Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 46 (2005), No. 1, 207-214.

Primeness in Near-rings of Continuous Functions 2

G. L. Booth

University of Port Elizabeth
Port Elizabeth 6000, South Africa

Abstract. This paper is a continuation of work done by the present author together with P. R. Hall [1]. We characterise the prime and equiprime radicals of $N_0(G)$ for certain topological groups G. Various results are obtained concerning primeness and strongly primeness for the sandwich near-ring $N_0(G, X, \theta)$.

MSC 2000: 16Y30, 22A05

1. Preliminaries

In this paper, all near-rings will be right distributive. All of the near-rings N considered in this paper will also be *zero-symmetric*, that is $x0 = 0$ for all $x \in N$. (The identity $0x = 0$ follows of course from the right distributivity.) The notation "$A \lhd N$" means "A is an ideal of N". We refer to Pilz [12] for all undefined concepts concerning near-rings.

It is well-known (cf. McCoy [11]) that there are a number of equivalent definitions for primeness in associative rings. These definitions do not coincide for zero-symmetric near-rings. Consequently, a number of generalisations of primeness have appeared in the literature of near-rings. The classical notion is given in [12]: A near-ring N is called *prime* (resp. *semiprime*) if $A, B \lhd N$ (resp. $A \lhd N$), $AB = 0$ implies $A = 0$ or $B = 0$ (resp. $A^2 = 0$ implies $A = 0$). N is called 3-*prime* (resp. 3-*semiprime*) if $x, y \in N$ (resp. $x \in N$) $xNx = 0$ implies $x = 0$ or $y = 0$ (resp. $xNy = 0$ implies $x = 0$). An ideal I of N is called prime (resp. 3-prime) if the factor near-ring N/I is prime (resp. 3-prime).

A *radical map* is a mapping ρ which assigns to evey near-ring N an ideal $\rho(N)$ of N such that (i) if $f : N \to R$ is a surjective near-ring homomorphism, then $f(\rho(R)) \subseteq \rho(S)$ and (ii) $\rho(N/\rho(N)) = 0$ for every near-ring N. If in addition (iii) $\rho(\rho(N)) = \rho(N)$ and (iv) $I \lhd N$,

0138-4821/93 \$ 2.50

$\rho(I) = I$ implies that $I \subseteq \rho(N)$ for all near-rings N, then ρ is called a *Kurosh-Amitsur radical* (*KA-radical*). The prime radical $\mathcal{P}(N)$ (resp 3-prime radical $\mathcal{P}_3(N)$) is the intersection of the prime (resp. 3-prime) ideals of N. It is clear that $\mathcal{P}$ and $\mathcal{P}_3$ are radical maps. Kaarli and Kriis [8] have shown that $\mathcal{P}$ is not a KA-radical. It is not known whether $\mathcal{P}_3$ is a KA-radical, but this is widely considered to be unlikely.

N is called *equiprime* (cf. Booth, Groenewald and Veldsman [2]) if $a, x, y \in N$, $anx = any$ for all $n \in N$ implies $a = 0$ or $x = y$. An ideal I of N is called equiprime if the factor near-ring N/I is equiprime. The *equiprime radical* of N, $\mathrm{P}_e(N)$, is the intersection of the equiprime ideals of N. It is shown in [2] that $\mathcal{P}_e$ is a KA-radical, and is moreover *ideal-hereditary*, that is $\mathcal{P}_e(I) = I \cap \mathcal{P}_e(N)$ for all ideals I of N.

Prior to the study of equiprime near-rings, the only well-known ideal-hereditary KA-radicals in this class were the Jacobson-type radical $\mathcal{J}_2$ and the Brown-McCoy radical $\mathcal{B}$. Their scarcity lead Kaarli [7] to conjecture that all such radicals were based on either $\mathcal{J}_2$ or $\mathcal{B}$. The study of equiprime near-rings leads to the discovery of a considerable number of new ideal-herditary radicals (cf. [3]) which are independent of both $\mathcal{J}_2$ and $\mathcal{B}$. It is well-known that equiprime $\Longrightarrow$ 3-prime $\Longrightarrow$ prime $\Longrightarrow$ semiprime for near-rings, and that these implications are strict. For further details on these generalisations of primeness to near-rings and their associated radicals, the reader may consult Groenewald's survey paper [5] and its references.

Strongly prime rings were defined by Handelman and Lawrence [6]. There are two generalisations of the concept to near-rings. A near-ring N is *strongly prime* [4] if $0 \neq a \in N$ implies that there exists a finite subset F of N such that $aFx = 0$ implies $x = 0$, for all $x \in N$. N is *strongly equiprime* [3] if $0 \neq a \in N$ implies that there exists a finite subset F of N such that $x, y \in N, afx = afy$ for all $f \in F$ implies $x = y$. Clearly strongly equiprime $\Longrightarrow$ strongly prime $\Longrightarrow$ prime and strongly equiprime $\Longrightarrow$ equiprime. These implications are all strict and strongly prime $\Longrightarrow$ prime is strict even for associative rings. The radicals associated with the classes of strongly prime and strongly equiprime near-rings are denoted $\mathcal{S}$ and $\mathcal{S}_e$, respectively. We consider it unlikely that $\mathcal{S}$ is a KA-radical. However $\mathcal{S}_e$ is an ideal-hereditary KA-radical [3].

In the sequel G will denote a T_1 (and hence completely regular) additive topological group. The set of zero-preserving continuous self-maps of G forms a zero-symmetric near-ring with respect to addition and composition of functions, and is denoted $N_0(G)$. If the topology on G is discrete, $N_0(G)$ is the set of all zero-preserving self-maps of G, and is denoted $M_0(G)$ in this case. It is easily shown that $M_0(G)$ is equiprime. In order to avoid trivial cases, all topological groups will be assumed to contain more than one element. Composition of functions will be denoted by juxtaposition, e.g. ab rather than $a \circ b$ (with the function b acting first). For surveys of work done on near-rings of continuous functions, [9] and [10] can be consulted.

2. Primeness in $N_0(G)$

In [1] it was shown that $N_0(G)$ is equiprime if the topology on G is either 0-dimensional or arcwise connected. In contrast, an example was given of a topological group such that $N_0(G)$ is not semiprime. The main result of this section sharpens [1, Proposition 1.1].

Lemma 2.1. *Let H be the connected component of G which contains 0. If H is open, then the quotient topology on G/H is discrete.*

Proof. Let φ be the canonical mapping of G onto G/H and let $\mathcal{T}$ and $\mathcal{T}^*$ denote the topology on G and the quotient topology on G/H, respectively. Let $g \in G$. Then $\varphi^{-1}(\{g+H\}) = g+H \in \mathcal{T}$ since $H \in \mathcal{T}$. Hence $\{g+H\} \in \mathcal{T}^*$. Hence $\mathcal{T}^*$ contains all one-point subsets of G/H, and so is discrete. □

Proposition 2.2. *Let G be a disconnected topological group, with open components which are arcwise connected and which contain more than one element. Let H be the component of G which contains 0, $I := \{a \in N_0(G) \mid a(G) \subseteq H\}$ and $J := \{a \in N_0(G) \mid a(H) = 0\}$. Then $\mathcal{P}(N_0(G)) = \mathcal{P}_e(N_0(G)) = I \cap J$.*

Proof. In the proof of [1, Proposition 1.1], it was shown that $I \lhd N_0(G)$. It is clear that $J \lhd N_0(G)$. Let $a \in J$, $b \in I$, $g \in G$. Then $(ba)(g) = b(a(g)) = 0$ since $a(g) \in H$. Hence $JI = 0$, so $(I \cap J)^2 = 0 \subseteq \mathcal{P}(N_0(G))$. Since $\mathcal{P}(N_0(G))$ is the intersection of the prime ideals of $N_0(G)$, it is a semiprime ideal of $N_0(G)$. Hence

$$I \cap J \subseteq \mathcal{P}(N_0(G)). \tag{1}$$

We claim that $N_0(G)/I \cong N_0(G/H)$. We define $\theta : N_0(G) \to N_0(G/H)$ as follows. Let $\theta(a) : G/H \to G/H$ be the mapping defined by $(\theta(a))(g+H) := a(g)+H$ for all $a \in N_0(G)$, $g \in G$. Then $\theta(a)$ is well-defined, for let $g_1, g_2 \in G$ be such that $g_1 + H = g_2 + H$. Then g_1 and g_2 are contained in the same coset of H in G, i.e. in the same connected component of G. By continuity of a, $a(g_1)$ and $a(g_2)$ are in the same component, whence $a(g_1)+H = a(g_2)+H$, and so θ is well-defined. It is clear that $(\theta(a))\varphi(g) = (\varphi a)(g)$ for all $g \in G$, where $\varphi : G \to G/H$ is the canonical homomorphism. Hence $(\theta(a))\varphi = \varphi a$. Let U be open in G/H. Then $\varphi^{-1}((\theta(a))^{-1})(U) = (\theta(a)\varphi)^{-1}(U) = (\varphi a)^{-1}(U) = a^{-1}\varphi^{-1}(U)$. By definition of the quotient topology, $\varphi^{-1}(U)$ is open in G, and by the continuity of a, $a^{-1}\varphi^{-1}(U)$ is open in G. Again by the definition of the quotient topology, this implies that $((\theta(a))^{-1})(U)$ is open in G/H. Hence $\theta(a)$ is continuous. Moreover, $(\theta(a))(H) = (\theta(a))(0+H) = \theta(0) + H = H$ since $a(H) \subseteq H$. Hence $\theta(a) \in N_0(G/H)$. Clearly θ is a near-ring homomorphism. Let $b \in N_0(G/H)$. Let $G/H = \{C_i \mid i \in I\}$ and choose a coset representative g_i of C_i for each $i \in I$. In the case $C_j = H$, choose $g_j = 0$. Let $b(C_i) = C_{k_i}$ for each $i \in I$. We define $a : G \to G$ as follows: If $g \in C_i$ and, let $b(g) := g_{k_i}$. It follows from the fact that constant functions are continuous and that the components of G are open that a is continuous. Moreover $a(0) = 0$, since $b(H) = H$ and by our choice of coset representative for H. Hence $a \in N_0(G)$. Thus $\theta : N_0(G) \to N_0(G/H)$ is onto, so $N_0(G/H) \cong N_0(G)/\ker\theta$. Clearly, $\ker\theta = I$, so $N_0(G/H) \cong N_0(G)/I$. Moreover, the topology on G/H is discrete by Lemma 2.1, so $N_0(G/H) = M_0(G/H)$ and hence is equiprime. Hence I is an equiprime ideal of $N_0(G)$ and so

$$\mathcal{P}_e(N_0(G)) \subseteq I. \tag{2}$$

Now let $a \in N_0(G)$ and let $\lambda(a)$ be the restriction of a to H. Since a maps H into itself, $\lambda(a) \in N_0(H)$. It is also clear that $\lambda : N_0(G) \to N_0(H)$ is a near-ring homomorphism.

Moreover, if $b \in N_0(H)$, let a be defined by

$$a(g) := \begin{cases} b(g) & \text{if } g \in H \\ 0 & \text{if } g \in G \backslash H \,. \end{cases}$$

Since G has open components, a is continuous and so is in $N_0(G)$. Moreover, $b = \lambda(a)$, so $\lambda : N_0(G) \to N_0(H)$ is onto. It is clear that $\ker \lambda = J$ and hence $N_0(H) \cong N_0(G)/J$. Since H is arcwise connected, it follows from [1, Proposition 3.2] that $N_0(H)$ is equiprime and so J is an equiprime ideal of $N_0(G)$. Consequently

$$\mathcal{P}_e(N_0(G)) \subseteq J. \tag{3}$$

Combining (1), (2), (3) and the fact that $\mathcal{P}(N_0(G)) \subseteq \mathcal{P}_e(N_0(G))$ we obtain

$$I \cap J \subseteq \mathcal{P}(N_0(G)) \subseteq \mathcal{P}_e(N_0(G)) \subseteq I \cap J$$

and the proof is complete. □

3. Sandwich near-rings

Let X and G be a topological space and a topological group respectively, and let $\theta : G \longrightarrow X$ be a continuous map. The *sandwich near-ring* $N_0(G, X, \theta)$ is the set $\{a : X \longrightarrow G \mid a$ is continuous and $a\theta(0) = 0\}$. Addition is pointwise and multiplication is defined by $a \cdot b := a\theta b$. It is clear that $N_0(G, X, \theta)$ is a zerosymmetric near-ring with respect to these operations. If the topologies on X and G are discrete $N_0(G, X, \theta)$ consists of all mappings $a : X \to G$ satisfying $a\theta(0) = 0$. In this case we denote the near-ring by $M_0(G, X, \theta)$. In this section we will assume that both G and X have more than one element. The closure of a subset A of X will be denoted $\mathrm{cl}(A)$.

Lemma 3.1. *Let X and G be a completely regular topological space and an arcwise connected topological group, respectively. If $N_0(G, X, \theta)$ is 3-semiprime, then* $\mathrm{cl}(\theta(G)) = X$.

Proof. Suppose that $\mathrm{cl}(\theta(G)) \neq X$. Let $x \in X \backslash \mathrm{cl}(\theta(G))$. Since X is completely regular, there exists a continuous map $\alpha : X \to [0, 1]$ such that $\alpha(\mathrm{cl}(\theta(G))) = 0$ and $\alpha(x) = 1$. Let $0 \neq g \in G$. Since G is arcwise connected, there exists a continuous map $\beta : [0, 1] \to G$ such that $\beta(0) = 0$ and $\beta(1) = g$. Let $a := \beta\alpha$. Then a is continuous, and $a(y) = 0$ for all $y \in \mathrm{cl}(\theta(G))$. Moreover $a(x) = g$, so $a \neq 0$. Clearly $a \in N_0(G, X, \theta)$. Let $n \in N_0(G, X, \theta)$. If $y \in X$, then $a\theta n(y) = 0$, since $\theta n(y) \in \theta(G) \subseteq \mathrm{cl}(\theta(G))$. Hence $a \cdot n = 0$, whence $a \cdot n \cdot a = 0$ for all $n \in N_0(G, X, \theta)$. Since $a \neq 0$, $N_0(G, X, \theta)$ is not 3-semiprime, and the proof is complete. □

Proposition 3.2. *Let X and G be a completely regular topological space and an arcwise connected topological group, respectively, and let $\theta : G \to X$ be a continuous map such that $\theta^{-1}\theta(0) = \{0\}$. Then the following are equivalent:*

(a) $\mathrm{cl}(\theta(G)) = X$.
(b) $N_0(G, X, \theta)$ *is 3-prime.*
(c) $N_0(G, X, \theta)$ *is 3-semiprime.*

Proof. (a)$\Longrightarrow$(b): Let $c := \theta(0)$ and let $0 \neq a, b \in N_0(G, X, \theta)$. Then there exist $x, y \in X$ such that $a(x) \neq 0$, $b(y) \neq 0$. We may assume without loss of generality that $x \in \theta(G)$. For by continuity of a, there exists an open set U of X containing x, such that $a(z) \neq 0$ for all $z \in U$. Since $\mathrm{cl}(\theta(G)) = X$, $U \cap \theta(G) \neq \emptyset$. Hence we may choose $x \in U \cap \theta(G)$ such that $a(x) \neq 0$. Let $g \in G$ be such that $x = \theta(g)$. Note that $g \neq 0$, since by the definition of $N_0(G, X, \theta)$ this would imply that $a(\theta(0)) = 0$, i.e. $a(x) = 0$, which contradicts our assumption that $a(x) \neq 0$.

Since $b(y) \neq 0$ and $\theta^{-1}\theta(0) = \{0\}$, $\theta b(y) \neq c$. Let $d := \theta b(y)$. Since X is completely regular, it is T_1. Hence the set $F := \{c\}$ is closed and $d \notin F$. Again since X is completely regular, there exists a continuous map $\alpha : X \to [0, 1]$ such that $\alpha(F) = 0$ and $\alpha(d) = 1$. Since G is arcwise connected, there exists a continuous map $\beta : [0, 1] \to G$ such that $\beta(0) = 0$ and $\beta(1) = g$. Let $n := \beta\alpha$. Clearly, n is continuous. Moreover, $n(\theta(0)) = n(c) = \beta\alpha(c) = \beta(0) = 0$. Hence $n \in N_0(G, X, \theta)$. Also $n(d) = g$. Furthermore, $a\theta n\theta b(y) = a\theta n(d) = a\theta(g) = a(x) \neq 0$. It follows that $a \cdot n \cdot b \neq 0$. Hence $N_0(G, X, \theta)$ is 3-prime.

(b)$\Longrightarrow$(c): Obvious.

(c)$\Longrightarrow$(a): Follows from Lemma 3.1. □

The condition $\theta^{-1}\theta(0) = \{0\}$ cannot be omitted from the hypothesis of Proposition 3.2, as the following example shows.

Example 3.3. Let $X := G := \mathbb{R}$, both with the usual topology. Define $\theta : X \to G$ by

$$\theta(x) = \begin{cases} x - 1 & x \geq 1 \\ 0 & -1 < x < 1 \\ x + 1 & x \leq 1. \end{cases}$$

Then θ is continuous and surjective whence it holds trivially that $\mathrm{cl}(\theta(G)) = X$. Let $a(x) := \sin x$ for all $x \in X$. Then $0 \neq a \in N_0(G, X, \theta)$ and $a\theta n\theta a(x) = 0$ and hence $a \cdot n \cdot a = 0$ for all $n \in N_0(G, X, \theta)$. Thus $N_0(G, X, \theta)$ is not 3-semiprime.

Proposition 3.4. *Let X and G be a completely regular topological space and an arcwise connected topological group, respectively, and let $\theta : G \to X$ be a continuous, injective map. Then the following are equivalent:*

(a) $\mathrm{cl}(\theta(G)) = X$.

(b) *$N_0(G, X, \theta)$ is equiprime.*

(c) *$N_0(G, X, \theta)$ is 3-semiprime.*

Proof. (a)$\Longrightarrow$(b): Let $a, b, c \in N_0(G, X, \theta)$ be such that $a \neq 0$ and $b \neq c$. Then there exist $x, y \in X$ such that $a(x) \neq 0$ and $b(y) \neq c(y)$. As in the proof of Proposition 3.2 it may be shown that there exists $0 \neq g \in G$ such that $x = \theta(g)$. Since $b(y) \neq c(y)$ and θ is injective, $\theta b(y) \neq \theta c(y)$. Let $x_0 := \theta(0)$, $x_1 := \theta b(y)$ and $x_2 := \theta c(y)$. Either $x_1 \neq 0$ or $x_2 \neq 0$. Assume the latter. Since X is T_1 the set $F := \{x_0, x_1\}$ is closed, where $x_0 := \theta(0)$ and $x_2 \notin F$. Since X is completely regular, there exists a continuous map $\alpha : X \to [0, 1]$ such that $\alpha(F) = 0$ and $\alpha(x_2) = 1$. Since G is arcwise connected, there exists a continuous map $\beta : [0, 1] \to G$ such that $\beta(0) = 0$ and $\beta(1) = g$. Let $n := \beta\alpha$. Clearly, n is continuous. Moreover, $n(\theta(0)) = n(x_0) = \beta\alpha(x_0) = \beta(0) = 0$. Hence $n \in N_0(G, X, \theta)$. Also $n(x_1) = 0$

and $n(x_2) = g$. Furthermore, $a\theta n\theta b(y) = a\theta n(x_1) = a\theta(0) = 0$ and $a\theta n\theta c(y) = a\theta n(x_2) = a\theta(g) = a(x) \neq 0$. It follows that $a \cdot n \cdot b \neq a \cdot n \cdot c$. Hence $N_0(G, X, \theta)$ is equiprime.

(b)$\Longrightarrow$(c): Obvious.

(c)$\Longrightarrow$(a): Follows from Lemma 3.1. $\square$

If θ is not injective, conditions (a), (b) and (c) of Proposition 3.4 need not to be equivalent.

Example 3.5. Let $G := \mathbb{R}$ and $X := [0, \infty)$, both with the usual topology and let $\theta(g) := g^2$ for all $g \in G$. Then $\theta : G \to X$ is a surjection, so $\mathrm{cl}(\theta(G)) = X$ holds trivially. Clearly θ is not injective. Clearly this example satisfies the conditions of Proposition 3.2, so $N_0(G, X, \theta)$ is 3-semiprime. Now let $b(x) = x$ and $c(x) = -x$ for all $x \in X$. Then $b, c \in N_0(G, X, \theta)$ and $\theta b(x) = \theta c(x) = x$ for all $x \in X$. Let $0 \neq a \in N_0(G, X, \theta)$. If $n \in N_0(G, X, \theta)$, then $a\theta n\theta b(x) = a\theta n\theta c(x)$ for all $x \in X$. Hence $a \cdot n \cdot b = a \cdot n \cdot c$ for all $n \in N_0(G, X, \theta)$, but $a \neq 0$ and $b \neq c$. Thus $N_0(G, X, \theta)$ is not equiprime, so (a) and (c) of Proposition 3.4 hold, while (b) does not hold in this case.

Proposition 3.6. *Suppose that X is a 0-dimensional, T_0 space and that $\theta : G \to X$ is injective and that $\mathrm{cl}(\theta(G)) = G$. Then $N_0(G, X, \theta)$ is strongly prime if and only if the topology on X is discrete.*

Proof. Suppose that the topology on X is discrete. Then all mappings of X into G are continuous. Hence $N_0(G, X, \theta) = M_0(G, X, \theta)$. Moreover, $\theta(G) = \mathrm{cl}(\theta(G)) = X$, i.e. θ is surjective. It follows from [13, Proposition 9.1] that $N_0(G, X, \theta) \cong N_0(G)$, where G has the discrete topology. Hence by [1, Proposition 2.2(b)], $N_0(G, X, \theta)$ is strongly prime.

Conversely, suppose that the topology on X is not discrete. Let $c := \theta(0)$. Then $f(c) = 0$ for all $f \in N_0(G, X, \theta)$. Let U be a nonempty clopen set in X which does not contain c. Let $0 \neq g \in G$ and let $a : X \to G$ be defined by

$$a(x) := \begin{cases} g & x \in U \\ 0 & x \in X \backslash U\,. \end{cases}$$

Then $a \in N_0(G, X, \theta)$ and $a \neq 0$. Let $F := \{f_1, \ldots, f_n\}$ be a finite subset of $N_0(G, X, \theta)$. Since $X \backslash U$ is clopen and f_i is continuous $(\theta f_i)^{-1}(X \backslash U)$ is clopen. Let $V_i := f_i^{-1}(X \backslash U) \backslash U$. Then V_i is clopen and $c \in V_i$. Let $V := \bigcap_{i=1}^{n} V_i$. Then V is clopen and $c \in V$. Since X is T_0 and 0-dimensional, it is T_2. Since X is not discrete, V is infinite. Let W be a clopen set in X such that $c \notin W$ and $W \cap V \neq \emptyset$. Then $W \cap V$ is a clopen set in X. Since $\mathbf{c}(\theta(G)) = X$, $\theta(G) \cap W \cap V \neq \emptyset$. Let $d \in \theta(G) \cap W \cap V$ and let $h \in G$ be such that $d = \theta(h). \in V$. Then $\theta f_i(d) \in X \backslash U$ for $1 \leq i \leq n$. Define $0 \neq b \in N_0(G, X, \theta)$ by

$$b(x) := \begin{cases} h & x \in U \\ 0 & x \in X \backslash U\,. \end{cases}$$

If $x \in U$, then $a\theta f_i \theta b(x) = a\theta f_i \theta(h) = a\theta f_i(d) = 0$, since $\theta f_i(d) \in X \backslash U$. If $x \in X \backslash U$, then $a\theta f_i \theta b(x) = a\theta f_i \theta(0) = a\theta f_i(c) = a\theta 0) = a(c) = 0$. Thus $a\theta f_i \theta b = 0$, i.e. $a \cdot f_i \cdot b = 0$, $1 \leq i \leq n$, whence $a \cdot F \cdot b = 0$. Hence $N_0(G, X, \theta)$ is not strongly prime. $\square$

Corollary 3.7. *Suppose that X is a 0-dimensional, T_0 space and that $\theta : G \to X$ is injective and that $\mathrm{cl}(\theta(G)) = G$. Then $N_0(G, X, \theta)$ is strongly equiprime if and only if X is finite.*

Proof. Suppose that X is finite. Since X is T_0 and 0-dimensional, it is T_2 and hence discrete. As in the proof of Proposition 3.6, $N_0(G,X,\theta) = M_0(G,X,\theta)$ and hence from [13, Proposition 9.1] $N_0(G,X,\theta) \cong M_0(G)$. Now $\operatorname{card} G = \operatorname{card}(\theta(G)) \leq \operatorname{card} X$. Hence G is finite. It follows from [1, Proposition 2.2(c)] that $N_0(G,X,\theta)$ is strongly equiprime.

Conversely, suppose that $N_0(G,X,\theta)$ is strongly equiprime. Then $N_0(G,X,\theta)$ is strongly prime, and hence by Proposition 3.6, the topology on X is discrete. As in the proof of Proposition 3.6 we have that $N_0(G,X,\theta) \cong N_0(G)$, where G has the discrete (and hence 0-dimensional) topology. It follows from [1, Proposition 2.2(c)] that G is finite. Hence $\theta(G)$ is finite, and since X is discrete, $\theta(G) = \operatorname{cl}(\theta(G)) = X$ and hence X is finite. □

Proposition 3.8. *Suppose that X is a completely regular space, that G is arcwise connected and, that the topology on G has a base $\mathcal{B}$ consisting of arcwise connected open sets. Then $N_0(G,X,\theta)$ is not strongly prime (and hence not strongly equiprime).*

Proof. We consider the cases $\operatorname{cl}(\theta(G)) = G$ and $\operatorname{cl}(\theta(G)) \neq G$ separately. If $\operatorname{cl}(\theta(G)) \neq G$, it follows from Lemma 3.1 that $N_0(G,X,\theta)$ is not 3-semiprime and hence not strongly prime.

Suppose that $\operatorname{cl}(\theta(G)) = G$. Let $c := \theta(0)$. Let U be an open set in X containing c whose closure $\operatorname{cl}(U)$ is not X. The $X\backslash\operatorname{cl}(U)$ is nonempty and open. Since $\operatorname{cl}(\theta(G)) = X$, $(G\backslash\operatorname{cl}(U)) \cap \theta(G)$ contains an element, d, say. Let $g \in G$ be such that $d = \theta(g)$. Since X is completely regular, there exists a continuous function $\alpha : X \longrightarrow [0,1]$ such that $\alpha(\operatorname{cl}(U)) = 0$ and $\alpha(d) = 1$. Since G is arcwise connected, there exists a continuous function $\beta : [0,1] \to G$ such that $\beta(0) = 0$ and $\beta(1) = g$. Let $a := \beta\alpha$. Then $0 \neq a \in N_0(G,X,\theta)$ and $a(U) = 0$.

Now let $F := \{f_1, \dots, f_n\}$ be a finite subset of $N_0(G)$. Let $V_i := (\theta f_i)^{-1}(U)$ for $1 \leq i \leq n$ and $V := \bigcap_{i=1}^n V_i$. Note that $c \in V$. If $V = X$, $a\theta f_i = 0$ for $1 \leq i \leq n$ so $a \cdot F \cdot b = 0$ for any $0 \neq b \in N_0(G,X,\theta)$ and we are done. Suppose that $V \neq G$. Let W be an element of $\mathcal{B}$ such that $0 \in W \subseteq \theta^{-1}(V)$. We have that $W \neq \{0\}$, since then G would be discrete. Since G is arcwise connected, this would imply that G consists of one element, which contradicts the assumption of this paper. Hence $W\backslash\{0\}$ is nonempty. Let $h \in W\backslash\{0\}$ and let $e := \theta(h)$. Since X is completely regular, there exists a continuous function $\lambda : X \longrightarrow [0,1]$ such that $\lambda(c) = 0$ and $\lambda(e) = 1$. Since W is arcwise connected, there exists a continuous function $\mu : [0,1] \longrightarrow W$ with $\mu(0) = 0$ and $\mu(1) = h$. Let $b := \mu\lambda$. Then $0 \neq b \in N_0(G,X,\theta)$, $b(e) = h$ and $b(X) \subseteq W \subseteq \theta^{-1}(V)$. It follows that $a\theta f_i \theta b = 0$ for $1 \leq i \leq n$ so $a \cdot F \cdot b = 0$, but $b \neq 0$. Hence $N_0(G,X,\theta)$ is not strongly prime. □

References

[1] Booth, G. L.; Hall, P. R.: *Primeness in near-rings of continuous functions.* Beitr. Algebra Geom. **45**(1) (2004), 21–27. Zbl pre02096227

[2] Booth, G. L.; Groenewald, N. J.; Veldsman, S.: *A Kurosh-Amitsur prime radical for near-rings.* Commun. Algebra **18**(9) (1990), 3111–3122. Zbl 0706.16025

[3] Booth, G. L.; Groenewald, N. J.; Veldsman, S.: *Strongly equiprime near-rings.* Quaest. Math. **14**(4) (1991), 483–489. Zbl 0756.16012

[4] Groenewald, N. J.: *Strongly prime near-rings.* Proc. Edinburgh Math. Soc. **31**(3) (1988), 337–343. Zbl 0682.16028

[5] Groenewald, N. J.: *Prime near rings and special radicals.* East-West J. Math. **3**(2) (2001), 147–162. Zbl 1006.16060

[6] Handelman, D.; Lawrence, J.: *Strongly prime rings.* Trans. Amer. Math. Soc. **211** (1975), 209–223. Zbl 0345.16004

[7] Kaarli, K.: *Survey on the radical theory of near-rings.* Contrib. General Algebra **4**, 45–62. Teubner, Stuttgart 1987. Zbl 0647.16029

[8] Kaarli, K.; Kriis, T.: *On the prime radical of near-rings.* Tartu Riikl. Ul Toimetised **764** (1987), 23–29.

[9] Magill, K. D.: *Near-rings of continuous self-maps: a brief survey and some open problems.* Proc. Conf. San Bernadetto del Tronto 1981, 25–47, (1982). Zbl 0516.16027

[10] Magill, K. D.: *A survey of topological nearrings and nearrings of continuous functions.* Proc. Tenn. Top. Conf., World Scientific Pub. Co., Singapore 1997, 121–140. Zbl 0915.54033

[11] McCoy, N. H.: *Prime ideals in general rings.* Amer. J. Math. (1949), 823–833. Zbl 0035.01804

[12] Pilz, G.: *Near-rings.* 2nd ed., North-Holland, Amsterdam 1983. Zbl 0521.16028

[13] Veldsman, S: *On equiprime near-rings.* Commun. Algebra **20**(9) (1992), 2569–2587. Zbl 0795.16034

Received October 10, 2003