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Using the Frattini Subgroup and Independent Generating Sets to Study RWPRI Geometries

Claude Archer Philippe Cara[1] Jan Krempa

Université Libre de Bruxelles, C.P.165/11 -Physique et Mathématique Faculté des Sciences Appliquées
avenue F.D. Roosevelt 50, 1050 Bruxelles, Belgium
e-mail: carcher@ulb.ac.be

Vrije Universiteit Brussel, Department of Mathematics
Pleinlaan 2, B-1050 Brussel, Belgium
e-mail: pcara@vub.ac.be

Warsaw University, Institute of Mathematics
Banacha 2, 02-097 Warszawa, Poland
e-mail: jkrempa@mimuw.edu.pl

Abstract. In [4], Cameron and Cara showed a relationship between independent generating sets of a group G and RWPRI geometries for G. We first notice a connection between such independent generating sets in G and those in the quotient $G/\Phi(G)$, where $\Phi(G)$ is the Frattini subgroup of G. This suggests a similar connection for RWPRI geometries. We prove that there is a one-to-one correspondence between the RWPRI geometries of G and those of $G/\Phi(G)$. Hence only RWPRI geometries for Frattini free groups have to be considered. We use this result to show that RWPRI geometries for p-groups are direct sums of rank one geometries. We also give a new test which can be used when one wants to enumerate RWPRI geometries by computer.

[1]Corresponding author; postdoctoral fellow of the Fund for Scientific Research-Flanders (Belgium) (F.W.O.-Vlaanderen).

1. Geometries

1.1. Basic definitions and notation

After Tits [11], there is a standard way to define an (incidence) geometry from a group and a collection of subgroups. In this section, we recall this construction.

Let $I = \{1, \dots, n\}$ be a finite set whose elements are called *types*. Let G be a group together with a finite nonempty family of distinct subgroups $(G_i)_{i\in I}$. The (coset) *pregeometry* $\Gamma = \Gamma(G, (G_i)_{i\in I})$ is defined as follows. The set X of *elements* of Γ consists of all cosets $G_i g$, $g \in G$, $i \in I$. An incidence relation $*$ is defined on X by:

$$G_i g_1 * G_j g_2 \Longleftrightarrow G_i g_1 \cap G_j g_2 \neq \emptyset.$$

The *type function* t on Γ is $X \longrightarrow I \colon G_i g \longmapsto i$ and we call $|I| = n$ the *rank* of Γ. The group G acts on Γ as an automorphism group. Indeed, by right multiplication, $g \in G$ maps $G_i g_1$ to $G_i g_1 g$ and this action preserves each type as well as the incidence between elements. For each type i, the action of G on the elements of type i is transitive and G_i is the stabilizer of the element G_i of type i.

A *flag* is a set of pairwise incident elements and a flag containing an element of each type is called a *chamber*. The *type* of a flag F is simply the image $t(F)$ of F under the type function. We call the cardinality of $t(F)$ the *rank* of F.

The *residue* Γ_F of a flag F is the pregeometry induced on the set X_F of all elements of type $I \setminus t(F)$ incident with each element of F.

1.2. More axioms

As such, the structure of a coset pregeometry is too general. In order to have a structure that is more similar to classical geometries more axioms are needed. We follow the set of axioms proposed by the team of Buekenhout in [3].

A pregeometry Γ is said to be *flag-transitive* (FT) provided that G acts transitively on all flags of any given type $J \subseteq I$. We call Γ a *(coset) geometry* if every flag of Γ is contained in a chamber.

For $J \subseteq I$, we put $G_J := \bigcap_{j\in J} G_j$. If Γ is a flag-transitive geometry, every flag of type $J \subseteq I$ is the image under G of the flag $F_J := \{G_j : j \in J\}$. The stabilizer of F_J is G_J and the residue of F_J is isomorphic to the coset geometry

$$\Gamma_{F_J} = \Gamma(G_J, (G_{J\cup\{k\}} : k \in I \setminus J)).$$

The *Borel subgroup* of Γ is the subgroup $B := G_I = \bigcap_{i\in I} G_i$.

We call Γ *firm* (F) provided that every non maximal flag is contained in at least two chambers. The geometry Γ is said to be *residually connected* (RC) whenever the incidence graph $(X_F, *_F)$ of each residue of rank ≥ 2 is connected.

We call Γ *primitive* (PRI) if the action of G is primitive on the elements of any given type (i.e. all G_i are maximal in G). We call Γ *weakly primitive* (WPRI) provided G acts primitively on the set of elements of type i in Γ for some $i \in I$. The geometry Γ is said to be *residually weakly primitive* (RWPRI) whenever the residue Γ_F of any flag F is weakly primitive for the group induced on Γ_F by the stabilizer G_F of F. Similarly we define *residually primitive* (RPRI) where we require that every residue is PRI.

The reader can find a complete survey of the origins of these concepts in the Handbook of Incidence Geometry [2].

1.3. Group theoretic formulations

When dealing with coset geometries, we have to translate the axioms mentioned above into group theory. Assuming flag-transitivity allows us to do this easily. Detailed proofs can be found in [5].

(F) The subgroups G_J, for $J \subseteq I$, are all distinct.

(RC) If $J \subseteq I$ and $|J| < |I| - 1$, then $G_J = \langle G_{J\cup\{k\}} : k \in I \setminus J\rangle$.

(FT) If a family $(G_j x_j : j \in J)$ of right cosets has pairwise non-empty intersection, then there is an element of G lying in all these cosets.

Since the action of G on the cosets of G_i is primitive if and only if G_i is a maximal subgroup of G, the RWPRI condition means that the group G_J acts primitively on the elements of at least one type in the residue of the standard flag $F_J = \{G_j \mid j \in J\}$ of type J. Hence the coset geometry is residually weakly primitive if and only if the following condition holds:

(RWPRI) For any $J \subset I$, there exists $k \in I \setminus J$ such that $G_{J\cup\{k\}}$ is a maximal subgroup of G_J.

2. Independent generating sets

2.1. Definitions

Let $S = \{s_i : i \in I\}$ be a family of elements of a group G. For $J \subseteq I$, let $G_J = \langle s_i : i \notin J\rangle$; we abbreviate $G_{\{i\}}$ to G_i. We say that S is *independent* if $s_i \notin G_i$ for all $i \in I$. A family of elements which generates G is independent if and only if it is a minimal generating set (that is, no proper subset generates G).

Like in [4], we also define a relativized version. Let B be a subgroup of G. A family $S = \{s_i : i \in I\}$ of elements of G, is *independent relative to* B if $s_i \notin \langle B, s_j : j \neq i\rangle$, and it is an *independent generating set relative to* B if in addition $\langle B \cup S\rangle = G$.

2.2. Independent generating sets and the Frattini subgroup

The Frattini subgroup $\Phi(G)$ of a group G is defined as the intersection of all maximal subgroups of G. We briefly recall the connection between $\Phi(G)$ and generating sets for G. An element $x \in G$ is a *nongenerator* if for every subset S of G such that $\langle x, S\rangle = G$ we have $\langle S\rangle = G$. An important property is that the set of all nongenerators is exactly $\Phi(G)$ (see [10], p. 156). Hence $\langle \Phi(G) \cup S\rangle = G$ if and only if $\langle S\rangle = G$.

Theorem 2.1. *Let B be a subgroup of a group G and let $\Phi := \Phi(G)$. A subset $\{s_i : i \in I\}$ is an independent generating set of G relative to B if and only if $\{\Phi s_i : i \in I\}$ is an independent generating set of G/Φ relative to $\Phi B/\Phi$.*

Proof. Observe that every subset of cardinality $|I|$ in G/Φ may be written as $\tilde{S} := \{\Phi s_i : i \in I\}$, where $S := \{s_i : i \in I\}$ is a subset of G. First we prove the equivalence for the generating property. Obviously $\langle S \cup B\rangle = G$ implies that $\tilde{S} \cup \Phi B/\Phi$ generates G/Φ.

Conversely, if $\tilde{S} \cup \Phi B/\Phi$ generates G/Φ then every coset Φg in G is a product of cosets in $\tilde{S} \cup \Phi B/\Phi$. This means that $\Phi g = \Phi h$, where h is a product of elements of $S \cup B$. Hence for all $g \in G$ we have $g \in \langle \Phi \cup S \cup B\rangle$ and thus, by the non generating property of Φ, we get $G = \langle S \cup B\rangle$. This shows that $\tilde{S} \cup \Phi B/\Phi$ generates G/Φ if and only if $S \cup B$ generates G.

We still have to prove the independence. First remark that if $\langle X, H\rangle = T$ for a subset X and a subgroup H of a group T, then

$$X \not\subseteq H \Leftrightarrow H \neq T. \qquad (\star)$$

Let $G_i := \langle B, s_j : j \neq i\rangle$ and let $\tilde{G_i} := \langle \Phi B/\Phi, \Phi s_j : j \neq i\rangle$. Notice that $\tilde{G_i} = \langle \Phi \cup G_i\rangle/\Phi$. Thus $\{\Phi s_i : i \in I\}$ being an independent generating set of G/Φ relative to $\Phi B/\Phi$ means $\Phi s_i \notin \tilde{G_i}$ which is thus equivalent to $\Phi s_i \not\subseteq \langle \Phi, G_i\rangle$. Since we have shown previously that $\langle \Phi s_i, \langle \Phi, G_i\rangle/\Phi\rangle = G/\Phi$ if and only if $\langle s_i, \langle \Phi, G_i\rangle\rangle = G$, we can use $(\star)$ with $T = G/\Phi$ to replace $\Phi s_i \not\subseteq \langle \Phi, G_i\rangle$ by $\langle \Phi, G_i\rangle \neq G$. This is equivalent to $G_i \neq G$ (by the non generating property) and by $(\star)$ again (with $T = G$), this happens if and only if $s_i \notin G_i$. □

2.3. Independent generating sets and RWPri geometries

In [4], Cameron and Cara have shown that any firm RWPri coset geometry gives rise to an independent generating set S relative to the Borel subgroup.

Their construction is the following. Let $\Gamma = \Gamma(G, (G_i)_{i\in I})$. Choose elements s_i, for $i \in I$, so that s_i fixes the elements G_j for $j \neq i$ but moves the element G_i. In other words, $s_i \in G_{I\setminus\{i\}}$ where G_J denotes the stabilizer of the standard flag F_J. Then $S := \{s_i \mid i \in I\}$ is an independent set relative to the Borel subgroup B. Furthermore if the coset geometry Γ happens to be RWPri, then $S \cup B$ also generates the whole group G and hence $\{s_i : i \in I\}$ is an independent generating set for G relative to B.

Moreover this construction yields a *strongly* independent set of G relative to B, i.e. $G_J \cap G_K = G_{J\cup K}$ for all $J, K \subseteq I$. Nevertheless, the converse is not true. If $\{s_i : i \in I\}$ is a strongly independent generating set for G relative to B, and we put $G_i := \langle B, s_j : j \neq i\rangle$, then conditions (F) and (RC) hold, but (FT) and (RWPri) may fail.

A natural question

Theorem 2.1 states a correspondence between independent generating sets (IGS for short) in G and in $G/\Phi(G)$. Since a part of the IGS of G (respectively $G/\Phi(G)$) yields the firm RWPri geometries of G (respectively $G/\Phi(G)$), it is natural to ask whether the correspondence also holds between firm RWPri geometries in G and in $G/\Phi(G)$. This problem is the main motivation for this paper and we will solve it in next section.

3. New applications to RWPRI geometries

3.1. Bijection between firm RWPRI geometries of G and of $G/\Phi(G)$

In general for a normal subgroup N in G, any RWPRI geometry of G/N lifts to an RWPRI geometry of G whose Borel subgroup contains N. This is due to the bijection between subgroups (respectively maximal subgroups) of G_J/N and subgroups (respectively maximal subgroups) of G_J containing N. However, not all geometries for G come from a single quotient G/N because we cannot be sure that all G_i contain the fixed normal subgroup N. We show, with the (F) axiom, that for $N = \Phi(G)$, all RWPRI geometries of G can be obtained from the quotient $G/\Phi(G)$.

The following theorem also holds when the group G is infinite.

Theorem 3.1. *Let $\Gamma = \Gamma(G, (G_i)_{i\in I})$ be a firm* RWPRI *coset geometry. Then $\Phi(G) \subset G_i$ for all $i \in I$.*

Proof. For $J \subseteq I$, let $G_J := \bigcap_{j\in J} G_j$. By the RWPRI condition and the firm axiom (F), there is a chain of subgroups

$$B = G_I \subset \cdots \subset G_{\{i_1,i_2,i_3\}} \subset G_{\{i_1,i_2\}} \subset G_{i_1} \subset G$$

where every inclusion is strict and maximal. To simplify notation we relabel the indexes, replacing i_k by k and $\{1,\ldots,k\}$ by $\overline{k}$. The chain is now written as $B \subset \cdots \subset G_{\overline{3}} \subset G_{\overline{2}} \subset G_{\overline{1}} \subset G =: G_{\overline{0}}$ and we prove that $\Phi = \Phi(G) \subset G_i$, $\forall i \in I$.

We proceed by induction. Since G_1 is maximal in G we have $\Phi \subset G_{\overline{1}}$. Assume that up to an index k we have $\Phi \subset G_{\overline{m}}$ for all $m < k$, then if $\Phi \not\subseteq G_{\overline{k}}$, we derive a contradiction as follows. Assume that the following holds:

$$\forall m < k, \quad \Phi G_{\overline{m}\cup\{k\}} = G_{\overline{m}} \quad \text{implies} \quad \Phi G_{\overline{m-1}\cup\{k\}} = G_{\overline{m-1}}\,. \tag{1}$$

We will prove statement (1) in the last paragraph. We claim that the first part of (1) holds for $m = k-1$. Indeed, by our induction hypothesis, Φ is a normal subgroup of $G_{\overline{k-1}} = \cap_{m<k} G_m$ and the subgroup $\Phi G_{\overline{k-1}\cup\{k\}} = \Phi G_{\overline{k}}$ is strictly larger than $G_{\overline{k}}$ since $\Phi \not\subseteq G_{\overline{k}}$. Hence maximality of $G_{\overline{k}}$ in $G_{\overline{k-1}}$ implies $\Phi G_{\overline{k}} = G_{\overline{k-1}}$. Now we use statement (1) from $m = k-1$ up to $m = 1$, we obtain $\Phi G_k = G$. As Φ is the Frattini subgroup of G, this is only possible when $G_k = G$ which contradicts axiom (F).

It remains to prove statement (1). Assume $\Phi G_{\overline{m}\cup\{k\}} = G_{\overline{m}}$ for some $m < k$. Then $G_{\overline{m}} = \Phi G_{\overline{m}\cup\{k\}} \subset \Phi G_{\overline{m-1}\cup\{k\}} \subset \Phi G_{\overline{m-1}}$ and $\Phi G_{\overline{m-1}} = G_{\overline{m-1}}$ since $\Phi \subset G_{\overline{m-1}}$ (here $m-1 < k$). Now $G_{\overline{m}} \subset \Phi G_{\overline{m-1}\cup\{k\}} \subset G_{\overline{m-1}}$ together with the maximality of $G_{\overline{m}}$ in $G_{\overline{m-1}}$ implies

$$\text{either (A)} : \Phi G_{\overline{m-1}\cup\{k\}} = G_{\overline{m-1}} \quad \text{or} \quad (B) : G_{\overline{m}} = \Phi G_{\overline{m-1}\cup\{k\}}.$$

As (A) is what we want to prove, let us show that (B) does not occur. (B) implies $G_{\overline{m-1}\cup\{k\}} \subset \Phi G_{\overline{m-1}\cup\{k\}} = G_{\overline{m}}$. Since $G_{\overline{m-1}\cup\{k\}} = G_k \cap G_{\overline{m-1}}$, we can say that $G_k \cap G_{\overline{m-1}}$ is a subgroup of $G_k \cap G_{\overline{m}}$. The inclusion $G_{\overline{m}} \subset G_{\overline{m-1}}$ then shows that $G_{\overline{m-1}\cup\{k\}}$ and $G_{\overline{m}\cup\{k\}}$ are equal. This contradicts axiom (F), since $\overline{m-1} \cup \{k\} \neq \overline{m} \cup \{k\}$ when $m \neq k$. $\square$

A geometry where all stabilizers G_i contain a normal subgroup K of G is isomorphic to a geometry of the quotient group G/K. More precisely:

Proposition 3.2. [2] *If $\Gamma(G,(G_i)_{i\in I})$ is a pregeometry and if K is a normal subgroup of G such that $K \leq G_i$ for every $i \in I$, then*

$$\Gamma(G,(G_i)_{i\in I}) \cong \Gamma(G/K,(G_i/K)_{i\in I}).$$

From Theorem 3.1 we now conclude

Corollary 3.3. *A firm* RWPRI *coset geometry $\Gamma(G,(G_i)_{i\in I})$ is isomorphic to*

$$\Gamma(G/\Phi(G),(G_i/\Phi(G))_{i\in I}).$$

The groups of the form $G/\Phi(G)$ are called *Frattini free groups* and they are quite exceptional among finite groups (see section 4.1). By the corollary we obtain:

Theorem 3.4. *Every firm* RWPRI *geometry is isomorphic to a firm* RWPRI *geometry of a Frattini free group.*

3.2. Firm RWPRI geometries are trivial for *p*-groups

3.2.1. Direct sums of geometries

If in a rank 2 pregeometry Γ each element of type i is incident with every element of type j, the geometry $\Gamma(G,(G_i,G_j))$ is called a *direct sum* $\Gamma_i \oplus \Gamma_j$. More generally if the type set I is a union $I = I_1 \cup \cdots \cup I_r$ of disjoint subsets such that each element of type $i \in I_k$ is incident with every element of type $j \in I_l$ whenever $k \neq l$, we write Γ as a direct sum $\Gamma = \Gamma_1 \oplus \cdots \oplus \Gamma_r$, where the summand $\Gamma_k = \Gamma(G,(G_i)_{i\in I_k})$.

In fact, it can be proved easily that the structure and the properties of a pregeometry Γ are fully determined by those of its summands $\Gamma_1,\ldots,\Gamma_r$. A flag F of Γ is a union $F = F_1 \cup \cdots \cup F_r$ of (possibly empty) flags of the summands. The residue Γ_F of F is the direct sum $\Gamma_{F_1} \oplus \Gamma_{F_2} \oplus \cdots \oplus \Gamma_{F_r}$ of the residues in the corresponding summands (where F_k is the intersection of F with $t^{-1}(I_k)$). In the same way, a chamber is a union of disjoint chambers. Γ is residually connected if and only if the summands have this property. For these reasons, direct sum decompositions of pregeometries have a great importance (see Valette [12] for further details and [1] for the following well-known proposition).

Proposition 3.5. *$\Gamma(G,(G_1,G_2))$ is a direct sum if and only if $G_1G_2 = G$.*

3.2.2. Firm RWPRI geometries for *p*-groups

Let us recall that for a finite p-group P, the quotient $P/\Phi(P)$ is elementary abelian and has the structure of a vector space over $\mathbb{F}_p$. Proving the following lemma is an easy exercise.

Lemma 3.6. *Let G_1 and G_2 be proper subgroups of $\mathbb{Z}_p^k$ with $G_2 \not\subseteq G_1$. If G_1 is maximal and $G_1 \cap G_2$ is maximal in G_1, then G_2 is a maximal subgroup of $\mathbb{Z}_p^k$.*

Proposition 3.7. *For a finite p-group G, a firm coset geometry is* RWPRI *if and only if it is* RPRI.

Proof. Since RPRI implies RWPRI, it is sufficient to show that RWPRI implies RPRI. By Theorem 3.1 it is enough to show the property in $G/\Phi(G)$, which is elementary abelian. So we may assume without restriction that $G = \mathbb{Z}_p^k$. Write G_J for $\cap_{j\in J} G_j$. For a given subset J of I, the RWPRI and firm property, allows us (after relabeling) to achieve that all inclusions in the following chain are strict and maximal

$$B = G_I \subset \cdots \subset G_{J\cup\{1,2\}} \subset G_{J\cup\{1\}} \subset G_J\ .$$

Write $\tilde{G}_K$ for $G_{J\cup K}$ and $\overline{k}$ for $\{1,\ldots,k\}$ (we also put $\overline{0} := \emptyset$). Observe that $\tilde{G}_{\overline{k}} = \tilde{G}_{\{k\}\cup\overline{k-2}} \cap \tilde{G}_{\overline{k-1}}$. As a subgroup of an elementary abelian group, $\tilde{G}_{\overline{k-2}}$ is also elementary abelian and we may apply Lemma 3.6. Thus a pair of strict maximal inclusions $\tilde{G}_{\overline{k}} \subset \tilde{G}_{\overline{k-1}} \subset \tilde{G}_{\overline{k-2}}$ implies that $\tilde{G}_{\{k\}\cup\overline{k-2}}$ is proper maximal in $\tilde{G}_{\overline{k-2}}$.

Induction on m with $k \geq m \geq 2$ yields that $\tilde{G}_{\{k\}\cup\overline{k-m}}$ is a proper maximal subgroup of $\tilde{G}_{\overline{k-m}}$ so that finally $G_{J\cup\{k\}} = \tilde{G}_{\{k\}}$ is maximal in $\tilde{G}_{\overline{0}} = G_J$ and this for all k. □

Theorem 3.8. *A firm* RWPRI *coset geometry for a p-group is a direct sum of* PRI *geometries of rank* 1.

Proof. Again by Theorem 3.1 it is sufficient to show the property in $G = \mathbb{Z}_p^k$. The previous theorem ensures that all stabilizers G_i are maximal subgroups of G and hence $(k-1)$-dimensional subspaces. For $i \neq j$ Grassmann's dimension formula yields

$$\dim(G_i + G_j) = \dim G_i + \dim G_2 - \dim(G_i \cap G_j) = k - 1 + k - 1 - (k-2) = k.$$

Hence $G_i + G_j$ must be equal to G. Proposition 3.5 terminates the proof. □

4. Implications for RWPRI geometries

4.1. Reduction to Frattini free groups

Let us first remark that the Frattini subgroup of $G/\Phi(G)$ is the trivial subgroup $\{\Phi(G)\}$. Such a group for which the Frattini subgroup is the identity, is called a Frattini free group. The following theorem describes the structure of such groups as semi-direct products (see [9]). We say that a group K acts semi-simply on an abelian group A if the intersection of all maximal K-normal subgroups of A is trivial.

Theorem 4.1. *Let F be a finite Frattini free group with socle $S = A \times B$ where A (resp. B) is a direct product of abelian (resp. non-abelian) simple groups. Then $F = A \rtimes K$ where K is a subgroup of $Aut(S) = Aut(A) \times Aut(B)$ which contains $B = Inn(S)$ and acts semi-simply on A.*

4.1.1. Frattini free groups are scarce

Considering only geometries on Frattini free groups reduces considerably the number of groups to take into account. According to the SmallGroups library in GAP (see [8] and [7]), there are $49,500,460,704$ finite groups of order less than $1536 = 3 * 512$ and among them only 7818 are Frattini free (a proportion less than $100 - 99,9999\%$). This is easily understandable since p-groups form the overwhelming majority of finite groups up to order 2000. There are for instance more than $49 \cdot 10^9$ groups of order 2^{10} and the library of Eick, Besche and O'Brien ([8]) suggests even that the proportion of p-groups among all finite groups up to order n tends to 1 when n tends to infinity. Nevertheless, although there could exist billions of groups of order p^k, there is only one Frattini free group of order p^k, namely the elementary abelian group.

4.2. The Φ-test for a residue

Suppose we want to test whether a collection of subgroups $\{G_i, i \in I\}$ defines an RWPRI geometry $\Gamma(G, (G_i)_{i\in I})$. According to Theorem 3.1, a first obvious test is to check whether $\Phi(G) \subset G_i$ for all $i \in I$.

The original aim of RWPRI coset geometries was to obtain a geometrical interpretation of sporadic simple groups. All these groups have a trivial Frattini subgroup and hence $\Phi(G) \subset G_i$ is certainly true. However, RWPRI property must hold for any residue and the groups G_J involved in residues are not, in general, Frattini free.

Let F be a flag of Γ. The residue of F must be an RWPRI geometry $\Gamma_F = \Gamma(G_{t(F)}, (G_i \cap G_{t(F)})_{i\in I\setminus t(F)})$. Therefore $\Phi(G_{t(F)})$ must be included in every $G_i \cap G_{t(F)}$ but in general $\Phi(G)$ is not the Frattini subgroup of $G_{t(F)}$. Sometimes $\Phi(G_{t(F)})$ is not even contained in $\Phi(G)$, so that a trivial $\Phi(G)$ does not imply a trivial $\Phi(G_{t(F)})$. Even if $\Phi(G) \subset G_i$ for all $i \in I$, there is no guarantee that $\Phi(G_{t(F)})$ is contained in every $G_i \cap G_{t(F)}$. Hence this provides a new test for every subgroup G_J. We refer to this as the Φ-test.

Therefore, even in the geometric study of sporadic simple groups, the Φ-test can be a useful tool.

4.2.1. How to compute $\Phi(G)$?

For finite soluble groups there exist specific methods for computing the Frattini subgroup without computing all maximal subgroups (see [6]). For non soluble groups, Eick suggests to use the fact that $\Phi(G)$ is contained in the Fitting subgroup of G.

4.3. Other reduction in some cases

In order to reduce the geometries of a group G to geometries of a quotient G/K (see Proposition 3.2), we would like to determine the largest G-normal subgroup K of the Borel subgroup $B = \cap_{i\in I} G_i$. By definition this group is $K := Core_G(B) = \cap_{i\in I} Core_G(G_i)$. For firm, RWPRI geometries, we have shown that $\Phi(G) \subset K$ and it is easy to find examples where $\Phi(G) = K$ (if $G/\Phi(G)$ is simple for instance).

In some cases K is a larger subgroup of G. For example in [1] we have proved that an RPRI geometry that is not a direct sum must be a geometry $\Gamma(G, (G_i)_{i\in I})$ where G belongs to a very specific family of Frattini free groups, namely that of primitive groups.

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