted arrow category* of $\mathbb{B}$ in [14], and the *category of factorizations* of $\mathbb{B}$ in [3]. Objects of $\mathbb{B}^{\#}$ are morphisms of $\mathbb{B}$, whereas $\hom_{\mathbb{B}^{\#}}(b, b')$ consists of pairs (b_1, b_2) with $b_1 b b_2 = b'$. A *natural system* on $\mathbb{B}$ with values in a category $\mathbf{C}$ is a functor $D : \mathbb{B}^{\#} \to \mathbf{C}$. It is thus a collection of $\mathbf{C}$-objects $(D_b)_{b:X \to Y}$ of $\mathbf{C}$, indexed by morphisms of $\mathbb{B}$, together with $\mathbf{C}$-morphisms $b_1(\ ) : D_b \to D_{b_1 b}$ and $(\ )b_2 : D_b \to D_{b b_2}$, for all composable morphisms b_1, b, b_2 in $\mathbb{B}$, such that certain evident diagrams commute. In other words, one must have

$$\begin{aligned}
(b_1 b_2) x_3 &= b_1 (b_2 x_3), \\
(b_1 x_2) b_3 &= b_1 (x_2 b_3), \\
(x_1 b_2) b_3 &= x_1 (b_2 b_3)
\end{aligned}$$

for any composable $\xrightarrow{b_3}\xrightarrow{b_2}\xrightarrow{b_1}$ and any $x_i : X \to D_{b_i}$.

We will use the following notion from [3]: for a natural system D on a category $\mathbb{B}$ with values in abelian groups, a *linear extension* of $\mathbb{B}$ by D is an object of $\mathbf{Cat}/_{\!=}\mathbb{B}$, i.e. a functor $P : \mathbb{E} \to \mathbb{B}$ that is identity on objects, together with transitive and effective actions $D_b \times P^{-1}(b) \to P^{-1}(b)$, $(x, e) \mapsto x + e$, for all $b : X \to Y$ in $\mathbb{B}$, such that for any composable morphisms e_1, e_2 in $\mathbb{E}$ and any $x_i \in D_{P(e_i)}$, $i = 1, 2$, one has

$$(x_1 + e_1)(x_2 + e_2) = (x_1 P(e_2) + P(e_1)x_2) + e_1e_2.$$

An example of a linear extension by a natural system D is given by the *trivial* linear extension $\mathbb{B} \rtimes D$ with $\hom_{\mathbb{B}\rtimes D}(X, Y) = \coprod_{b:X\to Y} D_b$, composition $x_1x_2 = x_1b_2 + b_1x_2$ for $x_1 \in D_{b_1}$, $x_2 \in D_{b_2}$, identities $0 \in D_{1_X}$, $X \in \mathbb{B}$, and the actions $D_b \times D_b \to D_b$ given by the group law in D_b.

For natural systems D of abelian groups, there are cohomology groups $H^*(\mathbb{B}; D)$ of $\mathbb{B}$ with coefficients in D, having the usual properties, such that $H^2(\mathbb{B}; D)$ classifies linear extensions of $\mathbb{B}$ by D. See [3].

Proposition 1.6. *For any small category $\mathbb{B}$, assigning to a natural system D on $\mathbb{B}$ the trivial extension $\mathbb{B} \rtimes D \to \mathbb{B}$ determines an equivalence between the category of natural systems of abelian groups on $\mathbb{B}$ and the category of internal abelian groups in $\mathbf{Cat}/_{\!=}\mathbb{B}$. Moreover linear extensions of $\mathbb{B}$ are the same as torsors in $\mathbf{Cat}/_{\!=}\mathbb{B}$, i.e. those objects which, as morphisms in $\mathbf{Cat}$, are abelian in the sense of* 1.4*; more precisely, for any natural system D on $\mathbb{B}$ linear extensions of $\mathbb{B}$ by D are in one-to-one correspondence with $(\mathbb{B} \rtimes D \to \mathbb{B})$-torsors in $\mathbf{Cat}/_{\!=}\mathbb{B}$.*

Proof. The group structure on $\mathbb{B} \rtimes D$ is given as follows: the zero is the functor $\mathbb{B} \to \mathbb{B} \rtimes D$ which sends a morphism $b : X \to Y$ to $0 \in D_b$. The addition functor $(\mathbb{B} \rtimes D) \times_{\mathbb{B}} (\mathbb{B} \rtimes D) \to \mathbb{B} \rtimes D$ is given by addition in the groups D_b and similarly for inverses. Conversely, any abelian group $P : \mathbb{A} \to \mathbb{B}$ in $\mathbf{Cat}/_{\!=}\mathbb{B}$ determines a natural system with $D_b = P^{-1}(b)$. These correspondences are evidently functorial and can be easily checked to define mutually inverse equivalences.

Similarly, given any linear extension $\mathbb{E} \to \mathbb{B}$, of $\mathbb{B}$ by a natural system D, its transitive and effective actions combine into a functor $(\mathbb{B} \rtimes D) \times_{\mathbb{B}} \mathbb{E} \to \mathbb{E}$ which can be checked to form a $\mathbb{B} \rtimes D$-torsor. And conversely, any torsor furnishes the required action for a linear extension. $\square$

In particular, linear extensions can be defined in terms of the subtraction map. One sees easily that the corresponding identities are

$$e_1e_2 - e_1e_2' = P(e_1)(e_2 - e_2'), \quad e_1e_2 - e_1'e_2 = (e_1 - e_1')P(e_2), \qquad (\bullet)$$

for $P(e_i) = P(e_i')$, $i = 1, 2$. In view of 1.3 and 1.6, linear extensions can be also defined in terms of commutative associative Maltsev operations, without mentioning any natural system. Namely, linear extension structures on an object $\mathbb{E} \to \mathbb{B}$ of $\mathbf{Cat}/_{\!=}\mathbb{B}$ with global support are in one-to-one correspondence with functors $\mathbb{E} \times_{\mathbb{B}} \mathbb{E} \times_{\mathbb{B}} \mathbb{E} \to \mathbb{E}$ over $\mathbb{B}$ which are commutative associative Maltsev operations.

There is another context in which natural systems arise as internal abelian groups.

Proposition 1.7. *For any $\mathbb{B}$, there is an equivalence of categories*

$$\mathbf{Ab}(\mathbf{Cat}/\!\!_=\mathbb{B}) \cong \mathbf{Ab}(\mathbf{Set}^{\mathbb{B}^{\mathrm{op}}\times\mathbb{B}}/\hom_{\mathbb{B}}),$$

i.e. the category of natural systems of abelian groups on $\mathbb{B}$ is equivalent to the category of internal abelian groups in the slice over $\hom_{\mathbb{B}}$ of the category of set-valued bifunctors on $\mathbb{B}$. Under this equivalence, the inclusion

$$\hom_{\mathbb{B}}^* : \mathbf{Ab}(\mathbf{Set}^{\mathbb{B}^{\mathrm{op}}\times\mathbb{B}}) \to \mathbf{Ab}(\mathbf{Set}^{\mathbb{B}^{\mathrm{op}}\times\mathbb{B}}/\hom_{\mathbb{B}})$$

of constant internal groups, carrying D to the projection $\hom_{\mathbb{B}}\times D \to \hom_{\mathbb{B}}$, becomes identified with the inclusion into natural systems, carrying a bifunctor $D : \mathbb{B}^{\mathrm{op}}\times\mathbb{B} \to \mathbf{Ab}$ to the natural system with $D_{b:X\to Y} = D(X,Y)$. □

This (as well as 1.6, in fact) is a consequence of general facts from [2] (see 1.5 and 4.11 there).

We will call the particular natural systems arising, as above, from bifunctors, and linear extensions by them *untwisted*.

Thus natural systems on $\mathbb{B}$ can be identified with (trivial) abelian extensions of $\hom_{\mathbb{B}}$ in $\mathbf{Set}^{\mathbb{B}^{\mathrm{op}}\times\mathbb{B}}$, in the sense of 1.4, and moreover untwisted natural systems correspond to trivial central extensions in that category. A natural question then arises – what should be analog of 1.5 in this context, that is, which torsors in $\mathbf{Cat}/\!\!_=\mathbb{B}$ correspond to untwisted linear extensions under the equivalence of 1.6. The answer is given by the following

Proposition 1.8. *Let $P : \mathbb{E} \to \mathbb{B}$ be a full functor bijective on objects, with a torsor structure in $\mathbf{Cat}/\!\!_=\mathbb{B}$ given by the functor $m : \mathbb{E}\times_{\mathbb{B}}\mathbb{E}\times_{\mathbb{B}}\mathbb{E} \to \mathbb{E}$ over $\mathbb{E}$. Then, the linear extension corresponding to it by* 1.6 *is untwisted if and only if m can be extended to a collection of commutative associative Maltsev operations*

$$m_{X,Y} : \hom_{\mathbb{E}}(X,Y) \underset{\hom_{\mathbb{B}}(X,Y)}{\times} \hom_{\mathbb{E}}(X,Y)\times\hom_{\mathbb{E}}(X,Y) \to \hom_{\mathbb{E}}(X,Y),$$

such that

$$P(m_{X,Y}(f_1,f_2,f)) = P(f)$$

and

$$gm_{X,Y}(f_1,f_2,f) = m_{X,Z}(gf_1,gf_2,gf),\ m_{X,Y}(f_1,f_2,f)h = m_{T,Y}(f_1h,f_2h,fh)$$

for any $f_1,f_2,f\in\hom_{\mathbb{E}}(X,Y)$ with $P(f_1)=P(f_2)$ and any $g\in\hom_{\mathbb{E}}(Y,Z)$, $h\in\hom_{\mathbb{E}}(T,X)$.

Proof. The "only if" part follows since as soon as one has a linear extension by a bifunctor D, all the groups D_b, $D_{b'}$ are naturally identified for any $b,b'\in\hom_{\mathbb{B}}(X,Y)$. Hence one can define $m_{X,Y}(f_1,f_2,f) = (f_1-f_2)+f$ by identifying $f_1-f_2\in D_{P(f_i)}$ with the corresponding element of $D_{P(f)}$. The above identities then follow easily from the corresponding identities for linear extensions.

For the "if" part, let D be the natural system of abelian groups corresponding to P according to 1.3 and 1.6. Thus for $b\in\hom_{\mathbb{B}}(X,Y)$, elements of D_b have the form f_1-f_2, with $P(f_1) = P(f_2) = b$, and $f_1 - f_2$ determines the same element as $m(f_1, f_2, f) - f$, for any other f with $P(f) = b$. Then using the $m_{X,Y}$ above we can define a collection of isomorphisms $\varphi_{b,P(f)} : D_b \to D_{P(f)}$ for $f \in \hom_{\mathbb{E}}(X, Y)$, $b \in \hom_{\mathbb{B}}(X, Y)$, via

$$\varphi_{b,P(f)}(f_1 - f_2) = m_{X,Y}(f_1, f_2, f) - f.$$

This is correctly defined since $m_{X,Y}(m(f_1, f_2, f_3), f_3, f) = m_{X,Y}(f_1, f_2, f)$ by associativity and Maltsev identity. And it does not really depend on f. Indeed for any other f' with $P(f') = P(f)$ one has, by the same identities, $m(m_{X,Y}(f_1, f_2, f), f, f') = m_{X,Y}(f_1, f_2, f')$. But this is the same as $m_{X,Y}(f_1, f_2, f) - f = m_{X,Y}(f_1, f_2, f') - f'$ – recall (from the proof of 1.3 that $m(g, f, f') = g'$ is equivalent to $g - f = g' - f'$ as soon as $P(f) = P(g) = P(f') = P(g')$.

Furthermore,

$$\varphi_{b,b} = \mathrm{identity}_{D_b},$$

as $m_{X,Y}(f_1, f_2, f) - f = m(f_1, f_2, f) - f = f_1 - f_2$ for $P(f_1) = P(f_2) = P(f)$, and

$$\varphi_{b',b''}\varphi_{b,b'} = \varphi_{b,b''},$$

as $m_{X,Y}(m_{X,Y}(f_1, f_2, f'), f', f'') - f'' = m_{X,Y}(f_1, f_2, f'') - f''$ for $P(f_1) = P(f_2) = b$, $P(f') = b'$, $P(f'') = b''$.

Finally for any $a : Y \to Z$, $b, b' : X \to Y$ in $\mathbb{B}$, and any f_1, f_2 with $P(f_1) = P(f_2) = b$ one has, using the equations ($\bullet$) above:

$$\begin{aligned} a\varphi_{b,b'}(f_1 - f_2) = a(m_{X,Y}(f_1, f_2, f) - f) = g m_{X,Z}(f_1, f_2, f) - gf \\ = m_{X,Z}(gf_1, gf_2, gf) - gf = \varphi_{ab,ab'}(gf_1 - gf_2) = \varphi_{ab,ab'}(a(f_1 - f_2)) \end{aligned}$$

for any g with $P(g) = a$, and similarly, for any $c : T \to X$ in $\mathbb{B}$,

$$(\varphi_{b,b'}(f_1 - f_2))c = \varphi_{bc,b'c}((f_1 - f_2)c).$$

We thus have constructed an isomorphism of the natural system D with the one obtained from the bifunctor $\bar{D}$, where

$$\bar{D}(X, Y) = \left(\bigoplus_{b \in \hom_{\mathbb{B}}(X,Y)} D_b \right) / \sim,$$

with $\sim$ being the equivalence relation identifying any $x \in D_b$ with $\varphi_{b,b'}(x) \in D_{b'}$, for all $b, b' \in \hom_{\mathbb{B}}(X, Y)$. □

2. Theories

Recollections on algebraic theories

Everywhere in the sequel, **Set** will denote the category of sets. The opposite of its full subcategory with finite sets $\{1, \dots, n\}$ $(n \geqslant 0)$ as objects, will be denoted $\mathbb{S}$. Its objects will be redenoted by $X^0 = 1$, $X^1 = X$ (the *generator*), $X^2, X^3, \dots$, and the morphisms from $\mathbb{S}(X^n, X)$ by $x_1, \dots, x_n$.

A *finitary algebraic theory*, or simply theory, is a small category $\mathbb{T}$ equipped with a functor $\mathbb{S} \to \mathbb{T}$ which is identity on objects and preserves finite products. This functor will be usually suppressed from the notations, and objects and morphisms of $\mathbb{S}$ will be identified with their images under it – an usual abuse of notation with algebras.

A *model* of a theory $\mathbb{T}$ in a category $\mathbf{C}$ is a finite product preserving functor from $\mathbb{T}$ to $\mathbf{C}$. These functors and their natural transformations form the category of models $\mathbb{T}$-$\mathbf{mod}(\mathbf{C})$. For $\mathbf{C} = \mathbf{Set}$, this will be abbreviated to just $\mathbb{T}$-$\mathbf{mod}$. Since representable functors preserve any available limits, there is a full embedding $I_{\mathbb{T}} : \mathbb{T}^{\mathrm{op}} \to \mathbb{T}$-$\mathbf{mod}$.

A model M of $\mathbb{T}$ is in fact nothing but an object $M(X)$ with operations $u_M : M(X)^n \to M(X)$ for each element of $u \in \hom_{\mathbb{T}}(X^n, X)$, satisfying identities prescribed by category structure of $\mathbb{T}$. By this reason, elements of $\mathbb{T}(n) = \hom_{\mathbb{T}}(X^n, X)$ will be called *n-ary operations of* $\mathbb{T}$. Thus for any theory $\mathbb{T}$, the category $\mathbb{T}$-$\mathbf{mod}$ is a variety of universal algebras. In particular, $\mathbb{T}$-$\mathbf{mod}$ is an exact category, regular epis are exactly surjective maps, etc. Conversely, for any variety $\mathbf{V}$, the opposite of the category of the algebras freely generated by the sets $\{1, ..., n\}$, $n \geqslant 0$, is an algebraic theory, whose category of models is equivalent to $\mathbf{V}$.

A morphism of theories $\mathbb{T}' \to \mathbb{T}$ is a model of $\mathbb{T}'$ in $\mathbb{T}$ which respects the structure functors from $\mathbb{S}$. So by definition, $\mathbb{S}$ is the initial object of the category **Theories** of finitary algebraic theories. For every morphism of theories $F : \mathbb{T}' \to \mathbb{T}$, the induced forgetful functor "compose with F", $U_F : \mathbb{T}$-$\mathbf{mod} \to \mathbb{T}'$-$\mathbf{mod}$, has a left adjoint, which we again denote by F, with the adjunction unit $\eta : \text{Identity} \to (-)_F := U_F F$ and counit $\varepsilon : FU_F \to \text{Identity}$. In particular, for $\mathbb{T}' = \mathbb{S}$, the corresponding adjoint pair will be denoted $\mathbb{T}(-) : \mathbf{Set} \rightleftarrows \mathbb{T}$-$\mathbf{mod} : U_{\mathbb{T}}$; models in the image of $\mathbb{T}(-)$ are *free*. The notation is justified by the fact that these left adjoints are compatible with the above embeddings in the sense that for any $F : \mathbb{T}' \to \mathbb{T}$, the square

$$\begin{tikzcd}
\mathbb{T}'^{\mathrm{op}} \arrow[r, "F"] \arrow[d, "I_{\mathbb{T}'}"] & \mathbb{T}^{\mathrm{op}} \arrow[d, "I_{\mathbb{T}}"] \\
\mathbb{T}'\text{-}\mathbf{mod} \arrow[r, "F"] & \mathbb{T}\text{-}\mathbf{mod}
\end{tikzcd}$$

commutes, so taking $\mathbb{T}' = \mathbb{S}$, the Yoneda embedding $I_{\mathbb{T}}$ identifies $\mathbb{T}^{\mathrm{op}}$ with the full subcategory of $\mathbb{T}$-$\mathbf{mod}$ consisting of free models generated by objects of $\mathbb{S}$, i.e. by finite cardinals. Moreover note that the functor $U_{\mathbb{T}}$ is representable by the free model on one generator, $U_{\mathbb{T}}(M) \cong \hom(\mathbb{T}(1), M)$ for any $\mathbb{T}$-model M. So in the sequel, we will interchangeably use notation $\mathbb{T}(n)$ for $\hom_{\mathbb{T}}(X^n, X)$, for $I_{\mathbb{T}}(X^n)$ and for $\mathbb{T}(\{1, \ldots, n\})$.

We note for future reference the following (doubtlessly well known) fact:

Proposition 2.1. *A morphism of theories $P : \mathbb{T}' \to \mathbb{T}$ is a full functor if and only if each $\eta_M : M \to M_P$ is surjective. In this case, U_P is full and faithful, and the corresponding full replete image of $\mathbb{T}$-$\mathbf{mod}$ under U_P is a subvariety of $\mathbb{T}'$-$\mathbf{mod}$, i.e. a full subcategory closed under subobjects, products and homomorphic images. Moreover, for any surjection*

*$q : M \twoheadrightarrow N$ in $\mathbb{T}'$-**mod**, the square*

$$\begin{tikzcd} M \arrow[r, two heads, "q"] \arrow[d, "\eta_M"] & N \arrow[d, "\eta_N"] \\ M_P \arrow[r, two heads, "q_P"] & N_P \end{tikzcd}$$

is pushout.

Proof. Consider the maps $\hom_{\mathbb{T}'}(X^n, X) \to \hom_{\mathbb{T}}(X^n, X)$, $n \geqslant 0$, induced by P. These are surjective iff P is full, and can be identified with $\mathbb{T}'$-homomorphisms $P_n : I_{\mathbb{T}'}(n) \to I_{\mathbb{T}}(n)$, namely with $U_{\mathbb{T}'}\eta_{\mathbb{T}'(n)} : U_{\mathbb{T}'}\mathbb{T}'(n) \to U_{\mathbb{T}'}(\mathbb{T}'(n)_P) = U_{\mathbb{T}}\mathbb{T}(n)$.

Thus P is full iff $\eta_{\mathbb{T}'(n)}$ is surjective for each n.

Next consider a free $\mathbb{T}'$-model $\mathbb{T}'(S)$, for some set S. Then for any $(s_1, \dots, s_n) : \{1, \dots, n\} \to S$ there is a commutative square

$$\begin{tikzcd} \mathbb{T}'(n) \arrow[r, "\mathbb{T}'(s_1,\dots,s_n)"] \arrow[d, "\eta_{\mathbb{T}'(n)}"] & \mathbb{T}'(S)\ ar[d]^{\eta_{\mathbb{T}'(S)}} \\ \mathbb{T}(n) \arrow[r, "\mathbb{T}(s_1,\dots,s_n)"] & \mathbb{T}(S), \end{tikzcd}$$

so the homomorphism $\eta_{\mathbb{T}'(S)}$ is given by colimit of a filtered diagram of homomorphisms of the form $\eta_{\mathbb{T}'(n)}$. As each of these homomorphisms is surjective, $\eta_{\mathbb{T}'(S)}$ is surjective too iff P is full. Finally for any model M one chooses a surjective homomorphism $q : \mathbb{T}'(S) \twoheadrightarrow M$. Both P and U_P preserve surjections: the former – since surjections are precisely coequalizers, and since P, being a left adjoint, preserves them; and the latter – since it commutes with the forgetful functors to sets. Thus $\eta_M q = q_P \eta_{\mathbb{T}'(S)}$ is surjective, so all η_M's are surjective iff P is full.

Now by adjunction, ηU_P is a split mono, with left inverse $U_P\varepsilon$. In our case ηU_P is also componentwise surjective, so these natural transformations are mutually inverse isomorphisms. As U_P obviously reflects isomorphisms, it follows that ε is an isomorphism, i.e. U_P is full and faithful.

Next, consider a mono $i : M \rightarrowtail U_P(N)$. Its composite with the isomorphism $\eta_{U_P(N)}$ is mono again. On the other hand $\eta_{U_P(N)} i = U_P P(i)\eta_M$, so η_M is also mono. As it is surjective, it is thus an isomorphism, i.e. M also belongs to the replete image of U_P.

Now to show that the image of U_P is also closed under quotients, suppose given a surjective homomorphism $q : U_P(N) \twoheadrightarrow M$ in $\mathbb{T}'$-**mod**. For M to belong to the image of U_P, for any u, u' in $\mathbb{T}'(n)$ with $P_n(u) = P_n(u')$, the maps $M(u), M(u') : M(X^n) \to M(X)$ must be equal. In fact since the P_n are surjective, this condition is also sufficient. But $M(u)q(X^n) = q(X)N(u) = q(X)N(u') = M(u')q(X^n)$, and since q is surjective, $M(u) = M(u')$.

Finally, let us prove the pushout property of the square above. Indeed, given homomorphisms $h : N \to N'$, $r : M_P \to N'$ with $hq = r\eta_M$, one has $\mathrm{Im}(h) = \mathrm{Im}(hq) = \mathrm{Im}(r\eta_M) = \mathrm{Im}(q)$. The image of U_P is closed under quotients, as we just proved, so any quotient of

$M_P = U_P P(M)$ is (isomorphic to) U_P of something. In particular so is $\mathrm{Im}(h)$, and by adjunction h factors through η_M, via some h'. Then $h'q_P = r$ since their composites with the epi η_M are easily seen to be equal. □

Remark. In fact, conditions of the above proposition are interrelated: it follows, for example, from 3.1 in [8], that the image of U_P is closed under subobjects iff η is surjective, and under quotients iff the indicated squares are pushouts.

Extensions of theories

Since for a theory $\mathbb{T}$ the category **Theories**/$\mathbb{T}$ is a subcategory of $\mathbf{Cat}/\!\!\!_{=}\mathbb{T}$, closed under finite products, internal groups and torsors in **Theories**/$\mathbb{T}$ are particular groups and torsors in $\mathbf{Cat}/\!\!\!_{=}\mathbb{T}$, hence by 1.6 they can be considered as particular natural systems and linear extensions over $\mathbb{T}$. It is easy to identify the property of natural systems which distinguishes these particular ones (see [10]):

Definition 2.2. *A natural system D on a category with finite products $\mathbb{T}$ is said to be cartesian if for any product diagram $p_i : X_1 \times \cdots \times X_n \to X_i$, $i = 1, \ldots, n$ and any $f : X \to X_1 \times \cdots \times X_n$, the maps $p_i(\ ) : D_f \to D_{p_i f}$, $i = 1, \ldots, n$ also form a product diagram.*

The equivalence of 1.7 restricted to cartesian natural systems yields

Proposition 2.3. *The category of cartesian natural systems of sets on a theory $\mathbb{T}$ is equivalent to the category*

$$\mathbb{T}\text{-}\mathbf{mod}(\mathbf{Set}^{\mathbb{T}^{\mathrm{op}}})/I_{\mathbb{T}},$$

with untwisted cartesian natural systems corresponding to objects in the image of

$$I_{\mathbb{T}}^* : \mathbb{T}\text{-}\mathbf{mod}(\mathbf{Set}^{\mathbb{T}^{\mathrm{op}}}) \to \mathbb{T}\text{-}\mathbf{mod}(\mathbf{Set}^{\mathbb{T}^{\mathrm{op}}})/I_{\mathbb{T}}.$$

Proof. Indeed, looking at the equivalence in 1.7 one sees that the category of cartesian natural systems of sets on a small category with finite products $\mathbb{T}$ is equivalent to the full subcategory of $\mathbf{Set}^{\mathbb{T}^{\mathrm{op}}\times\mathbb{T}}/\hom_{\mathbb{T}}$ on those natural transformations $p : B \to \hom_{\mathbb{T}}$ for which the natural system given by $b \mapsto p^{-1}(b)$ is cartesian. But it is straightforward to check that this happens iff B preserves finite products in the second variable. Thus when $\mathbb{T}$ is a theory this means that for any fixed object X^n, the functor $B(X^n, -)$ is a model of $\mathbb{T}$. So cartesian natural systems correspond to the full subcategory $(\mathbb{T}\text{-}\mathbf{mod})^{\mathbb{T}^{\mathrm{op}}} \cong \mathbb{T}\text{-}\mathbf{mod}(\mathbf{Set}^{\mathbb{T}^{\mathrm{op}}})$ of $(\mathbf{Set}^{\mathbb{T}})^{\mathbb{T}^{\mathrm{op}}} \cong \mathbf{Set}^{\mathbb{T}^{\mathrm{op}}\times\mathbb{T}} \cong (\mathbf{Set}^{\mathbb{T}^{\mathrm{op}}})^{\mathbb{T}}$. □

Corollary 2.4. *Every linear extension $P : \mathbb{T} \to \mathbb{T}_R$ of the theory $\mathbb{T}_R$ of (left) modules over a ring R is untwisted.*

Proof. The category $\mathbb{T}_R$-**mod** $= R$-**mod** is abelian, hence so is $\mathbb{T}_R\text{-}\mathbf{mod}(\mathbf{Set}^{\mathbb{T}_R^{\mathrm{op}}}) = (\mathbb{T}_R\text{-}\mathbf{mod})^{\mathbb{T}_R^{\mathrm{op}}}$; but as it is well known, for any additive category $\mathcal{A}$, and any of its objects A, the functors

$$\mathcal{A} \xleftarrow{\text{forget}} \mathbf{Ab}(\mathcal{A}) \xrightarrow{A^*} \mathbf{Ab}(\mathcal{A}/A)$$

are equivalences of categories. In our case this gives that every object of $\mathbf{Ab}((\mathbb{T}_R\text{-}\mathbf{mod})^{\mathbb{T}_R^{\mathrm{op}}}/I_{\mathbb{T}_R})$ is isomorphic to a projection $I_{\mathbb{T}_R}\times T\to I_{\mathbb{T}_R}$ for some $T:\mathbb{T}_R^{\mathrm{op}}\to\mathbb{T}_R\text{-}\mathbf{mod}$. Translating this fact along the equivalence of 1.7 one obtains that any cartesian abelian natural system on $\mathbb{T}_R$ is isomorphic to one of the form $D_{u:X^m\to X^n}\equiv\mathrm{Hom}_R(R^n,T(X^m))$, for some functor $T:\mathbb{T}_R^{\mathrm{op}}\to R\text{-}\mathbf{mod}$. Evidently this is an untwisted natural system. □

One has (cf. [10], (6.1))

Proposition 2.5. *A natural system of abelian groups D on a category with finite products $\mathbb{T}$ is cartesian iff for any linear extension $P:\mathbb{T}'\to\mathbb{T}$ of $\mathbb{T}$ by D, the category $\mathbb{T}'$ also has finite products, and P preserves them.* □

In particular, linear extensions of an algebraic theory $\mathbb{T}$ by a cartesian natural system D are morphisms of theories, and equivalence classes of such extensions form an abelian group isomorphic to $H^2(\mathbb{T};D)$.

There are lots of examples of linear extensions of theories in [10]. Let us mention those which we will encounter in this paper.

Examples 2.6. 1. Consider the functor from theories to monoids given by $\mathbb{T}\mapsto\hom_{\mathbb{T}}(X,X)$. This functor has a full and faithful right adjoint assigning to a monoid M, the theory $\mathbb{T}_M$ of M-sets. Thus the category of monoids can be identified with a full subcategory of **Theories** closed under limits there. In particular, groups, torsors, herds, natural systems, linear extensions, etc. of monoids (considered as categories with one object) can be identified with those of the corresponding theories. In other words, a morphism of theories $P:\mathbb{T}_N\to\mathbb{T}_M$ induced by a homomorphism of monoids $p:N\to M$ is a linear extension iff p, considered as a functor between categories with one object, is a linear extension – i.e. p is an abelian extension in the category of monoids. The corresponding natural system on M consists of abelian groups D_x, for x in M, and actions $x(-):D_y\to D_{xy}$, $(-)y:D_x\to D_{xy}$. It can be also considered as an "M-graded M-M-bimodule". The corresponding extensions of theories are untwisted iff all the D_x are equal.

2. Any homomorphism of rings $p:S\to R$ gives rise to a morphism $P:\mathbb{T}_S\to\mathbb{T}_R$ from the theory of (left) S-modules to that of R-modules. This morphism is a linear extension iff p is a *singular extension*, i.e. $\mathrm{Ker}(p)=B$ is a square zero ideal in S. In [10], an isomorphism is obtained

$$H^2(\mathbb{T}_R;D_B)\cong H^2(R;B)$$

from the group of (untwisted) linear extensions of $\mathbb{T}_R$ by the bifunctor given by

$$D_B(X^n,X^k)=\mathrm{Hom}_{R\text{-}\mathbf{mod}}(\mathbb{T}_R(k),B\underset{R}{\otimes}\mathbb{T}_R(n))\cong(B^{\oplus n})^k,$$

to the second MacLane cohomology group of R with coefficients in B.

3. It is proved in [10] that for each n there is a linear extension from the theory of $(n+1)$-nilpotent groups to that of n-nilpotent ones; similarly for groups replaced by Lie rings, associative rings without unit, or associative commutative rings without unit.

4. For a left module M over a ring R, let $M/(R\text{-}\mathbf{mod})$ be the coslice category of modules under M, with objects of the form $M \to N$ and obvious commutative triangles as morphisms. Let $P : M/(R\text{-}\mathbf{mod}) \to R\text{-}\mathbf{mod}$ be the functor sending $f : M \to N$ to $\mathrm{Coker}(f)$. It has a right adjoint U_P given by $U_P(N) = 0 : M \to N$. It is then easy to see that this adjoint pair is induced by a morphism of theories $P : \mathbb{T}_{R;M} \to \mathbb{T}_R$, where $\mathbb{T}_{R;M}$ is the opposite of the full subcategory of $M/(R\text{-}\mathbf{mod})$ on objects of the form $(1,0) : M \to M \oplus R^n$ for $n \geqslant 0$. In particular, $M/(R\text{-}\mathbf{mod})$ is equivalent to $\mathbb{T}_{R;M}\text{-}\mathbf{mod}$.

Now this P in fact presents $\mathbb{T}_{R;M}$ as a trivial linear extension of $\mathbb{T}_R$, by the bifunctor H_M given by composition

$$\mathbb{T}_R^{\mathrm{op}} \times \mathbb{T}_R \xrightarrow{\text{projection}} \mathbb{T}_R \xrightarrow{I_{\mathbb{T}_R}^{\mathrm{op}}} (R\text{-}\mathbf{mod})^{\mathrm{op}} \xrightarrow{\mathrm{Hom}_R(-,M)} \mathbf{Ab},$$

that is,

$$H_M(X^n, X^k) = \mathrm{Hom}_R(R^k, M) \cong M^k.$$

Indeed, the trivial extension $P : \mathbb{T}_R \rtimes H_M \to \mathbb{T}_R$ can be easily calculated; one has

$$\hom_{\mathbb{T}_R \rtimes H_M}(X^n, X^k) = \mathrm{Hom}_R(R^k, M \oplus R^n).$$

One can represent the latter group also as

$$\hom_{M/(R\text{-}\mathbf{mod})}(M \xrightarrow{(1,0)} M \oplus R^k, M \xrightarrow{(1,0)} M \oplus R^n),$$

which is precisely $\hom_{\mathbb{T}_{R;M}}(X^n, X^k)$.

The Maltsev case

A *Maltsev theory* is a theory $\mathbb{T}$ for which there is a Maltsev operation on the generating object X. The corresponding variety, i.e. the category of models, will be also called Maltsev in this case. It is a classical result of Maltsev that such varieties are precisely those in which join of congruences coincides with their composition.

Proposition 2.7. *Let a morphism of theories $P : \mathbb{T}' \to \mathbb{T}$ be a linear extension of $\mathbb{T}$ by a natural system D. If $\mathbb{T}$ is a Maltsev theory, then $\mathbb{T}'$ is also Maltsev.*

Proof. We will prove that for any $m : X^3 \to X$ in $\mathbb{T}'$ such that $P(m)$ is Maltsev, there is a Maltsev $m' : X^3 \to X$ with $M(m') = P(m)$.

Let $x_1, x_2, \ldots : X^n \to X$ be the projections. Now $P(m(x_1, x_2, x_2)) = P(x_1)$ and $P(m(x_1, x_1, x_2)) = P(x_2)$, so since P is a $\mathbb{T} \rtimes D$-torsor, the elements $x_1 - m(x_1, x_2, x_2) \in D_{x_1}$ and $x_2 - m(x_1, x_1, x_2) \in D_{x_2}$ are defined. Denoting by $x : X \to X$ the identity, let

$$m' = (x_2 - m(x_1,x_1,x_2))(x_1, P(m)) + (m(x,x,x) - x)P(m) + (x_1 - m(x_1,x_2,x_2))(P(m), x_3) + m.$$

One then has

$$\begin{aligned} &m'(x_1, x_1, x_2) \\ &\quad = (x_2 - m(x_1,x_1,x_2))(x_1,x_2) + (m(x,x,x) - x)x_2 + (x_1 - m(x_1,x_2,x_2))(x_2,x_2) \\ &\quad + m(x_1,x_1,x_2) = (x_2 - m(x_1,x_1,x_2)) + (m(x_2,x_2,x_2) - x_2) + (x_2 - m(x_2,x_2,x_2)) \\ &\quad + m(x_1,x_1,x_2) = x_2 \end{aligned}$$

and

$$\begin{aligned}
&m'(x_1,x_2,x_2)\\
&\quad=(x_2-m(x_1,x_1,x_2))(x_1,x_1)+(m(x,x,x)-x)x_1+(x_1-m(x_1,x_2,x_2))(x_1,x_2)\\
&\quad+m(x_1,x_2,x_2)=(x_1-m(x_1,x_1,x_1))+(m(x_1,x_1,x_1)-x_1)+(x_1-m(x_1,x_2,x_2))\\
&\quad+m(x_1,x_2,x_2)=x_1. \qquad\square
\end{aligned}$$

3. Commutators and nilpotence

Commutator calculus has been extended to general varieties of algebraic systems by several people – first by J. D. H. Smith [19] for Maltsev varieties, then extended to more general cases by Gumm, Hagemann and Herrmann, McKenzie and others (see [5] for precise information).

Taking the point of view of category theory enables one to make more apparent the invariant properties of the commutator calculus, i.e. properties which do not depend on a particular choice of basic operations for the algebras of the variety. A good example of such approach is [16]. Also in [8] a notion of central extension is derived from abstract categorical version of Galois theory, and it is shown in [9] that central extensions in the sense of commutator calculus can be described in this way too.

For our paper, categorical reformulation of the commutator calculus given in [16] is most suitable. Let us recall it briefly.

In a category with kernel pairs and coequalizers one may generate from any pair of morphisms $X \rightrightarrows M$ a congruence on M, as the kernel pair of the coequalizer of this pair. In particular, given two congruences $p',p'':R\rightrightarrows M$ and $q',q'':S\rightrightarrows M$, one denotes by $r',r'':\Delta_{R,S}\to R$ the congruence on R defined to be the kernel pair of the coequalizer of the morphisms (diagonal)q', (diagonal)$q'':S\rightrightarrows R$. Also let $R'\rightrightarrows R$ be the kernel pair of p'. Then in [16], the commutator $[R,S]$ is defined to be the image under p'' of the intersection $\Delta_{R,S}\cap R'$. It is then proved in [16] that this agrees with the definition of commutators from [5] at least for Maltsev varieties, so in particular all the properties of the commutators from [5] hold.

If $[R,S]=\Delta_M$ (the smallest congruence diagonal: $M\rightarrowtail M^2$), then the congruences R and S are said to *centralize* each other. In fact, $[R,S]$ is the smallest among those congruences T on M for which the congruences R/T and S/T (on M/T) centralize each other. A congruence R on M is called *abelian* if it centralizes itself, i.e. $[R,R]=\Delta_M$, and *central* if it is centralized by the largest congruence ∇_M (the identity: $M^2\rightarrowtail M^2$), i.e. $[\nabla_M,R]=\Delta_M$. The *center* $\zeta(M)$ of M is the largest central congruence; it always exists (in fact more generally for any R and S always exists the largest congruence T with $[T,R]\leqslant S$). One then defines, generalizing the usual notions, a central series for a model M to be a chain of congruences $\Delta_M=R_0\leqslant R_1\leqslant\cdots\leqslant R_n=\nabla_M$ such that for all i one has $R_{i+1}/R_i\leqslant\zeta(M/R_i)$ (equivalently, $[\nabla_M,R_{i+1}]\leqslant R_i$), i.e. R_{i+1}/R_i is a central congruence on M/R_i. A model is called *abelian* if $\zeta(M)=\nabla_M$ (equivalently, $[\nabla_M,\nabla_M]=\Delta_M$) and n *stage nilpotent*, or just n-nilpotent, if it has a central series of length n. This happens iff either the *upper central series* $\Delta_M=\zeta^0(M)\leqslant\zeta^1(M)\leqslant\zeta^2(M)\leqslant\cdots$ ends with $\zeta^n(M)=\nabla_M$ or the *lower central series* $\nabla_M=\Gamma^0(M)\geqslant\Gamma^1(M)\geqslant\Gamma^2(M)\geqslant\cdots$ ends with $\Gamma^n(M)=\Delta_M$;

here, $\zeta^{n+1}(M)/\zeta^n(M) = \zeta(M/\zeta^n(M))$ and $\Gamma^{n+1}(M) = [\nabla_M, \Gamma^n(M)]$. Just as in the case of groups, algebras, etc., the Γ^n are functorial (and the ζ^n are not). A theory is called n-nilpotent (abelian for $n = 1$) if all of its models are.

Let us give another equivalent construction of the commutator in Maltsev varieties.

Proposition 3.1. *For any congruences R, S on an object A in a Maltsev variety, there is a pushout square*

$$\begin{tikzcd} R \sqcup_A S \arrow[r, "p_{R,S}"] \arrow[d, "q_{R,S}"] & A \arrow[d] \\ R \sqcap_A S \arrow[r, "m_{R,S}"] & A/[R,S] \end{tikzcd} \ , \tag{†}$$

where $p_{R,S} : R \sqcup_A S \to A$ is induced by the pair $((x,y) \mapsto x) : R \to A$, $((x,y) \mapsto y) : S \to A$ and $q_{R,S} : R \sqcup_A S \to R \sqcap_A S$ is induced by the pair $((x,y) \mapsto ((x,y),(y,y))) : R \to R \sqcap_A S$, $((x,y) \mapsto ((x,x),(x,y))) : S \to R \sqcap_A S$.

Proof. Observe that $R \sqcap_A S$ consists of elements of the form $((x,y),(y,z))$ with $(x,y) \in R$ and $(y,z) \in S$. Let m be any Maltsev operation in our variety, then

$$\begin{aligned} ((x,y),(y,z)) &= (m((x,y),(y,y),(y,y)), m((y,y),(y,y),(y,z))) \\ &= m(((x,y),(y,y)),((y,y),(y,y)),((y,y),(y,z))) \\ &= m(q_{R,S} i_R(x,y), q_{R,S} i(y,y), q_{R,S} i_S(y,z)) \\ &= q_{R,S} m(i_R(x,y), i(y,y), i_S(y,z)), \end{aligned}$$

where i_R, i_S are the canonical coproduct inclusions and i stands for any of them.

This shows first of all that $q_{R,S}$ is surjective, so if one forms a pushout square as above, one gets for the right vertical map the quotient $A \twoheadrightarrow A/T$ for some congruence T on A. Moreover it follows that the induced homomorphism $R \sqcap_A S \to A/T$ in this pushout maps any element $((x,y),(y,z)) = q_{R,S} m(i_R(x,y), i(y,y), i_S(y,z))$ to the T-equivalence class of the element $p_{R,S} m(i_R(x,y), i(y,y), i_S(y,z))$, which equals $m(p_{R,S} i_R(x,y), p_{R,S} i(y,y), p_{R,S} i_S(y,z))$ $= m(x,y,z)$.

It follows that one has a commutative diagram

$$\begin{tikzcd} & & R \sqcup_A S \arrow[rr, "p_{R,S}"] \arrow[d, "q_{R,S}"] \arrow[ddll] & & A \arrow[d] \arrow[ddl] \\ & & R \sqcap_A S \arrow[urr, dashed, "m"] \arrow[rr] \arrow[ddll] & & A/T \arrow[dl, equal] \\ R/T \sqcup_{A/T} S/T \arrow[rrr, "p_{R/T,S/T}"] \arrow[d, "q_{R/T,S/T}"'] & & & A/T & \\ R/T \sqcap_{A/T} S/T \arrow[urrr, dashed, "m"] & & & & \end{tikzcd} \ , \tag{‡}$$

where dashed lines denote maps which are not necessarily homomorphisms. Indeed, we just showed that the triangle

$$\begin{tikzcd} & A \arrow[d] \\ R \sqcap_A S \arrow[ur, dashed, "m"] \arrow[r] & A/T \end{tikzcd}$$

commutes; whereas the parallelogram

$$\begin{tikzcd} & & A \arrow[d] \\ & R \sqcap_A S \arrow[ur, dashed, "m"] \arrow[dl] & A/T \\ R/T \sqcap_{A/T} S/T \arrow[urr, dashed, "m"] & & \end{tikzcd}$$

commutes simply because $A \twoheadrightarrow A/T$ is a homomorphism.

Now (‡) shows that composing the map $m : R/T \sqcap_{A/T} S/T \to A/T$ with a surjective homomorphism $R \sqcup_A S \twoheadrightarrow R/T \sqcap_{A/T} S/T$ produces a homomorphism $R \sqcup_A S \to A/T$. It then follows that $m : R/T \sqcap_{A/T} S/T \to A/T$ is a homomorphism too. Thus by 3.3 R/T and S/T centralize each other, i.e. $[R, S] \subseteq T$.

Conversely, since $R/[R, S]$ and $S/[R, S]$ centralize each other, there is a homomorphism $m : (R/[R, S]) \sqcap_{A/[R,S]} (S/[R, S]) \to A/[R, S]$ as in 3.3. Composing it with the product of quotient maps $R \sqcap_A S \to (R/[R, S]) \sqcap_{A/[R,S]} (S/[R, S])$ gives a homomorphism $m_{R,S}$, and to say that it fits in a commutative square as (†) above is precisely the same as to say that m satisfies the Maltsev identities. This shows that there is a homomorphism $A/T \to A/[R, S]$ under A, i.e. that $T \subseteq [R, S]$. $\square$

Remark 3.2. Note that this proposition hints at another possibility of the construction of commutators in varieties more general than Maltsev: given congruences R and S on an algebra A, a new congruence $[R, S]$ is uniquely determined by requiring existence of a pushout square

$$\begin{tikzcd} R \sqcup_A S \arrow[r, "p_{R,S}"] \arrow[d, "q_{R,S}"] & A \arrow[d] \\ R \square_A S \arrow[r] & A/[R, S] \; . \end{tikzcd} \tag{†}$$

Here we used notation from 3.1 above and denoted by $R\Box_A S$ the image of $q_{R,S} : R \sqcup_A S \to R \sqcap_A S$.

It is not difficult to determine syntactic content of the above definition of the commutator: the above $[R, S]$ is the smallest congruence with the property that for any two operations $u(x_1, \dots, x_m, y_1, \dots, y_n)$, $v(x_1, \dots, x_p, y_1, \dots, y_q)$ in our variety and any $a_1, \dots, a_m$, $b_1, \dots, b_n$, $c_1, \dots, c_p$ and $d_1, \dots, d_q$ in A one has $u(a_1, \dots, a_m, b_1, \dots, b_n) \sim_{[R,S]} v(c_1, \dots, c_p, d_1, \dots, d_q)$ whenever there exist $a'_i \sim_R a_i$, $i = 1, \dots, m$, $b'_i \sim_S b_i$, $i = 1, \dots, n$, $c'_i \sim_R c_i$, $i = 1, \dots, p$ and $d'_i \sim_S d_i$, $i = 1, \dots, q$ satisfying the equalities $u(a'_1, \dots, a'_m, b_1, \dots, b_n) = v(c'_1, \dots, c'_p, d_1, \dots, d_q)$, $u(a_1, \dots, a_m, b'_1, \dots, b'_n) = v(c_1, \dots, c_p, d'_1, \dots, d'_q)$, and $u(a'_1, \dots, a'_m, b'_1, \dots, b'_n) = v(c'_1, \dots, c'_p, d'_1, \dots, d'_q)$.

This kind of condition has been considered by universal algebraists as a "better" behaved than the usual one for non-Maltsev varieties. In fact in [18] a whole infinite sequence of conditions is considered, each stronger than previous, and ours is the second in the row.

Fortunately we are confined to the realm of Maltsev varieties. In fact all we have to know about commutators is the following fact, which is crucial for what follows:

Proposition 3.3. *Congruences R and S on a model M of a Maltsev theory centralize each other if and only if there is a homomorphism m from the submodel*

$$R \sqcap_M S \cong \{(x, y, z) \in M^3 \,|\, (x, y) \in R, (y, z) \in S\}$$

of M^3 to M which satisfies $xSm(x, y, z)Rz$ for any $xRySz$ and is Maltsev, i.e. $m(x, y, y) = x$ for any xRy, and $m(y, y, z) = z$ for any ySz. Then restriction of any Maltsev operation p of the theory to that submodel coincides with this homomorphism, is associative and commutative. More precisely, $p(u, v, p(x, y, z)) = p(p(u, v, x), y, z)$ holds if uRv, ySz, and either vSx or xRy, while $p(x, y, z) = p(z, y, x)$ holds for any $xRySz$.

Proof. The first statement can be found in [16], see Lemma 2.11 there; we only prove the second (the proof is essentially the same as in [11]).

Take any Maltsev operation p. Then homomorphicity of m means

$$m(p(x_1, x_2, x_3), p(y_1, y_2, y_3), p(z_1, z_2, z_3)) = p(m(x_1, y_1, z_1), m(x_2, y_2, z_2), m(x_3, y_3, z_3)),$$

for any $x_iRy_iSz_i$ $(i = 1, 2, 3)$. Taking here $y_1 = z_1 = x_2 = y_2 = z_2 = x_3 = y_3$ then gives $m(x_1, y_2, z_3) = p(x_1, y_2, z_3)$ for any $x_1Ry_2Sz_3$. Now for any uRv, $xRySz$ one has

$$\begin{aligned} p(u, v, p(x, y, z)) &= p(m(u, v, v), m(v, v, v), m(x, y, z)) \\ &= m(p(u, v, x), p(v, v, y), p(v, v, z)) = m(p(u, v, x), y, z) = p(p(u, v, x), y, z), \end{aligned}$$

and similarly for $uRvSx$, ySz. Whereas taking any $xRySz$,

$$\begin{aligned} p(z, y, x) &= p(m(y, y, z), m(y, y, y), m(x, y, y)) \\ &= m(p(y, y, x), p(y, y, y), p(z, y, y)) = m(x, y, z) = p(x, y, z). \end{aligned}$$

□

Corollary 3.4. *A congruence R on a model M of a Maltsev theory $\mathbb{T}$ is abelian if and only if the morphism $M \to M/R$ is an abelian extension in the sense of* 1.4. *Furthermore R is central if and only if $M \to M/R$ is (the trivial case $M = \varnothing$ excluded).*

Proof. The first statement is immediate by 1.3 and 3.3, and the second by 1.5. □

In fact, this is a special case of a statement which deals with arbitrary R, S centralizing each other. This requires generalizing herd structures to s. c. *herdoids*; see [16].

Corollary 3.5. *A congruence R on a non-empty model M of a Maltsev theory $\mathbb{T}$ is central if and only if there is an internal abelian group A in $\mathbb{T}$-**mod**, an action of A on M, and an isomorphism $\varphi : R \to A \times M$ fitting in the commutative triangle*

$$\begin{tikzcd} R \arrow[dr] \arrow[dd, "\varphi"', "\cong"] & \\ & M \times M \\ A \times M. \arrow[ur, "\text{(projection, action)}"'] & \end{tikzcd}$$

Proof. This follows easily from 3.4, in view of 1.5. □

Since linear, respectively untwisted, extensions of a theory $\mathbb{T}$ by cartesian natural systems (resp. bifunctors) are, by 1.6, none other than abelian, resp. central, objects of **Theories**/$\mathbb{T}$, one might expect that they are related to abelian, resp. central, extensions in $\mathbb{T}$-**mod**. For Maltsev theories, a link between these notions is provided by

Theorem 3.6. *For a morphism $P : \mathbb{T}' \to \mathbb{T}$ of Maltsev theories, the following conditions are equivalent:*

i) *P is a linear (respectively, untwisted) extension;*

ii) *for all $\mathbb{T}'$-models M the homomorphisms $\eta_M : M \to M_P$ are abelian (respectively, central) extensions in $\mathbb{T}'$-**mod** and, moreover, the following condition is satisfied:*

   ♢ *for any Maltsev operation p in $\mathbb{T}'$, any $u, v : \mathbb{T}'(n) \to \mathbb{T}'(k)$ in $\mathbb{T}'$-**mod** with $P(u) = P(v)$, and any $x, y \in \mathbb{T}'(n)$ with $\eta_{\mathbb{T}'(n)}(x) = \eta_{\mathbb{T}'(n)}(y)$, one has*

$$p(u(x), v(x), v(y)) = u(y).$$

Proof. i) ⇒ ii): By 1.6, the morphism P is a linear extension if and only if it is full and there is a functor $m : \mathbb{T}' \times_{\mathbb{T}} \mathbb{T}' \times_{\mathbb{T}} \mathbb{T}' \to \mathbb{T}'$ over $\mathbb{T}'$ which is a commutative associative Maltsev operation in **Theories**/$\mathbb{T}$. Identifying $\mathbb{T}'$ with the opposite of the category of finitely generated free models via the Yoneda embedding, the action of m on $\hom(X^n, X)$ may be viewed as a commutative associative Maltsev operation $m_n : \mathbb{T}'(n) \times_{\mathbb{T}(n)} \mathbb{T}'(n) \times_{\mathbb{T}(n)} \mathbb{T}'(n) \to$

$\mathbb{T}'(n)$ over $\mathbb{T}(n)$. Now functoriality of m means under the above identification that for any $u, v, w \in \hom_{\mathbb{T}\textbf{-mod}}(\mathbb{T}'(n), \mathbb{T}'(k)) \approx \mathbb{T}'(k)^n$ with $P(u) = P(v) = P(w)$, and any $x, y, z \in \hom_{\mathbb{T}\textbf{-mod}}(\mathbb{T}'(i), \mathbb{T}'(n)) \approx \mathbb{T}'(n)^i$ with $P(x) = P(y) = P(z)$ one has

$$m_k^i(ux, vy, wz) = m_k^n(u, v, w)m_n^i(x, y, z). \tag{$\circ$}$$

This then shows that each m_n is a homomorphism: taking $i = 1$ and $x = y = z$ in ($\circ$) gives

$$m_k(ux, vx, wx) = m_k^n(u, v, w)m_n(x, x, x) = m_k^n(u, v, w)x$$

which, in terms of $\mathbb{T}'$ again, means

$$m_k(x(u_1, \dots, u_k), x(v_1, \dots, v_k), x(w_1, \dots, w_k)) = x(m_k(u_1, v_1, w_1), \dots, m_k(u_n, v_n, w_n)).$$

So $\eta_{\mathbb{T}'(n)}$ is a linear extension, and m_n coincides with the restriction of any Maltsev operation on $\mathbb{T}'(n)$. If moreover P is untwisted, then, as in 1.5, this m_n is defined on $\mathbb{T}'(n) \times_{\mathbb{T}(n)} \mathbb{T}'(n) \times \mathbb{T}'(n)$, and $\eta_{\mathbb{T}'(n)}$ is central.

Now taking $i = 1$, $v = w$ and $x = y$ in ($\circ$) gives

$$m_k(u(x), v(x), v(z)) = m_k^n(u, v, v)m_n(x, x, z) = u(z),$$

which, since the m's coincide with the restrictions of Maltsev operations, gives $\diamondsuit$.

Now for a general free model $\mathbb{T}'(S)$, the homomorphism $\eta_{\mathbb{T}'(S)}$ is a colimit of a filtered diagram of those of the form $\eta_{\mathbb{T}'(n)}$, just as in the proof of 2.1. Since filtered colimits commute with finite limits and are created by the forgetful functors, it follows that the collection of the m_n on the $\mathbb{T}'(n)$ over $\mathbb{T}(n)$ give rise to one m_S on $\mathbb{T}'(S)$ over $\mathbb{T}(S)$, so $\eta_{\mathbb{T}'(S)}$ is abelian, resp. central, whenever all the $\eta_{\mathbb{T}'(n)}$ are.

Finally, consider any $\mathbb{T}'$-model M. Let us choose a surjective homomorphism $q : \mathbb{T}'(S) \twoheadrightarrow M$, so that $M = \mathbb{T}'(S)/R_M$ for some congruence R_M on $\mathbb{T}'(S)$. By 2.1, both $\eta_{\mathbb{T}'(S)}$ and η_M are surjective, so also $\mathbb{T}(S) = \mathbb{T}'(S)/R_{\mathbb{T}}$, $M_P = \mathbb{T}'(S)/R$, for certain congruences $R_{\mathbb{T}}$, R, with $R_M \subseteq R$. In fact, the pushout condition from 2.1 shows that $R = R_M \vee R_{\mathbb{T}}$ in the lattice of congruences on $\mathbb{T}'(S)$. And since $\mathbb{T}'$ is Maltsev, in fact $R = R_M \circ R_{\mathbb{T}}$. Thus for any $u_1, u_2, \dots \in \mathbb{T}'(S)$, one has $u_1 R u_2 R \dots$ iff $u_1 R_M v_1 R_{\mathbb{T}} u_2 R_M v_2 R_{\mathbb{T}} \dots$, for some $v_1, v_2, \dots$. We then conclude that for any $x_1, x_2, \dots \in M$, one has $\eta_M(x_1) = \eta_M(x_2) = \cdots$ iff there are $u_i \in \mathbb{T}'(S)$ with $x_i = q(u_i)$ $(i = 1, 2, \dots)$ and $\eta_{\mathbb{T}'(S)}(u_1) = \eta_{\mathbb{T}'(S)}(u_2) = \cdots$.

We then define the Maltsev operation $m_M : M \times_{M_P} M \times_{M_P} M \to M$ over M_P (respectively $M \times_{M_P} M \times M \to M$ for untwisted P) by $m_M(x_1, x_2, x_3) = qm_S(u_1, u_2, u_3)$, for some (u_1, u_2, u_3) in $\mathbb{T}'(S) \times_{\mathbb{T}(S)} \mathbb{T}'(S) \times_{\mathbb{T}(S)} \mathbb{T}'(S)$ (respectively, in $\mathbb{T}'(S) \times_{\mathbb{T}(S)} \mathbb{T}'(S) \times \mathbb{T}'(S)$) with $q(u_i) = x_i$ – which exist by the preceding argument. This is legitimate since for any other choice v_i one would have $v_i R_M u_i$, hence $m_S(v_1, v_2, v_3) R_M m_S(u_1, u_2, u_3)$. This since R_M is a submodel of $\mathbb{T}'(S)^2$, while m_S, by 3.3, coincides with the restriction of any Maltsev operation. Homomorphicity, Maltsev identities, associativity and commutativity of m_M now follow from those of m_S.

ii) $\Rightarrow$ i): By 1.3, 1.6, and 1.8, to prove that P is a linear (resp., untwisted) extension, it suffices to construct a family of commutative associative Maltsev operations, denoted $(u, v, w) \mapsto u - v + w$, from

$$\hom_{\mathbb{T}'}(X^n, X^k) \underset{\hom_{\mathbb{T}}(X^n, X^k)}{\times} \hom_{\mathbb{T}'}(X^n, X^k) \underset{\hom_{\mathbb{T}}(X^n, X^k)}{\times} \hom_{\mathbb{T}'}(X^n, X^k)$$

– respectively, from

$$\hom_{\mathbb{T}'}(X^n,X^k)\underset{\hom_{\mathbb{T}}(X^n,X^k)}{\times}\hom_{\mathbb{T}'}(X^n,X^k)\times\hom_{\mathbb{T}'}(X^n,X^k)$$

– to $\hom_{\mathbb{T}'}(X^n,X^k)$, which define a functor $P\times P\times P\to P$ over $\mathbb{T}$ in **Theories**/$\mathbb{T}$, (resp., and also satisfy the conditions as in 1.8). For that, choose some Maltsev operation $p\in\hom_{\mathbb{T}'}(X^3,X)$, and put

$$\bar u-\bar v+\bar w=(p(u_1,v_1,w_1),\dots,p(u_k,v_k,w_k)),$$

for any $\bar u=(u_1,\dots,u_k),\bar v=(v_1,\dots,v_k),\bar w=(w_1,\dots,w_k)\in\hom_{\mathbb{T}'}(X^n,X^k)$. Functoriality then amounts to

$$p(u\bar u,v\bar v,w\bar w)=p(up(\bar u,\bar v,\bar w),vp(\bar u,\bar v,\bar w),wp(\bar u,\bar v,\bar w)),$$

for $u,v,w\in\hom_{\mathbb{T}'}(X^k,X)$ and $\bar u,\bar v,\bar w$ as above, whenever $P(\bar u)=P(\bar v)=P(\bar w)$ and $P(u)=P(v)=P(w)$. Whereas the conditions from 1.8 become

$$u(p(u_1,v_1,w_1),\dots,p(u_k,v_k,w_k))=p(u\bar u,u\bar v,u\bar w)\text{ and }p(u,v,w)\bar u=p(u\bar u,v\bar u,w\bar u),$$

if $P(\bar u)=P(\bar v)$ and $P(u)=P(v)$.

Let us use the Yoneda embedding to identify $\hom_{\mathbb{T}'}(X^i,X^j)$ with $\mathbb{T}'(i)^j$. Then, what we have to prove is this: for any $u,v,w\in\mathbb{T}'(k)$ and any homomorphisms $\bar u,\bar v,\bar w:\mathbb{T}'(k)\to\mathbb{T}'(n)$, the equality

$$p(\bar u(u),\bar v(v),\bar w(w))=p(\bar u(p(u,v,w)),\bar v(p(u,v,w)),\bar w(p(u,v,w)))$$

holds whenever $P(\bar u)=P(\bar v)=P(\bar w)$ and $\eta_{\mathbb{T}'(n)}(u)=\eta_{\mathbb{T}'(n)}(v)=\eta_{\mathbb{T}'(n)}(w)$ (resp., also

$$\begin{aligned}p(\bar u,\bar v,\bar w)(u)&=p(\bar u(u),\bar v(u),\bar w(u)),\\ \bar u(p(u,v,w))&=p(\bar u(u),\bar u(v),\bar u(w)),\end{aligned}$$

when $P(\bar u)=P(\bar v)$ and $\eta_{\mathbb{T}'(n)}(u)=\eta_{\mathbb{T}'(n)}(v)$).

The functoriality condition, using that $\bar u$, $\bar v$, $\bar w$ are homomorphisms and hence commute with the operation p, is equivalent to

$$p(\bar u(u),\bar v(v),\bar w(w))=p(p(\bar u(u),\bar u(v),\bar u(w)),p(\bar v(u),\bar v(v),\bar v(w)),p(\bar w(u),\bar w(v),\bar w(w))).$$

Now recall that for the given elements p is a commutative associative Maltsev homomorphism, so, switching to additive notation and using 1.2, the equality in question becomes

$$\bar u(u)-\bar v(v)+\bar w(w)=\bar u(u)-\bar u(v)+\bar u(w)-\bar v(w)+\bar v(v)-\bar v(u)+\bar w(u)-\bar w(v)+\bar w(w).$$

Let us replace $\bar u(u)-\bar v(v)+\bar w(w)$ by $\bar u(u)-\bar v(v)+\bar v(v)-\bar v(v)+\bar w(w)$ and then substitute, using $\diamondsuit$,

$$-\bar v(v)=-(\bar v(w)-\bar u(w)+\bar u(v))$$

in the first occurrence and

$$-\bar{v}(v) = -(\bar{w}(v) - \bar{w}(u) + \bar{v}(u))$$

in the second. We then obtain

$$\begin{aligned} &\bar{u}(u)-(\bar{v}(w) - \bar{u}(w) + \bar{u}(v)) + \bar{v}(v)-(\bar{w}(v) - \bar{w}(u) + \bar{v}(u)) + \bar{w}(w) \\ &= \bar{u}(u) - \bar{u}(v) + \bar{u}(w) - \bar{v}(w) + \bar{v}(v) - \bar{v}(u) + \bar{w}(u) - \bar{w}(v) + \bar{w}(w) \\ &= \bar{u}(u) - \bar{u}(v) + \bar{u}(w)-(\bar{v}(u) - \bar{v}(v) + \bar{v}(w)) + \bar{w}(u) - \bar{w}(v) + \bar{w}(w), \end{aligned}$$

as required.

As for the last two equalities, the first is trivial, and the second follows since $\bar{u}$ is a homomorphism. □

Outside the realm of Maltsev theories there exist linear extensions $P : \mathbb{T}' \to \mathbb{T}$ such that not all the η's are abelian.

Example. Let M be the multiplicative monoid $\{1, 0\}$, i.e. $1 \cdot 1 = 1$, $1 \cdot 0 = 0 \cdot 1 = 0 \cdot 0 = 0$. Consider the natural system D on it, in the sense explained in 2.6.1, given by $D_1 = 0$, $D_0 = \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$, with $0(x, y) = (y, y)$ and $(x, y)0 = (0, 0)$ for $(x, y) \in D_0$. Then the trivial linear extension of M by D is the monoid $M \rtimes D = \{(1, 0)\} \cup \{(0, (0, 0)), (0, (0, 1)), (0, (1, 0)), (0, (1, 1))\}$, with $(1, 0) \cdot (0, (x, y)) = (0, (x, y)) \cdot (1, 0) = (0, (x, y))$ and $(0, (x, y)) \cdot (0, (x', y')) = (0, (x, y)0 + 0(x', y')) = (0, (y', y'))$. For brevity, let us redenote this as $M \rtimes D = \{1, 00, 10, 01, 11\}$, so that 1 is the unit and

$$\begin{aligned} &00 \cdot 00 = 10 \cdot 00 = 01 \cdot 00 = 11 \cdot 00 = 00 \cdot 10 = 10 \cdot 10 = 01 \cdot 10 = 11 \cdot 10 = 00, \\ &00 \cdot 01 = 10 \cdot 01 = 01 \cdot 01 = 11 \cdot 01 = 00 \cdot 11 = 10 \cdot 11 = 01 \cdot 11 = 11 \cdot 11 = 11. \end{aligned}$$

As in 2.6.1, D extends uniquely to a natural system on $\mathbb{T}_M$ in such a way that the morphism $P : \mathbb{T}_{M \rtimes D} \to \mathbb{T}_M$, induced by the projection $M \rtimes D \to M$, is the trivial linear extension by that system. Now let $S = \{1, *0, 01, 11\} = M \rtimes D/(00 \sim 10)$ be the $M \rtimes D$-set obtained by identifying 00 and 10 in $M \rtimes D$, acting on itself from the left via $\cdot$; then clearly $S_P = (M \rtimes D)_P = M$, with $\eta_S(1) = 1$ and $\eta_S(*0) = \eta_S(01) = \eta_S(11) = 0$. Suppose η_S is abelian. Then there exists a Maltsev operation $m : S \times_M S \times_M S \to S$ over M. Since it must be a morphism of $M \rtimes D$-sets, one must have in particular $10 \cdot m(*0, 01, 11) = m(10 \cdot *0, 10 \cdot 01, 10 \cdot 11) = m(*0, 11, 11) = *0$. This is only possible if $m(*0, 01, 11) = *0$. Then m cannot be associative, since this would imply $11 = m(11, *0, *0) = m(11, m(*0, 01, 11), *0) = m(11, 11, m(01, *0, *0)) = 01$.

On the other hand, if all the η's are abelian, the morphism of (non-Maltsev) theories can still fail to be a linear extension – consider the morphism $\mathbb{S}. \to \mathbb{1}$ from the theory of pointed sets to the terminal theory. Clearly for any non-empty set S the map $S \to 1$ is abelian, as S can be equipped with an abelian group structure. But $\mathbb{S}.$ itself clearly cannot have any functorial Maltsev operation $m : \mathbb{S}. \times_{\mathbf{1}} \mathbb{S}. \times_{\mathbf{1}} \mathbb{S}. \to \mathbb{S}.$ since $m(f, g, h)m(f', g', h') = m(ff', gg', hh')$ implies that $ee' = m(e, 1, 1)m(1, 1, e') = m(e, 1, e') = m(1, 1, e')m(e, 1, 1) = e'e$ for any endomorphisms of any object, while $\mathbb{S}.$ has noncommutative endomorphism monoids.

Corollary 3.7. *Let $P : \mathbb{T}' \to \mathbb{T}$ be a morphism of Maltsev theories such that for each M in $\mathbb{T}'$-**mod** the morphism $\eta_M : M \to M_P$ is a central extension and moreover the subvariety $U_P(\mathbb{T}\text{-}\mathbf{mod})$ of $\mathbb{T}'$-**mod** contains all abelian $\mathbb{T}'$-models. Then P is an untwisted linear extension.*

Proof. In view of the previous theorem we just have to check the condition $\diamondsuit$. It is true in more generality, in fact: let $u, v : M' \to M$ be any homomorphisms in $\mathbb{T}'$-**mod** with $P(u) = P(v)$, and let $x, y \in M'$ be any elements with $\eta_{M'}(x) = \eta_{M'}(y)$. We thus have $(u, v) : M' \to R_M = M \times_{M_P} M$. Then by 3.5 we know that there is an action $A(M) \times M \to M$ of an internal abelian group $A(M)$ on M and an isomorphism $R_M \cong A(M) \times M$. Thus (u, v) give rise to a homomorphism $u - v : M' \to A(M)$. Now obviously any internal abelian group in $\mathbb{T}'$-**mod** is an abelian model of $\mathbb{T}'$, so by hypothesis $A(M)$ belongs to the image of U_P; hence $u - v$ factors through $\eta_{M'}$ and we obtain $(u - v)(x) = (u - v)(y)$. Using now the action of $A(M)$, 3.5 gives $u(x) - v(x) + z = u(y) - v(y) + z$ for any z in M, in particular taking $z = v(y)$ gives $\diamondsuit$. □

Remark. It is easy to identify the corresponding bifunctor on $\mathbb{T}$: one can show that it is given by
$$D(X^n, X^k) = \hom_{\mathbb{T}\text{-}\mathbf{mod}}(\mathbb{T}(k), A(\mathbb{T}'(n))) \cong A(\mathbb{T}'(n))^k.$$

Corollary 3.8. *A theory $\mathbb{T}$ is n-nilpotent Maltsev if and only if there is a tower of untwisted linear extensions of theories*
$$\mathbb{T} = \mathbb{T}_n \to \mathbb{T}_{n-1} \to \cdots \to \mathbb{T}_1$$
where $\mathbb{T}_1$ is an abelian Maltsev theory.

Proof. Fix a theory $\mathbb{T}$ and some n, and let $\mathbb{T}_{(n)}$ be the theory with
$$\hom_{\mathbb{T}_{(n)}}(X^k, X^l) = \hom_{\mathbb{T}\text{-}\mathbf{mod}}(\mathbb{T}(l)/\Gamma^n(\mathbb{T}(l)), \mathbb{T}(k)/\Gamma^n(\mathbb{T}(k))).$$
It is easy to see that there is a bijection
$$\hom_{\mathbb{T}\text{-}\mathbf{mod}}(M, N/\Gamma^n(N)) \approx \hom_{\mathbb{T}\text{-}\mathbf{mod}}(M/\Gamma^n(M), N/\Gamma^n(N))$$
for any models M, N, hence it follows that $\mathbb{T}_{(n)}$ is an n stage nilpotent theory and that sending a model M to $M/\Gamma^n(M)$ restricts to a morphism of theories $P : \mathbb{T} \to \mathbb{T}_{(n)}$. Moreover all models in the image of U_P are n-nilpotent, and for any M the morphism $\eta_M : M \to M_P$ is the quotient $q_n(M) : M \to M/\Gamma^n(M)$.

Now suppose $\mathbb{T}$ is itself $(n+1)$-nilpotent and Maltsev; then, the above construction gives a morphism of theories $\mathbb{T} \to \mathbb{T}_n$ whose structure maps $\hom_{\mathbb{T}}(X^k, X^l) \to \hom_{\mathbb{T}_n}(X^k, X^l)$ coincide with $q_n(\mathbb{T}(k))^l : \mathbb{T}(k)^l \twoheadrightarrow (\mathbb{T}(k)/\Gamma^n(\mathbb{T}(k)))^l$, with q_n as above. The fact that $\mathbb{T}(k)$ is $(n+1)$-nilpotent means according to 3.4 that q_n is a central extension in the sense of 1.4. Thus, 3.7 shows that $\mathbb{T} \to \mathbb{T}_n$ is an untwisted linear extension. By induction, one thus gets the “only if” part.

For the “if” part, suppose we are given an untwisted linear extension $P : \mathbb{T} \to \mathbb{T}_n$ of theories, and $\mathbb{T}_n$ is n-nilpotent and Maltsev. Then by 2.7 $\mathbb{T}$ is also Maltsev. Furthermore by 3.6, for any $\mathbb{T}$-model M the homomorphism $\eta_M : M \to M_P$ is a central extension, hence a quotient by a central congruence, by 3.4. Since $P(M)$ is n-nilpotent, it follows that M is $(n+1)$-nilpotent. $\square$

Remark. Let us also note in this connection that any nilpotent (in fact, also *solvable*) variety with modular congruence lattices is Maltsev – see 10.1 in [6].

4. Abelian theories

We finish with some information on the structure of abelian Maltsev theories and linear extensions between them. It follows from 3.3 that a model A of a Maltsev theory is abelian iff there is a homomorphism $m : A^3 \to A$ satisfying the Maltsev identities, and then any Maltsev operation on A coincides with m, is commutative and associative. It will be convenient to fix such an operation throughout and denote it by $m(x, y, z) = x +_y z$. Thus all models of an abelian theory are *abelian herds*, and all of their operations are homomorphisms of abelian herds. We will first deal with abelian theories without constants, i.e. nullary operations. To describe such theories, we need some definitions. Several similar considerations of these "affine" theories and their relationship with torsors can be found in the literature. To mention just few, see [15], [20], [12], [4]; also [1], kindly suggested by the referee. Abelian theories with constants are much more common and will be treated in the end.

Definition 4.1. *A left linear form consists of an associative ring R with unit, a left R-module M, and a homomorphism $\partial : M \to R$ of left R-modules.*

In fact usually we will omit the word "left", as it is customary with modules.

Definition 4.2. *An affinity over a linear form $\partial : M \to R$ is an abelian herd A together with maps $R \times A \times A \to A$ and $M \times A \to A$, denoted, respectively, $(r, a, b) \mapsto r_a b$ and $(x, a) \mapsto \varphi_a(x)$, such that the following identities hold:*

- *For each $a \in A$, the operations $(-)+_a(-)$ and $(-)_a(-)$ turn A into a left R-module (with zero a), and φ_a into a module homomorphism. In other words, for any $a, b, c, d \in A$, $r, s \in R$, $x, y \in M$ one has*

$$\begin{aligned}
&b +_a (c +_a d) = (b +_a c) +_a d,\\
&a +_a b = b,\\
&b +_a c = c +_a b,\\
&b -_a b = a,\\
&r_a(b +_a c) = r_a b +_a r_a c,\\
&(r + s)_a b = r_a b +_a s_a b,\\
&1_a b = b,\\
&r_a(s_a b) = (rs)_a b,\\
&\varphi_a(x + y) = \varphi_a(x) +_a \varphi_a(y),\\
&\varphi_a(rx) = r_a \varphi_a(x),
\end{aligned}$$

where we have denoted $b -_a c = b +_a ((-1)_a c)$.

- *("coordinate change") These structures are related by the identities*

$$
\begin{aligned}
&b +_{a'} c = ((b -_a a') +_a (c -_a a')) +_a a', \\
&r_{a'} b = r_a(b -_a a') +_a a', \\
&\varphi_{a'}(x) = \varphi_a(x) +_a (1 - \partial x)_a a'.
\end{aligned} \tag{$*$}
$$

A homomorphism between affinities A, A' is a map $f : A \to A'$ preserving all this, i.e. satisfying

$$
\begin{aligned}
&f(a +_b c) = f(a) +_{f(b)} f(c), \\
&f(r_a b) = r_{f(a)} f(b), \\
&f(\varphi_a(x)) = \varphi_{f(a)}(x).
\end{aligned}
$$

Obviously the category ∂-**aff** of affinities over a linear form ∂ is the category of models of a suitable abelian theory $\mathbb{T}_\partial$. Here is an explicit description of this theory.

Proposition 4.3. *The theory $\mathbb{T}_\partial$ of affinities over $\partial : M \to R$ can be described as follows:*

$$
\hom_{\mathbb{T}_\partial}(X^n, X) = \begin{cases} \varnothing, & n = 0; \\ M \times R^{n-1}, & n > 0. \end{cases}
$$

The projections $a_0, a_1, a_2, \ldots : X^n \to X$ are given, respectively, by the elements $\langle 0, 0, 0, \ldots \rangle$, $\langle 0, 1, 0, \ldots \rangle$, $\langle 0, 0, 1, \ldots \rangle$, $\ldots$; and, composition is given by

$$
\langle x, r_1, r_2, \ldots \rangle (\langle x_0, s_0, t_0, \ldots \rangle, \langle x_1, s_1, t_1, \ldots \rangle, \langle x_2, s_2, t_2, \ldots \rangle, \ldots) = \langle x', s', t', \ldots \rangle,
$$

where

$$
\begin{aligned}
x' &= x + (1 - \partial(x))x_0 + r_1(x_1 - x_0) + r_2(x_2 - x_0) + \cdots, \\
s' &= (1 - \partial(x))s_0 + r_1(s_1 - s_0) + r_2(s_2 - s_0) + \cdots, \\
t' &= (1 - \partial(x))t_0 + r_1(t_1 - t_0) + r_2(t_2 - t_0) + \cdots, \\
&\cdots \qquad\qquad .
\end{aligned}
$$

Proof. Take as basic operations the ternary $(-) +_{(-)} (-)$, the family of binaries $r_{(-)}(-)$ indexed by $r \in R$, and unaries $\varphi_{(-)}(x)$ indexed by $x \in M$. Using the above identities, one can write any composite of these operations in the form

$$
\langle x, r, s, \ldots \rangle (a, b, c, \ldots) = \varphi_a(x) +_a r_a b +_a s_a c +_a \cdots
$$

in a unique way. The rest is straightforward verification. $\square$

Define now a morphism of left linear forms from $\partial : M \to R$ to $\partial' : M' \to R'$ to be an equivariant homomorphism, i.e. a pair $(f : R \to R', g : M \to M')$ of additive maps such that the obvious square commutes, that f is a unital ring homomorphism, and that $g(rx) = f(r)g(x)$ holds for any $r \in R$, $x \in M$. This clearly defines the category **Lf** of left linear forms. We then have

Theorem 4.4. *The category of abelian Maltsev theories without nullary operations is equivalent to the category* **Lf** *of left linear forms.*

Proof. Define the functor $\mathbb{T}_{(-)}$: **Lf** $\to$ **Theories** by sending an object ∂ of **Lf** to the corresponding theory $\mathbb{T}_\partial$ described above in 4.3. It is clear from that description that any morphism in **Lf** determines a morphism of the corresponding theories.

Conversely, given an abelian Maltsev theory $\mathbb{T}$, define the left linear form $\partial_\mathbb{T} : M_\mathbb{T} \to R_\mathbb{T}$ as follows: let $M_\mathbb{T}$ be the set of all unary operations of $\mathbb{T}$, with the abelian group structure given by $x + y = m(x, \mathrm{id}, y)$, where m is the Maltsev operation of $\mathbb{T}$. Let $R_\mathbb{T}$ be the set of convex binary operations of $\mathbb{T}$, i.e. those binary operations r satisfying the identity $r(a, a) = a$. Define the ring structure on it by taking zero 0 to be the first projection, unit 1 the second projection, addition to be $r + s = m(r, 0, s)$, additive inverse $-r = m(0, r, 0)$, and multiplication $(rs)(a, b) = r(a, s(a, b))$. Let $R_\mathbb{T}$ act on $M_\mathbb{T}$ via $(rx)(a) = r(a, x(a))$, and let the crossing $M_\mathbb{T} \to R_\mathbb{T}$ be $(\partial_\mathbb{T} x)(a, b) = m(x(a), x(b), b)$. It is then straightforward to check that this defines a left linear form, that any morphism of theories gives rise to a morphism in **Lf** in a functorial way, and that if one starts from a theory of the form $\mathbb{T}_\partial$, then one recovers the original ∂ back. Finally for the second way round, observe that for any operation $u : X^n \to X$ in an abelian theory, with $n > 0$, one has

$$\begin{aligned} u(a, b, c, \dots) &= u(a +_a a +_a a +_a \cdots, a +_a b +_a a +_a \cdots, a +_a a +_a c +_a \cdots) \\ &= u(a, a, a, \dots) +_{u(a,a,a,\dots)} u(a, b, a, \dots) +_{u(a,a,a,\dots)} u(a, a, c, \dots) +_{u(a,a,a,\dots)} \cdots \\ &= u(a, a, a, \dots) +_a (a +_{u(a,a,a,\dots)} u(a, b, a, \dots)) +_a \\ &\qquad\qquad (a +_{u(a,a,a,\dots)} u(a, a, c, \dots)) +_a \cdots \\ &= x(a) +_a r(a, b) +_a s(a, c) +_a \cdots, \end{aligned}$$

with x in $M_\mathbb{T}$ and $r, s, \dots$ in $R_\mathbb{T}$. This implies easily that including $M_\mathbb{T}$ and $R_\mathbb{T}$ in $\mathbb{T}$ extends to an isomorphism of theories from $\mathbb{T}_{\partial_\mathbb{T}}$ to $\mathbb{T}$. $\square$

Remark. Construction of the ring $R_\mathbb{T}$ from an abelian Maltsev theory $\mathbb{T}$ is obviously well known to universal algebraists, in a slightly different context – see e. g. [4]. It is in fact closely related to the classical coordinatization construction for geometries. The reader might consult, e. g. [6] or [5] for that.

Using our description, we can now find out what kind of linear extensions exist between abelian theories. Indeed, since the Maltsev operation in abelian theories is unique, they are clearly closed under arbitrary finite limits, hence 4.4 together with 1.6 implies that abelian linear extensions of an abelian theory $\mathbb{T}$ can be identified with torsors under internal abelian groups in $\mathbf{Lf}/\partial_\mathbb{T}$. Thus we just have to describe torsors under a linear form $\partial : M \to R$. Consider one such, given by

$$\begin{tikzcd} K \arrow[r, tail, "j"] \arrow[d, "\delta"] & N \arrow[r, two heads, "q"] \arrow[d, "\partial'"] & M \arrow[d, "\partial"] \\ B \arrow[r, tail, "i"] & S \arrow[r, two heads, "p"] & R. \end{tikzcd} \tag{E}$$

Now by 1.3 we know that this torsor is equipped with a herd structure in $\mathbf{Lf}/\partial$. Thus we have a Maltsev homomorphism $m : \partial' \times_\partial \partial' \times_\partial \partial' \to \partial'$ over ∂; then, by an argument just as in 3.3, both in N and S one has

$$m(x,y,z) = m(x-y+y, y-y+y, y-y+z) = m(x,y,y) - m(y,y,y) + m(y,y,z) = x-y+z.$$

Thus it follows that (p,q) above is a torsor iff this map is a homomorphism. One sees easily that this happens iff $B^2 = BK = 0$. In such case, B becomes naturally an R-R-bimodule, K a left R-module, and restriction δ of ∂' to it – a module homomorphism, via $rb = sb$, $br = bs$, $rk = sk$, for any $b \in B$, $k \in K$, $r \in R$ and $s \in S$ with $p(s) = r$. Moreover there is an R-module homomorphism $B \otimes_R M \to K$, denoted $(b,m) \mapsto b \cdot m$, given by $b \cdot m = bn$ for any $n \in N$ with $q(n) = m$. It clearly satisfies $\delta(b \cdot m) = b\partial(m)$. On the whole, one gets a structure which can be described by

Definition 4.5. *For a left linear form $\partial : M \to R$, a ∂-bimodule consists of an R-R-bimodule B, a left R-module K, and R-linear maps $\delta : K \to B$ and $\cdot : B \otimes_R M \to K$ satisfying $\delta(b \cdot m) = b\partial m$ for any $b \in B$, $m \in M$. It will be denoted $\delta^{\cdot} = (B \otimes_R M \to K \to B)$.*

Examples of such ∂-bimodules include $R \otimes_R M \cong M \to R$, i.e. ∂ itself, $(B \otimes_R M \to B \otimes_R R \cong B = B)$, for any R-R-bimodule B, which we denote $\mathcal{C}(B)$, and $0 \to K \to 0$ for any left R-module K, which we denote $K[1]$.

It is easy to show that also conversely, internal groups in $\mathbf{Lf}/\partial$ are determined by ∂-bimodules $\delta^{\cdot}$ as above, and that as soon as the map $(x,y,z) \mapsto x - y + z$ is a homomorphism from $\partial' \times_\partial \partial' \times_d \partial'$ to ∂' over ∂, then there is a torsor structure on ∂' under the corresponding group. We summarize this as follows:

Proposition 4.6. *For a left linear form $\partial : M \to R$, internal groups in $\mathbf{Lf}/\partial$ are in one-to-one correspondence with ∂-bimodules. Moreover the underlying linear form of the corresponding group is $\delta \oplus \partial : K \oplus M \to B \oplus R$, with multiplicative structure $(b,r)(b',r') = (br' + rb', rr')$, $(b,r)(k,m) = (b \cdot m + rk, rm)$. Torsors under this group are in one-to-one correspondence with diagrams such as* (E) *above, where $B \rightarrowtail S \twoheadrightarrow R$ is a singular extension, i.e. the ideal $i(B)$ has zero multiplication in S and the induced R-R-bimodule structure coincides with the original one, and moreover $i(B)j(K) = 0$, the induced R-module structure on K is the original one, i.e. $j(p(s)k) = sj(k)$, and finally, the induced action $B \otimes_R M \to K$ coincides with the original one, i.e. $j(b \cdot q(n)) = i(b)n$.* □

Translating now all of the above from $\mathbf{Lf}$ to abelian theories, we conclude

Proposition 4.7. *For a left linear form $\partial : M \to R$, each internal group $\mathcal{A} = (\delta \oplus \partial \to \partial)$ in $\mathbf{Lf}/\partial$ corresponding to the ∂-bimodule $\delta^{\cdot} = (B \otimes_R M \to K \to B)$ as above, gives rise to a natural system $D^{\mathcal{A}}$ on the corresponding abelian theory $\mathbb{T}_\partial$. Explicitly, one has*

$$D^{\mathcal{A}}_{\langle x, r_1, \dots, r_{n-1}\rangle} = K \oplus B^{n-1},$$

with actions given by restricting those in 4.3 *for $\mathbb{T}_{\delta \oplus \partial}$ to $K \oplus B^{n-1} \subseteq (K \oplus M) \times (B \oplus R)^{n-1}$.*

A natural system on $\mathbb{T}_\partial$ is of this form iff all linear extensions by it are again abelian theories. □

In view of this, we will in what follows identify internal groups $\mathcal{A}$ in $\mathbf{Lf}/\partial$ with ∂-bimodules and with the corresponding natural systems $D^{\mathcal{A}}$ on $\mathbb{T}_\partial$. In particular, equivalence classes of extensions of $\mathbb{T}_\partial$ by $D^{\mathcal{A}}$ form, by [10], an abelian group isomorphic to $H^2(\mathbb{T}_\partial; D^{\mathcal{A}})$, which we can as well denote $H^2(M \to R;\ B \otimes_R M \to K \to B)$, or just by $H^2(\partial; \delta^{\cdot})$.

Now from [10] we know that any short exact sequence $\delta' \rightarrowtail \delta^{\cdot} \twoheadrightarrow \delta''$ induces the exact sequence

$$0 \to H^0(\partial; \delta') \to \cdots \to H^1(\partial; \delta'') \to H^2(\partial; \delta') \to H^2(\partial; \delta^{\cdot}) \to H^2(\partial; \delta'') \to \cdots$$

which one can use to reduce investigation of cohomologies, in particular linear extensions by a ∂-bimodule, to those by more "elementary" ones. In particular, observing the diagrams

$$\begin{tikzcd} \operatorname{Ker}(\delta) \arrow[r, tail] \arrow[d] & K \arrow[r, two heads] \arrow[d, "\delta"] & \operatorname{Im}(\delta) \arrow[d, tail, "\iota"] \\ 0 \arrow[r] & B \arrow[r, equal] & B, \end{tikzcd}$$

and

$$\begin{tikzcd} \operatorname{Im}(\delta) \arrow[r, tail, "\iota"] \arrow[d, tail] & B \arrow[r, two heads] \arrow[d, equal] & \operatorname{Coker}(\delta) \arrow[d] \\ B \arrow[r, equal] & B \arrow[r] & 0, \end{tikzcd}$$

one sees that there are short exact sequences of ∂-bimodules of the form $K'[1] \rightarrowtail \delta^{\cdot} \twoheadrightarrow \iota^{\cdot}$ and $\iota^{\cdot} \rightarrowtail \mathcal{C}(B) \twoheadrightarrow K''[1]$, so that linear extensions by any δ can be described in terms of those by bimodules of the form $K[1]$ and $\mathcal{C}(B)$.

Before dealing with these, just let us make a note about lower cohomologies – they can be expressed using derivations similarly to Hochschild cohomology.

Definition 4.8. *The group* $\operatorname{Der}(\partial; \delta^{\cdot})$ *of derivations of a linear form* $\partial : M \to R$ *with values in a* ∂*-bimodule* $\delta^{\cdot} = (B \otimes_R M \to K \to B)$ *consists of pairs of abelian group homomorphisms* $(d : R \to B, \nabla : M \to K)$ *satisfying*

$$\begin{aligned} d\partial &= \delta\nabla, \\ d(rs) &= d(r)s + rd(s), \\ \nabla(rm) &= d(r)m + r\nabla(m), \end{aligned}$$

under pointwise addition. Its subgroup $\operatorname{Ider}(\partial; \delta^{\cdot})$ *consists of inner derivations* $\operatorname{ad}(k) = (d_k, \nabla_k)$ *for* $k \in K$*, defined by*

$$d_k(r) = r\delta(k) - \delta(k)r, \quad \nabla_k(m) = \partial(m)k - \delta(k) \cdot m.$$

Then by analogy with well known classical facts, 4.7 and 4.10 of [10] readily give

Proposition 4.9. *For a linear form $\partial : M \to R$ and a ∂-bimodule $\delta^{\cdot} = (B \otimes_R M \to K \to B)$, one has an exact sequence*

$$0 \to H^0(\partial; \delta^{\cdot}) \to K \xrightarrow{\mathrm{ad}} \mathrm{Der}(\partial; \delta^{\cdot}) \to H^1(\partial; \delta^{\cdot}) \to 0.$$

In other words, there are isomorphisms

$$H^0(\mathbb{T}_\partial; \delta^{\cdot}) \cong \{c \in K \,|\, \forall m \in M \ (\partial m)c = (\delta c) \cdot m\}$$

and

$$H^1(\mathbb{T}_\partial; \delta^{\cdot}) \cong \mathrm{Der}(\partial; \delta^{\cdot}) / \mathrm{Ider}(\partial; \delta^{\cdot}).$$ □

Now for $\mathcal{C}(B)$, one has

Proposition 4.10. *For a linear form $\partial : M \to R$ and an R-R-bimodule B, there is an isomorphism*

$$H^2(\mathbb{T}_\partial; \mathcal{C}(B)) \cong H^2(R; B),$$

the latter being the MacLane cohomology group.

Proof. Observe the diagram

$$\begin{tikzcd}
B \arrow[r, tail] \arrow[d, equal] & N \arrow[r, two heads] \arrow[d] & M \arrow[d, "\partial"] \\
B \arrow[r, tail] & S \arrow[r, two heads] & R.
\end{tikzcd}$$

It shows that the right hand square is pullback, so that the upper row is completely determined by the lower one. Thus forgetting the upper row defines an isomorphism, with the inverse which assigns to a singular extension of R by B the pullback as above. □

Thus one arrives at a well studied situation here. As for the $K[1]$ case, we have

Proposition 4.11. *For a linear form $\partial : M \to R$ and a left R-module K, there is an isomorphism*

$$H^2(\mathbb{T}_\partial; K[1]) \cong \mathrm{Ext}^1_R(M, K).$$

Proof. This is obvious from the diagram

$$\begin{tikzcd}
K \arrow[r, tail] \arrow[d] & * \arrow[r, two heads, "p"] \arrow[d, "\partial p"] & M \arrow[d, "\partial"] \\
0 \arrow[r] & R \arrow[r, equal] & R.
\end{tikzcd}$$

□

But moreover the diagram

$$\begin{tikzcd}
\mathrm{Ker}(\partial) \arrow[r, tail] \arrow[d] & M \arrow[r, two heads] \arrow[d, "\partial"] & \mathrm{Im}(\partial) \arrow[d, tail] \\
0 \arrow[r] & R \arrow[r, equal] & R
\end{tikzcd}$$

shows that $\mathbb{T}_\partial$ is itself a linear extension of a theory corresponding to the linear form of type $\mathfrak{a} \rightarrowtail R$, where $\mathfrak{a}$ is a left ideal in R, by a natural system corresponding to an $(\mathfrak{a} \rightarrowtail R)$-bimodule of the form $K[1]$. And this is clearly the end: one obviously has

Proposition 4.12. *An abelian theory without constants cannot be represented nontrivially as a linear extension of another theory if and only if it is of the type $\mathbb{T}_{\mathfrak{a}\rightarrowtail R}$, for the left linear form determined by a left ideal $\mathfrak{a}$ in a ring R which does not have any nontrivial square zero two-sided ideals.*

Proof. The only nontrivial remark to make here is that for any square zero two-sided ideal $\mathfrak{b} \rightarrowtail R$, one gets an extension

$$\begin{tikzcd}
\mathfrak{k} \arrow[r, tail] \arrow[d, tail] & \mathfrak{a} \arrow[r, two heads] \arrow[d, tail] & \mathfrak{a}/\mathfrak{k} \arrow[d] \\
\mathfrak{b} \arrow[r, tail] & R \arrow[r, two heads] & R/\mathfrak{b}
\end{tikzcd}$$

for any left ideal $\mathfrak{k}$ with $\mathfrak{b}\mathfrak{a} \subseteq \mathfrak{k} \subseteq \mathfrak{b} \cap \mathfrak{a}$. □

Finally, consider an abelian theory $\mathbb{T}$ with constants. It has a largest subtheory $\mathbb{T}_0$ without constants, obtained from $\mathbb{T}$ by removing all morphisms $1 \to X^n$ for $n > 0$. The constants of $\mathbb{T}$ will then reappear in $\mathbb{T}_0$ as *pseudoconstants*, that is, those unary operations $p : X \to X$ satisfying the identity $p(a) = p(b)$. Conversely, if a theory without constants is obtained nontrivially in such way, it must have some pseudoconstants.

We know by 4.4 that $\mathbb{T}_0 = \mathbb{T}_\partial$ for some linear form $\partial : M \to R$. Now pseudoconstants in $\mathbb{T}_\partial$ correspond to elements p of M satisfying the identity $\varphi_a(p) = \varphi_b(p)$ (with a, b as variables). Using the identity $(*)$ from 4.2 this gives $(1 - \partial p)_a b = a$, i.e. $(\partial p)_a b = b$ for any a, b in any affinity. Then taking $a = \langle 0, 0\rangle$, $b = \langle 0, 1\rangle$ in $M \times R$ gives $\partial p = 1$. Now clearly there is a $p \in M$ with $\partial p = 1$ if and only if ∂ is surjective, in which case it is split by $\sigma(r) = rp$. Thus in this case our linear form is isomorphic to (projection): $\mathrm{Ker}(\partial) \oplus R \to R$. Let us fix one such p. We then may declare $\{p + t_0 \,|\, t_0 \in K\}$ to be the set of pseudoconstants corresponding to nullary operations, where K is either empty or any R-submodule of $\mathrm{Ker}(\partial)$. All choices will give equivalent categories of models, the only difference being that for $K = \varnothing$ the empty set is also allowed as a model. Each other model A shall then have at least one element a, and value of the unary operation p on A at a will then be $\varphi_a(p)$, which does not depend on a as we just saw. Denoting this element by 0_A fixes a canonical R-module structure on each non-empty model. Moreover each element of M becomes uniquely written as $m = k + rp$ with $r \in R$ and $k \in \mathrm{Ker}(\partial)$, so $\varphi_a(m) = \varphi_a(k) + r$, i.e. φ_a is completely determined by its restriction to $\mathrm{Ker}(\partial)$. Moreover by $(*)$ of 4.2 it is determined by φ_{0_A} alone. We see that, ignoring the possible empty model, the category $\mathbb{T}_\partial$-**mod** is equivalent to the coslice $(\mathrm{Ker}\,\partial)/R$-**mod**. We thus have proved

Proposition 4.13. *An abelian theory has at least one pseudoconstant if and only if the category of its models is equivalent to the category $K/(R$-**mod**$)$ of left R-modules under K, for some ring R and an R-module K, with the possible difference that the empty set is another model.* □

Now observing 2.6.4. we conclude

Corollary 4.14. *Any abelian theory with constants is isomorphic to $\mathbb{T}_{R;K}$ (defined in 2.6.4.) for some ring R and left R-module K.* □

Concerning linear extensions one observes that by 2.6.**4.**, any theory with constants $\mathbb{T}_{R;K}$ is a trivial untwisted linear extension of $\mathbb{T}_R$ by the bifunctor constructed there. Also observe that in any linear extension $\mathbb{T}' \to \mathbb{T}$ of abelian theories one has constants if and only if the other does.

On the other hand, a description similar to 4.13 is in fact possible for categories of models of abelian theories without constants too. For any left linear form $\partial : M \to R$, denote (temporarily) by ∂-**aff**$'$ the following category: objects are R-module homomorphisms $f : M \to N$; a morphism from $f' : M \to N'$ to $f : M \to N$ is a pair (g, n), where $g : N' \to N$ is an R-module homomorphism and $n \in N$ is an element such that $f(x) - gf'(x) = \partial(x)n$ holds for all $x \in M$. Composition is given by $(g, n)(g', n') = (gg', n + g(n'))$, and identities have form $(\mathrm{id}, 0)$. Equivalently, one might define objects as commutative triangles

$$\begin{tikzcd} M \arrow[dd, "\partial"'] \arrow[dr] & \\ & R \oplus N \arrow[dl, "\text{projection}"] \\ R & \end{tikzcd}$$

and morphisms as commutative diagrams

$$\begin{tikzcd} & M \arrow[dl] \arrow[dr] & \\ R \oplus N' \arrow[rr] \arrow[dr] & & R \oplus N \arrow[dl] \\ & R & \end{tikzcd}$$

in R-**mod**. One then has

Proposition 4.15. *The category* $\mathbb{T}_\partial$-**mod** *is equivalent to* ∂-**aff**$'$ *with an extra initial object added.*

Proof. Define a functor $\partial\text{-}\mathbf{aff}' \to \partial\text{-}\mathbf{aff}$ as follows: for $f : M \to N$, define an affinity structure on N by $a +_b c = a - b + c$, $r_a b = (1 - r)a + rb$, and $\varphi_a(x) = f(x) + (1 - \partial x)a$. And to a morphism (g, n) assign the homomorphism of affinities $N' \to N$ given by $n' \mapsto n + g(n')$. It is straightforward to check that this defines a full and faithful functor. Moreover any nonempty affinity is isomorphic to one in the image of this functor – just choose an element and use it as zero to define a module structure and a homomorphism from M according to the affinity identities. □

This allows to give an example, which looks pleasantly familiar

Example. Fix a field k, and let the category of cycles be defined as follows. Objects are pairs $((V, d), c)$, where (V, d) is a differential k-vector space and $c \in V$ is a cycle, i.e. $dc = 0$. A morphism from $((V, d), c)$ to $((V', d'), c')$ is a pair (φ, x), where $\varphi : V \to V'$ is a k-linear differential map and $x \in V'$ an element with $c' - \varphi(c) = dx$. With the evident identities and composition this forms a category which is clearly of the form $\mathbb{T}_\partial$-**mod**, for the linear form $\partial : \varepsilon k[\varepsilon] \rightarrowtail k[\varepsilon]$, where ε is an indeterminate element with $\varepsilon^2 = 0$.

Now obviously this example admits a linear extension structure over $\mathbb{T}_k$, since $\varepsilon k[\varepsilon]$ is a square zero ideal. But of course 4.12 provides lots of similar (less cute) examples without this property.

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Received June 4, 2003