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On the Genus of the Graph of Tilting Modules

Dedicated to Idun Reiten on the occasion of her 60th birthday

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Let Λ be a finite dimensional, connected, associative algebra with unit over a field k. Let n be the number of isomorphism classes of simple Λ-modules. By $\operatorname{mod}\Lambda$ we denote the category of finite dimensional left Λ-modules.

A module ${}_\Lambda T \in \operatorname{mod}\Lambda$ is called a *tilting module* if

(i) the projective dimension $\operatorname{pd}{}_\Lambda T$ of ${}_\Lambda T$ is finite, and

(ii) $\operatorname{Ext}^i_\Lambda(T,T) = 0$ for all $i > 0$, and

(iii) there is an exact sequence $0 \to {}_\Lambda\Lambda \to {}_\Lambda T^1 \to \cdots \to {}_\Lambda T^d \to 0$ with ${}_\Lambda T^i \in \operatorname{add}{}_\Lambda T$ for all $1 \le i \le d$.

Here $\operatorname{add}{}_\Lambda T$ denotes the category of direct sums of direct summands of ${}_\Lambda T$.

Tilting modules play an important role in many branches of mathematics such as representation theory of Artin algebras or the theory of algebraic groups.

Let $\bigoplus_{i=1}^{m} T_i$ be the decomposition of ${}_\Lambda T$ into indecomposable direct summands. We call ${}_\Lambda T$ *basic* if ${}_\Lambda T_i \not\cong {}_\Lambda T_j$ for all $i \ne j$. A basic tilting module has n indecomposable direct summands.

A direct summand ${}_\Lambda M$ of a basic tilting module ${}_\Lambda T$ is called an *almost complete tilting module* if ${}_\Lambda M$ has $n-1$ indecomposable direct summands.

Let $\mathcal{T}(\Lambda)$ be the set of all non isomorphic basic tilting modules over Λ. We associate with $\mathcal{T}(\Lambda)$ a quiver $\overrightarrow{\mathcal{K}(\Lambda)}$ as follows: The vertices of $\overrightarrow{\mathcal{K}(\Lambda)}$ are the tilting modules in $\mathcal{T}(\Lambda)$, and there is an arrow ${}_\Lambda T' \to {}_\Lambda T$ if ${}_\Lambda T$ and ${}_\Lambda T'$ have a common direct summand which is an almost complete tilting module and if $\mathrm{Ext}^1_\Lambda(T,T') \neq 0$. We call $\overrightarrow{\mathcal{K}(\Lambda)}$ the *quiver of tilting modules* over Λ. With $\mathcal{K}(\Lambda)$ we denote the underlying graph of $\overrightarrow{\mathcal{K}(\Lambda)}$. It has been recently shown [7] that $\mathcal{K}(\Lambda)$ is the Hasse diagram of a partial order of tilting modules which was basically introduced in [10]. From this it follows, that $\overrightarrow{\mathcal{K}(\Lambda)}$ has no oriented cycles.

If $\overrightarrow{\mathcal{K}(\Lambda)}$ is finite, then it is connected. Examples show that $\overrightarrow{\mathcal{K}(\Lambda)}$ may be rather complicated. One measure for the complicatedness of a graph G is its genus $\gamma(G)$. This is the minimal genus of an orientable surface on which G can be embedded.

The aim of these notes is to show that there are finite quivers of tilting modules of arbitrary genus. To be precise, we prove:

Theorem 1. *For all integers $r \geq 0$ there is a representation finite, connected algebra Λ_r such that $\gamma(\mathcal{K}(\Lambda_r)) = r$.*

The proof of the theorem is constructive. For each $r \in \mathbb{N}$ we give an explicit example of an algebra Λ_r and embed $\mathcal{K}(\Lambda_r)$ in an orientable surface of genus r. This gives an upper bound for $\gamma(\mathcal{K}(\Lambda_r))$. Then we use general results from graph theory to show that the bound is sharp. This will be done in Section 3. In Section 1 we recall some basic facts about tilting modules and embeddings of graphs. In Section 2 we introduce the algebras Λ_r and derive some properties of $\overrightarrow{\mathcal{K}(\Lambda_r)}$. For unexplained terminology and results from representation theory we refer to [1], and from graph theory to [8].

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1. Preliminaries

1.1. The construction of $\overrightarrow{\mathcal{K}(\Lambda)}$

Let ${}_\Lambda M$ be a direct summand of a tilting module. A basic Λ-module ${}_\Lambda X$ is called a *complement* to ${}_\Lambda M$ if ${}_\Lambda M \oplus {}_\Lambda X$ is a tilting module and if $\operatorname{add} M \cap \operatorname{add} X = 0$. It was proved in [5] that every direct summand of a tilting module has a distinguished complement ${}_\Lambda X$ which is characterized by the fact that there is no epimorphism ${}_\Lambda E \to {}_\Lambda X$ with ${}_\Lambda E \in \operatorname{add} {}_\Lambda M$. The module ${}_\Lambda X$ is unique up to isomorphism, and it is called the *source complement* to ${}_\Lambda M$. There is the dual concept of a source complement. A complement ${}_\Lambda Y$ to ${}_\Lambda M$ is called a *sink complement* to a direct summand ${}_\Lambda M$ of a tilting module, if there is no monomorphism ${}_\Lambda Y \to {}_\Lambda E$ with ${}_\Lambda E \in \operatorname{add} {}_\Lambda M$. In contrast to source complements, sink complements do not always exist. If ${}_\Lambda M$ has a sink complement then it is unique up to isomorphism [6]. The source and the sink complement to an almost complete tilting module ${}_\Lambda M$ coincide if and only if ${}_\Lambda M$ is not faithful [4]. The following result is basically contained in [4], compare [6].

Proposition 1. *Let ${}_\Lambda M$ be a faithful almost complete tilting module. Let ${}_\Lambda X$ be a complement to ${}_\Lambda M$ which is not the sink complement to ${}_\Lambda M$. Then*

(1) *there is a complement ${}_\Lambda Y$ to ${}_\Lambda M$ which is not isomorphic to ${}_\Lambda X$,*

(2) *there is an exact sequence $\eta : 0 \to {}_\Lambda X \to {}_\Lambda E \to {}_\Lambda Y \to 0$ with ${}_\Lambda E \in \operatorname{add} {}_\Lambda M$,*

(3) $\mathrm{Ext}^i_\Lambda(X,Y) = 0$ *for all* $i > 0$, *and* $\mathrm{Ext}^i_\Lambda(Y,X) = 0$ *for all* $i > 1$,

(4) *the module* ${}_\Lambda Y$ *is uniquely determined by the property* (2).

We call η the sequence connecting the complements ${}_\Lambda X$ and ${}_\Lambda Y$ to ${}_\Lambda M$. This result allows an alternative definition of the quiver $\overrightarrow{\mathcal{K}(\Lambda)}$ which is more useful for calculations. The vertices are the elements from $\mathcal{T}(\Lambda)$ as above. There is an arrow ${}_\Lambda T' \to {}_\Lambda T$ in $\overrightarrow{\mathcal{K}(\Lambda)}$ if ${}_\Lambda T' = {}_\Lambda M \oplus {}_\Lambda X$ and ${}_\Lambda T = {}_\Lambda M \oplus {}_\Lambda Y$ where ${}_\Lambda X$ and ${}_\Lambda Y$ are indecomposable, and if there is an exact sequence $0 \to {}_\Lambda X \to {}_\Lambda E \to {}_\Lambda Y \to 0$ with ${}_\Lambda E \in \operatorname{add}{}_\Lambda M$.

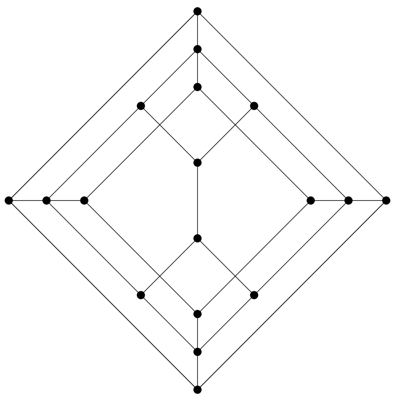
If $\overrightarrow{\mathcal{K}(\Lambda)}$ is finite, then it is connected. Then the definition of $\overrightarrow{\mathcal{K}(\Lambda)}$ yields an algorithm to construct $\overrightarrow{\mathcal{K}(\Lambda)}$. We write the tilting module ${}_\Lambda\Lambda$ as a direct sum of indecomposable modules ${}_\Lambda\Lambda = \bigoplus_{i=1}^{n} {}_\Lambda\Lambda_i$. Then ${}_\Lambda\Lambda_i$ is the source complement to ${}_\Lambda\Lambda[i] = \bigoplus_{j\neq i} {}_\Lambda\Lambda_j$. If ${}_\Lambda\Lambda_i$ is not the sink complement to ${}_\Lambda\Lambda[i]$ we construct the exact sequence $0 \to {}_\Lambda\Lambda_i \to {}_\Lambda E_i \to {}_\Lambda Y_i \to 0$ with ${}_\Lambda E_i \in \operatorname{add}{}_\Lambda\Lambda[i]$ connecting the complements ${}_\Lambda\Lambda_i$ and ${}_\Lambda Y_i$ to ${}_\Lambda\Lambda[i]$. In this way we construct all neighbors of ${}_\Lambda\Lambda$. We now proceed analogously with the neighbors of ${}_\Lambda\Lambda$ and all vertices we constructed. Since $\overrightarrow{\mathcal{K}(\Lambda)}$ is finite and connected and has no oriented cycles this algorithm stops when we constructed all basic tilting modules over Λ.

1.2. Embeddings of graphs

Let G be a connected, finite graph with p vertices and q edges. We think of G as embedded on a surface $\mathcal{S}$. Then G forms a polyhedron of genus $\gamma(G)$. From the Euler polyhedron formula Beinecke and Harary [3] deduce the following lower bound for $\gamma(G)$ which we shall use in Section 3.

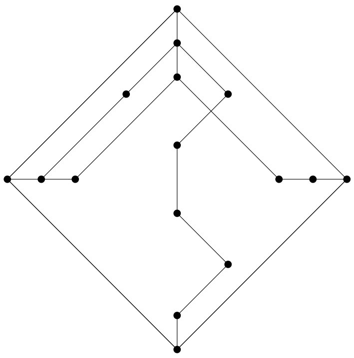
Proposition 2. *If G is connected and has no triangles, then* $\gamma(G) \geq \frac{1}{4}q - \frac{1}{2}(p-2)$.

In general this bound is not sharp. As an example we consider the following graph G which will become important in Section 3.

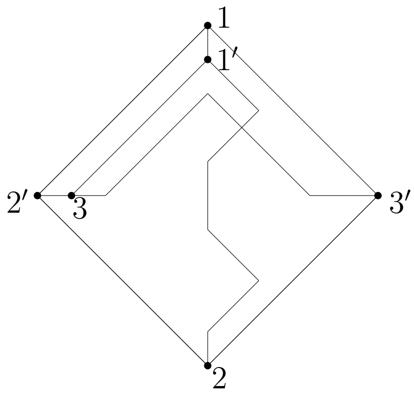


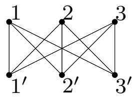
This graph has 18 vertices and 29 edges, hence the formula yields $\gamma(G) \geq -\frac{3}{4}$.

But G is not even planar, namely it contains the subgraph

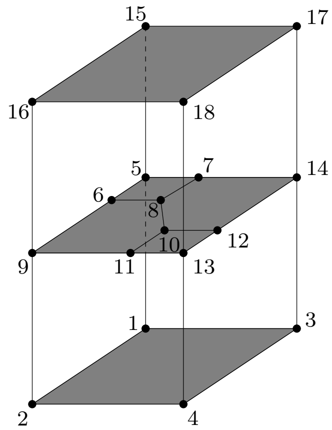


which is homeomorphic to

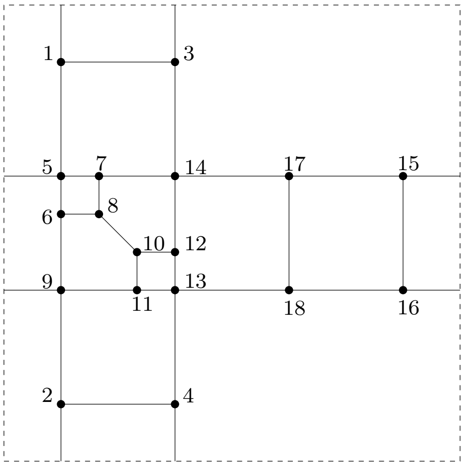


This graph is isomorphic to the complete bigraph $K_{3,3}$: 

. Kuratowski's theorem [9] implies $\gamma(G) \geq 1$. Conversely, we draw G differently and shade some of its faces:



We push a cylinder through the lower cube, close it under the upper square, adjust the vertices and edges accordingly and obtain an embedding of G on a torus. To be precise, the following figure shows an embedding of G on a torus:

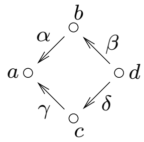


The parallel dotted lines have to be identified. Hence $\gamma(G) = 1$.

2. The algebras $\boldsymbol{\Lambda_r}$ and properties of $\overrightarrow{\mathcal{K}(\Lambda_r)}$

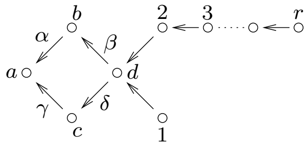
2.1. The algebras $\boldsymbol{\Lambda_r}$

Let Λ_1 be the path algebra of the quiver $\overrightarrow{\Delta}_1$:



bound by the relation $\alpha\beta = \gamma\delta$.

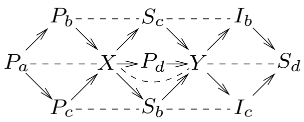
For all $r > 1$ let Λ_r be the path algebra of the quiver $\overrightarrow{\Delta}_r$:



bound by the relations $\alpha\beta = \gamma\delta$ and $\mathrm{rad}^2 = 0$, i.e. the composition of two consecutive arrows in $\overrightarrow{\Delta}_r \setminus \{a\}$ is zero.

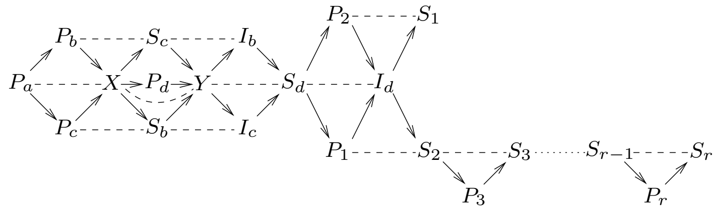
The Auslander-Reiten quivers $\overrightarrow{\Gamma}_{\Lambda_r}$ of Λ_r are as follows:

$\overrightarrow{\Gamma}_{\Lambda_1}$



and for $r > 1$

$\overrightarrow{\Gamma}_{\Lambda_r}$



Here S_x denotes the simple module corresponding to the vertex x, the module P_x is the projective cover of S_x and I_x denotes the injective hull of I_x. Moreover, X is the radical of $P_d = I_a$ and $Y = I_a/\operatorname{soc} I_a$, where $\operatorname{soc} I_a$ is the socle of I_a.

For all $1 \leq i \leq r$ we identify an indecomposable Λ_i-module ${}_{\Lambda_i}M$ with the corresponding Λ_j-module ${}_{\Lambda_j}M$, $j \geq i$, whose support is Λ_i. With this identification $\overrightarrow{\Gamma}_{\Lambda_i}$ is a full, convex subquiver of $\overrightarrow{\Gamma}_{\Lambda_j}$ for all $1 \leq i < j \leq r$.

We have $\operatorname{gl\,dim}\Lambda_i = i + 1$ for all $1 \leq i \leq r$, where $\operatorname{gl\,dim}\Lambda$ denotes the global dimension of an algebra Λ. The simple module S_d is the unique indecomposable module of projective dimension 2, the modules I_d, S_1, S_2 are the unique indecomposable modules of projective dimension 3, and for all $3 \leq j \leq r$ the module S_j is the unique indecomposable module of projective dimension $j + 1$. These observations show:

Remark 1. Let $1 \leq j \leq r - 1$. A non projective indecomposable module ${}_{\Lambda_j}X$ lies in $\operatorname{mod}\Lambda_j \setminus \operatorname{mod}\Lambda_{j-1}$ if and only if $\operatorname{pd}_{\Lambda_j} X = j + 1$.

2.2. Properties of the quiver $\overrightarrow{\mathcal{K}(\Lambda_r)}$

The following technical lemmas roughly describe the structure of the quiver $\overrightarrow{\mathcal{K}(\Lambda_r)}$. Let r be an integer, $r \geq 2$, and let $1 \leq i < j \leq r$. We decompose the projective module ${}_{\Lambda_j}\Lambda_j$ into ${}_{\Lambda_j}\Lambda_j = {}_{\Lambda_j}\Lambda_i \oplus {}_{\Lambda_j}P_{ij}$. Hence ${}_{\Lambda_j}P_{ij}$ is the maximal direct summand of ${}_{\Lambda_j}\Lambda_j$ with $\operatorname{add}{}_{\Lambda_j}P_{ij} \cap \operatorname{add}{}_{\Lambda_j}\Lambda_i = 0$.

Lemma 1. *Let $1 \leq i < j \leq r$, and let ${}_{\Lambda_i}T$ and ${}_{\Lambda_i}T'$ be tilting modules over Λ_i. Then*

(a) *${}_{\Lambda_j}T \oplus {}_{\Lambda_j}M$ is a tilting module over Λ_j if and only if ${}_{\Lambda_j}M = {}_{\Lambda_j}P_{ij}$.*

(b) *${}_{\Lambda_j}T' \oplus {}_{\Lambda_j}P_{ij} \to {}_{\Lambda_j}T \oplus {}_{\Lambda_j}P_{ij}$ is an arrow in $\overrightarrow{\mathcal{K}(\Lambda_j)}$ if and only if ${}_{\Lambda_i}T' \to {}_{\Lambda_i}T$ is an arrow in $\overrightarrow{\mathcal{K}(\Lambda_i)}$.*

Proof. (a) Since ${}_{\Lambda_j}P_{ij}$ is projective, $\operatorname{Ext}^k_{\Lambda_j}(P_{ij}, T) = 0$ for all $k > 0$. Since no indecomposable direct summand of ${}_{\Lambda_j}T$ is a successor of an indecomposable direct summand of ${}_{\Lambda_j}P_{ij}$ in the Auslander-Reiten quiver of Λ_j, it follows that $\operatorname{Ext}^k_{\Lambda_j}(T, P_{ij}) = 0$ for all $k > 0$. Hence ${}_{\Lambda_j}T \oplus {}_{\Lambda_j}P_{ij}$ is a tilting module. The module ${}_{\Lambda_j}P_{ij}$ is the source and the sink complement to ${}_{\Lambda_j}T$, hence the unique complement.

(b) There is an arrow ${}_{\Lambda_j}T' \oplus {}_{\Lambda_j}P_{ij} \to {}_{\Lambda_j}T \oplus {}_{\Lambda_j}P_{ij}$ if and only if $\operatorname{Ext}^1_{\Lambda_j}(T \oplus P_{ij}, T' \oplus P_{ij}) \neq 0$ and if ${}_{\Lambda_j}T \oplus {}_{\Lambda_j}P_{ij}$ and ${}_{\Lambda_j}T' \oplus {}_{\Lambda_j}P_{ij}$ have a common direct summand which is an almost complete tilting module. Equivalently, $\mathrm{Ext}^1_{\Lambda_i}(T,T') \neq 0$ and ${}_{\Lambda_i}T$ and ${}_{\Lambda_i}T'$ have a common direct summand which is an almost complete tilting module, hence if and only if there is an arrow ${}_{\Lambda_i}T' \to {}_{\Lambda_i}T$ in $\overrightarrow{\mathcal{K}(\Lambda_i)}$. $\square$

In particular, we may identify $\overrightarrow{\mathcal{K}(\Lambda_i)}$ with the full convex subquiver of $\overrightarrow{\mathcal{K}(\Lambda_r)}$, $1 \leq i < r$, with vertices ${}_{\Lambda_r}T \oplus {}_{\Lambda_r}P_{ir}$ where ${}_{\Lambda_i}T$ are the tilting modules over Λ_i. With this identification, the building blocks of $\overrightarrow{\mathcal{K}(\Lambda_r)}$ are the subquivers $\overrightarrow{\mathcal{K}(\Lambda_i)} \setminus \overrightarrow{\mathcal{K}(\Lambda_{i-1})}$ of $\overrightarrow{\mathcal{K}(\Lambda_r)}$ with $1 \leq i \leq r$. To simplify the notation we denote by $\overrightarrow{\mathcal{K}(\Lambda_1)} \setminus \overrightarrow{\mathcal{K}(\Lambda_0)}$ the subquiver $\overrightarrow{\mathcal{K}(\Lambda_1)}$ of $\overrightarrow{\mathcal{K}(\Lambda_r)}$. The next lemma gives an algebraic description of the vertices in $\overrightarrow{\mathcal{K}(\Lambda_i)} \setminus \overrightarrow{\mathcal{K}(\Lambda_{i-1})}$ of $\overrightarrow{\mathcal{K}(\Lambda_r)}$ with $1 < i \leq r$.

Lemma 2. *For all $1 < i \leq r$, the subquiver $\overrightarrow{\mathcal{K}(\Lambda_i)} \setminus \overrightarrow{\mathcal{K}(\Lambda_{i-1})}$ of $\overrightarrow{\mathcal{K}(\Lambda_r)}$ has as vertices all tilting modules of projective dimension $i+1$.*

Proof. With the previous lemma, ${}_{\Lambda_r}T \in \overrightarrow{\mathcal{K}(\Lambda_i)}$ if and only if ${}_{\Lambda_r}T = {}_{\Lambda_r}T' \oplus {}_{\Lambda_r}P_{ir}$ with ${}_{\Lambda_i}T'$ a tilting module over Λ_i. Using the lemma again, ${}_{\Lambda_i}T' \notin \overrightarrow{\mathcal{K}(\Lambda_{i-1})}$ if and only if there is an indecomposable, non projective direct summand ${}_{\Lambda_r}X$ of ${}_{\Lambda_r}T'$ with ${}_{\Lambda_r}X \in \operatorname{mod}\Lambda_i \setminus \operatorname{mod}\Lambda_{i-1}$. With the remark in 2.1, this holds if and only if $\operatorname{pd}_{\Lambda_i}X = i+1$. $\square$

Next we study arrows in $\overrightarrow{\mathcal{K}(\Lambda_r)}$ between vertices in different building blocks of $\overrightarrow{\mathcal{K}(\Lambda_r)}$.

Lemma 3. *Let $1 \leq i < j \leq r$. Let ${}_{\Lambda_r}T' \in \overrightarrow{\mathcal{K}(\Lambda_i)} \setminus \overrightarrow{\mathcal{K}(\Lambda_{i-1})}$ and ${}_{\Lambda_r}T \in \overrightarrow{\mathcal{K}(\Lambda_j)} \setminus \overrightarrow{\mathcal{K}(\Lambda_{j-1})}$ be tilting modules over Λ_r.*

(a) *There are no arrows ${}_{\Lambda_r}T \to {}_{\Lambda_r}T'$ in $\overrightarrow{\mathcal{K}(\Lambda_r)}$.*

(b) *If there is an arrow ${}_{\Lambda_r}T' \to {}_{\Lambda_r}T$ in $\overrightarrow{\mathcal{K}(\Lambda_r)}$ then $\operatorname{pd}_{\Lambda_r}T' = i+1$ and $\operatorname{pd}_{\Lambda_r}T = i+2$. In particular, $j = i+1$.*

Proof. (a) Assume there is an arrow ${}_{\Lambda_r}T \to {}_{\Lambda_r}T'$ in $\overrightarrow{\mathcal{K}(\Lambda_r)}$ with ${}_{\Lambda_r}T' \in \overrightarrow{\mathcal{K}(\Lambda_i)} \setminus \overrightarrow{\mathcal{K}(\Lambda_{i-1})}$ and ${}_{\Lambda_r}T \in \overrightarrow{\mathcal{K}(\Lambda_j)} \setminus \overrightarrow{\mathcal{K}(\Lambda_{j-1})}$ and $i < j$. Then ${}_{\Lambda_r}T' = {}_{\Lambda_r}\overline{T'} \oplus {}_{\Lambda_r}P_{ir}$ and ${}_{\Lambda_r}T = {}_{\Lambda_r}\overline{T} \oplus {}_{\Lambda_r}P_{jr}$, where ${}_{\Lambda_i}\overline{T'}$ and ${}_{\Lambda_j}\overline{T}$ are tilting modules over Λ_i respectively Λ_j. Note that ${}_{\Lambda_r}P_{jr}$ is a direct summand of ${}_{\Lambda_r}P_{ir}$. Then $0 \neq \mathrm{Ext}^1_{\Lambda_r}(T',T) = \mathrm{Ext}^1_{\Lambda_r}(\overline{T'},\overline{T}) = \mathrm{Ext}^1_{\Lambda_j}(\overline{T'},\overline{T})$. Since ${}_{\Lambda_r}T \in \overrightarrow{\mathcal{K}(\Lambda_j)} \setminus \overrightarrow{\mathcal{K}(\Lambda_{j-1})}$ there is an indecomposable direct summand ${}_{\Lambda_j}X \in \operatorname{mod}\Lambda_j \setminus \Lambda_{j-1}$ of ${}_{\Lambda_j}\overline{T}$ with $\mathrm{Ext}^1_{\Lambda_j}(\overline{T'},X) \neq 0$. This is a contradiction since ${}_{\Lambda_j}X$ is not a predecessor of an indecomposable direct summand of ${}_{\Lambda_j}T'$ in $\overrightarrow{\Gamma}_{\Lambda_j}$.

(b) Let ${}_{\Lambda_r}T' \to {}_{\Lambda_r}T$ in $\overrightarrow{\mathcal{K}(\Lambda_r)}$ be an arrow in $\overrightarrow{\mathcal{K}(\Lambda_r)}$ with ${}_{\Lambda_r}T' \in \overrightarrow{\mathcal{K}(\Lambda_i)} \setminus \overrightarrow{\mathcal{K}(\Lambda_{i-1})}$ and ${}_{\Lambda_r}T \in \overrightarrow{\mathcal{K}(\Lambda_j)} \setminus \overrightarrow{\mathcal{K}(\Lambda_{j-1})}$. Let $\eta : 0 \to {}_{\Lambda_r}X \to {}_{\Lambda_r}E \to {}_{\Lambda_r}Y \to 0$ be the corresponding sequence connecting the complements ${}_{\Lambda_r}X$ and ${}_{\Lambda_r}Y$, where ${}_{\Lambda_r}T' = {}_{\Lambda_r}X \oplus {}_{\Lambda_r}M$ and ${}_{\Lambda_r}T = {}_{\Lambda_r}Y \oplus {}_{\Lambda_r}M$. Since ${}_{\Lambda_r}T \in \overrightarrow{\mathcal{K}(\Lambda_j)} \setminus \overrightarrow{\mathcal{K}(\Lambda_{j-1})}$ it follows that ${}_{\Lambda_r}Y \in \operatorname{mod}\Lambda_j \setminus \operatorname{mod}\Lambda_{j-1}$, hence $\operatorname{pd}_{\Lambda_r}Y = j+1$. Let ${}_{\Lambda_r}Z \in \operatorname{mod}\Lambda_r$ with $\mathrm{Ext}^{j+1}_{\Lambda_r}(Y,Z) \neq 0$. We apply $\mathrm{Hom}_{\Lambda_r}(-,Z)$ to η and obtain $\operatorname{pd}_{\Lambda_r}X = j$. Since ${}_{\Lambda_r}T' \in \overrightarrow{\mathcal{K}(\Lambda_i)} \setminus \overrightarrow{\mathcal{K}(\Lambda_{i-1})}$ and $i \neq j$ we get $j = i+1$, the assertion. $\square$

As a consequence we obtain

Lemma 4. *Let $r > 1$.*

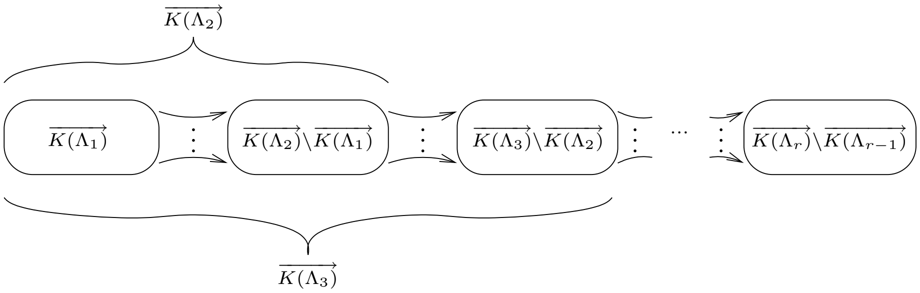
(a) *There is an arrow ${}_{\Lambda_r}T' \to {}_{\Lambda_r}T$ in $\overrightarrow{\mathcal{K}(\Lambda_r)}$ with ${}_{\Lambda_r}T' \in \overrightarrow{\mathcal{K}(\Lambda_1)}$ and ${}_{\Lambda_r}T \in \overrightarrow{\mathcal{K}(\Lambda_2)} \setminus \overrightarrow{\mathcal{K}(\Lambda_1)}$ if and only if ${}_{\Lambda_r}T' = {}_{\Lambda_r}S_d \oplus {}_{\Lambda_r}M$ and ${}_{\Lambda_r}T = {}_{\Lambda_r}I_d \oplus {}_{\Lambda_r}M$.*

(b) *Let $3 \leq i \leq r$. There is an arrow ${}_{\Lambda_r}T' \to {}_{\Lambda_r}T$ in $\overrightarrow{\mathcal{K}(\Lambda_r)}$ with ${}_{\Lambda_r}T' \in \overrightarrow{\mathcal{K}(\Lambda_{i-1})}$ and ${}_{\Lambda_r}T \in \overrightarrow{\mathcal{K}(\Lambda_i)} \setminus \overrightarrow{\mathcal{K}(\Lambda_{i-1})}$ if and only if ${}_{\Lambda_r}T' = {}_{\Lambda_r}S_{i-1} \oplus {}_{\Lambda_r}M$ and ${}_{\Lambda_r}T = {}_{\Lambda_r}S_i \oplus {}_{\Lambda_r}M$.*

Proof. (a) Let ${}_{\Lambda_r}T' \to {}_{\Lambda_r}T$ be an arrow in $\overrightarrow{\mathcal{K}(\Lambda_r)}$ with ${}_{\Lambda_r}T' \in \overrightarrow{\mathcal{K}(\Lambda_1)}$ and ${}_{\Lambda_r}T \in \overrightarrow{\mathcal{K}(\Lambda_2)} \setminus \overrightarrow{\mathcal{K}(\Lambda_1)}$. Then $\operatorname{pd}{}_{\Lambda_r}T = 3$ with Lemma 2 and $\operatorname{pd}{}_{\Lambda_r}T' = 2$. Then ${}_{\Lambda_r}S_d$ is a direct summand of ${}_{\Lambda_r}T$. Moreover, Lemma 1 shows that ${}_{\Lambda_r}P_1 \oplus {}_{\Lambda_r}P_2$ is a direct summand of ${}_{\Lambda_r}T$. Hence the sequence connecting the complements is the Auslander-Reiten sequence, which implies that ${}_{\Lambda_r}T' = {}_{\Lambda_r}S_d \oplus {}_{\Lambda_r}M$ and ${}_{\Lambda_r}T = {}_{\Lambda_r}I_d \oplus {}_{\Lambda_r}M$. Conversely, if ${}_{\Lambda_r}T' = {}_{\Lambda_r}S_d \oplus {}_{\Lambda_r}M$ and ${}_{\Lambda_r}T = {}_{\Lambda_r}I_d \oplus {}_{\Lambda_r}M$, the Auslander-Reiten sequence starting in ${}_{\Lambda_r}S_d$ lies in $\operatorname{add}({}_{\Lambda_r}T \oplus {}_{\Lambda_r}T')$. Hence we obtain an arrow ${}_{\Lambda_r}T' \to {}_{\Lambda_r}T$ in $\overrightarrow{\mathcal{K}(\Lambda_r)}$ with ${}_{\Lambda_r}T' \in \overrightarrow{\mathcal{K}(\Lambda_1)}$ and ${}_{\Lambda_r}T \in \overrightarrow{\mathcal{K}(\Lambda_2)} \setminus \overrightarrow{\mathcal{K}(\Lambda_1)}$.

(b) Let $3 \leq i \leq r$ and let ${}_{\Lambda_r}T' \to {}_{\Lambda_r}T$ be an arrow in $\overrightarrow{\mathcal{K}(\Lambda_r)}$ with ${}_{\Lambda_r}T' \in \overrightarrow{\mathcal{K}(\Lambda_{i-1})}$ and ${}_{\Lambda_r}T \in \overrightarrow{\mathcal{K}(\Lambda_i)} \setminus \overrightarrow{\mathcal{K}(\Lambda_{i-1})}$. Then $\operatorname{pd}{}_{\Lambda_r}T = i+1$ and $\operatorname{pd}{}_{\Lambda_r}T' = i$. Since $2 < i$ it follows that ${}_{\Lambda_r}S_{i-1}$ is a direct summand of of ${}_{\Lambda_r}T'$ and ${}_{\Lambda_r}S_i$ is a direct summand of of ${}_{\Lambda_r}T$. Conversely, if ${}_{\Lambda_r}T' = {}_{\Lambda_r}S_{i-1} \oplus {}_{\Lambda_r}M$ and ${}_{\Lambda_r}T = {}_{\Lambda_r}S_i \oplus {}_{\Lambda_r}M$ then the Auslander-Reiten sequence starting in ${}_{\Lambda_r}S_{i-1}$ lies in $\operatorname{add}({}_{\Lambda_r}T \oplus {}_{\Lambda_r}T')$. This yields an arrow ${}_{\Lambda_r}T' \to {}_{\Lambda_r}T$ in $\overrightarrow{\mathcal{K}(\Lambda_r)}$ with ${}_{\Lambda_r}T' \in \overrightarrow{\mathcal{K}(\Lambda_{i-1})}$ and ${}_{\Lambda_r}T \in \overrightarrow{\mathcal{K}(\Lambda_i)} \setminus \overrightarrow{\mathcal{K}(\Lambda_{i-1})}$. $\square$

To summarize our observations in this section we obtain the following structure of $\overrightarrow{\mathcal{K}(\Lambda_r)}$:



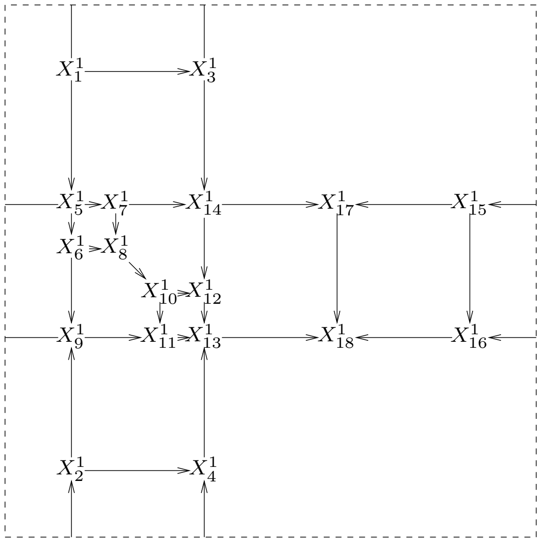
There are arrows from vertices in $\overrightarrow{\mathcal{K}(\Lambda_i)} \setminus \overrightarrow{\mathcal{K}(\Lambda_{i-1})}$ to vertices in $\overrightarrow{\mathcal{K}(\Lambda_j)} \setminus \overrightarrow{\mathcal{K}(\Lambda_{j-1})}$ if and only if $j = i+1$.

3. The proof of the theorem

3.1. An embedding of $\overrightarrow{\mathcal{K}(\Lambda_r)}$

We use induction on r to embed $\overrightarrow{\mathcal{K}(\Lambda_r)}$ on a surface of genus r.

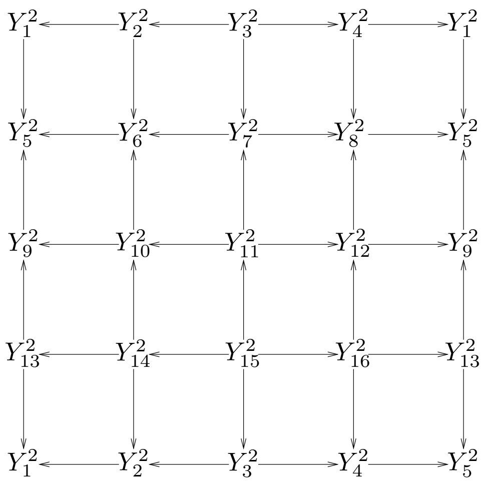
Let $r = 1$. Direct calculation shows that $\overrightarrow{\mathcal{K}(\Lambda_1)}$ equals



where the parallel dotted lines have to be identified. We saw in 1.2 that the underlying graph $\overline{\mathcal{K}(\Lambda_1))}$ of $\overrightarrow{\mathcal{K}(\Lambda_1)}$ has genus 1, hence can be embedded on a torus T_1. The vertices of $\overrightarrow{\mathcal{K}(\Lambda_1)}$ are the tilting modules

$$
\begin{array}{lll}
X_1^1 = P_a \oplus P_b \oplus P_c \oplus P_d, & X_2^1 = P_a \oplus P_c \oplus I_c \oplus P_d, & X_3^1 = P_a \oplus P_b \oplus I_b \oplus P_d, \\
X_4^1 = P_a \oplus I_b \oplus I_c \oplus P_d, & X_5^1 = P_b \oplus P_c \oplus X \oplus P_d, & X_6^1 = P_c \oplus S_c \oplus X \oplus P_d, \\
X_7^1 = P_b \oplus S_b \oplus X \oplus P_d, & X_8^1 = S_b \oplus S_c \oplus X \oplus P_d, & X_9^1 = P_c \oplus I_c \oplus S_c \oplus P_d, \\
X_{10}^1 = S_b \oplus S_c \oplus Y \oplus P_d, & X_{11}^1 = S_c \oplus I_c \oplus Y \oplus P_d, & X_{12}^1 = S_b \oplus I_b \oplus Y \oplus P_d, \\
X_{13}^1 = I_b \oplus I_c \oplus Y \oplus P_d, & X_{14}^1 = P_b \oplus I_b \oplus S_b \oplus P_d, & X_{15}^1 = P_b \oplus P_c \oplus S_d \oplus P_d, \\
X_{16}^1 = P_c \oplus I_c \oplus S_d \oplus P_d, & X_{17}^1 = P_b \oplus I_b \oplus S_d \oplus P_d, & X_{18}^1 = I_b \oplus I_c \oplus S_d \oplus P_d.
\end{array}
$$

Let $r = 2$. The quiver $\overrightarrow{\mathcal{K}(\Lambda_1)}$ is the full convex subquiver of $\overrightarrow{\mathcal{K}(\Lambda_2)}$ with vertices ${}_{\Lambda_2}X_i^2 = {}_{\Lambda_2}X_i^1 \oplus {}_{\Lambda_2}P_{12}$, where ${}_{\Lambda_2}P_{12} = {}_{\Lambda_2}P_1 \oplus {}_{\Lambda_2}P_2$. The quiver $\overrightarrow{\mathcal{K}(\Lambda_2)} \setminus \overrightarrow{\mathcal{K}(\Lambda_1)}$ is the full convex subquiver of $\overrightarrow{\mathcal{K}(\Lambda_2)}$ with vertices the tilting modules of projective dimension 3. Direct calculations show that $\overrightarrow{\mathcal{K}(\Lambda_2)} \setminus \overrightarrow{\mathcal{K}(\Lambda_1)}$ is

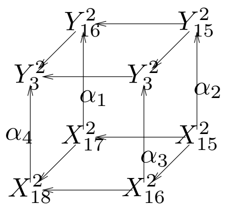


where we identify along the parallel horizontal, respectively vertical lines. It follows that $\overrightarrow{\mathcal{K}(\Lambda_2)} \setminus \overrightarrow{\mathcal{K}(\Lambda_1)}$ can be embedded on a torus T_2. The vertices of $\overrightarrow{\mathcal{K}(\Lambda_2)} \setminus \overrightarrow{\mathcal{K}(\Lambda_1)}$ are the tilting modules

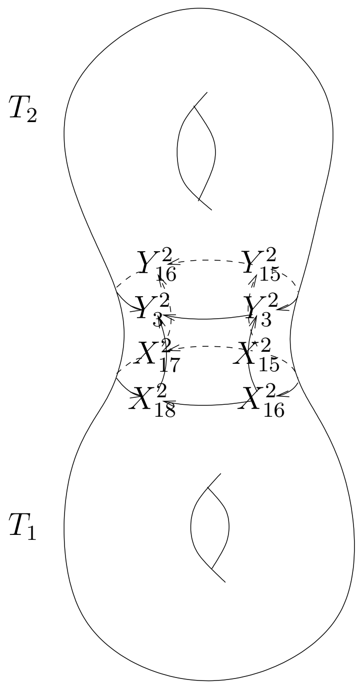
$$\begin{array}{ll}
Y_1^2 = P_d \oplus I_d \oplus I_c \oplus I_b \oplus P_1 \oplus S_1, & Y_2^2 = P_d \oplus I_d \oplus I_c \oplus P_c \oplus P_1 \oplus S_1, \\
Y_3^2 = P_d \oplus I_d \oplus I_c \oplus P_c \oplus P_1 \oplus P_2, & Y_4^2 = P_d \oplus I_d \oplus I_b \oplus I_c \oplus P_1 \oplus P_2, \\
Y_5^2 = P_d \oplus I_d \oplus I_c \oplus I_b \oplus S_2 \oplus S_1, & Y_6^2 = P_d \oplus I_d \oplus I_c \oplus P_c \oplus S_2 \oplus S_1, \\
Y_7^2 = P_d \oplus I_d \oplus I_c \oplus P_c \oplus S_2 \oplus P_2, & Y_8^2 = P_d \oplus I_d \oplus I_c \oplus I_b \oplus S_2 \oplus P_2, \\
Y_9^2 = P_d \oplus I_d \oplus P_b \oplus I_b \oplus S_2 \oplus S_1, & Y_{10}^2 = P_d \oplus I_d \oplus P_b \oplus P_c \oplus S_2 \oplus S_1, \\
Y_{11}^2 = P_d \oplus I_d \oplus P_b \oplus P_c \oplus S_2 \oplus P_2, & Y_{12}^2 = P_d \oplus I_d \oplus P_b \oplus I_b \oplus S_2 \oplus P_2, \\
Y_{13}^2 = P_d \oplus I_d \oplus P_b \oplus I_b \oplus P_1 \oplus S_1, & Y_{14}^2 = P_d \oplus I_d \oplus P_b \oplus P_c \oplus P_1 \oplus S_1, \\
Y_{15}^2 = P_d \oplus I_d \oplus P_b \oplus P_c \oplus P_1 \oplus P_2, & Y_{16}^2 = P_d \oplus I_d \oplus P_b \oplus I_b \oplus P_1 \oplus P_2.
\end{array}$$

The subquivers $\overrightarrow{Q}_1 : \begin{array}{ccc} X_{17}^2 & \longleftarrow & X_{15}^2 \\ \downarrow & & \downarrow \\ X_{18}^2 & \longleftarrow & X_{16}^2 \end{array}$ and $\overrightarrow{Q}_2 : \begin{array}{ccc} Y_{16}^2 & \longleftarrow & Y_{15}^2 \\ \downarrow & & \downarrow \\ Y_4^2 & \longleftarrow & Y_3^2 \end{array}$

bound squares on T_1 respectively T_2. In $\overrightarrow{\mathcal{K}(\Lambda_2)}$ they are joint as follows:



We cut out the interiors of $\overrightarrow{Q}_1$ on T_1 and $\overrightarrow{Q}_2$ on T_2 and insert a cylinder connecting T_1 and T_2. We obtain a surface of genus 2 on which $\overrightarrow{\mathcal{K}(\Lambda_2)}$ can be embedded:



Let $r > 2$. We abbreviate the injective Λ_r-module by ${}_{\Lambda_r}P_d \oplus {}_{\Lambda_r}I_d$ by ${}_{\Lambda_r}I$ and the projective-injective Λ_r-module $\bigoplus_{i=3}^{r} {}_{\Lambda_r}P_i$ by ${}_{\Lambda_r}P_{2r}$. Direct calculations show that $\overrightarrow{\mathcal{K}(\Lambda_r)} \setminus \overrightarrow{\mathcal{K}(\Lambda_{r-1})}$ is

$$
\begin{tikzcd}
 & Z_5 & & Z_8 \arrow[ll] \\
Z_6 \arrow[ur] & & Z_7 \arrow[ll] \arrow[ur] & \\
 & Z_1 \arrow[uu] & & Z_4 \arrow[ll] \arrow[uu] \\
Z_2 \arrow[ur] \arrow[uu] & & Z_3 \arrow[ll] \arrow[ur] \arrow[uu] &
\end{tikzcd}
$$

with

$$
\begin{array}{ll}
Z_1 = I \oplus P_{2r} \oplus S_r \oplus I_c \oplus P_c \oplus S_1, & Z_2 = I \oplus P_{2r} \oplus S_r \oplus P_b \oplus I_c \oplus S_1, \\
Z_3 = I \oplus P_{2r} \oplus S_r \oplus P_b \oplus P_c \oplus P_2, & Z_4 = I \oplus P_{2r} \oplus S_r \oplus I_c \oplus P_c \oplus P_2, \\
Z_5 = I \oplus P_{2r} \oplus S_r \oplus I_c \oplus I_b \oplus S_1, & Z_6 = I \oplus P_{2r} \oplus S_r \oplus P_b \oplus I_b \oplus P_2 \\
Z_7 = I \oplus P_{2r} \oplus S_r \oplus P_b \oplus I_b \oplus P_2, & Z_8 = I \oplus P_{2r} \oplus S_r \oplus I_c \oplus I_b \oplus P_2.
\end{array}
$$

We assume by induction that $\overrightarrow{\mathcal{K}(\Lambda_{r-1})}$ is embedded on a surface $\mathcal{S}_{r-1}$ of genus $r-1$ such that

a)

$$
\overrightarrow{Q}_1 : \begin{array}{ccc} Z_5' & \longleftarrow & Z_1' \\ \uparrow & & \uparrow \\ Z_6' & \longleftarrow & Z_2' \end{array} \quad \text{and} \quad \overrightarrow{Q}_2 : \begin{array}{ccc} Z_4' & \longrightarrow & Z_8' \\ \uparrow & & \uparrow \\ Z_3' & \longrightarrow & Z_7' \end{array}
$$

or

b)

$$
\overrightarrow{Q}_3 : \begin{array}{ccc} Z_6' & \longleftarrow & Z_7' \\ \uparrow & & \uparrow \\ Z_2' & \longleftarrow & Z_3' \end{array} \quad \text{and} \quad \overrightarrow{Q}_4 : \begin{array}{ccc} Z_5' & \longleftarrow & Z_8' \\ \uparrow & & \uparrow \\ Z_1' & \longleftarrow & Z_4' \end{array}
$$

bound squares on $\mathcal{S}_{r-1}$. Here Z_i', $1 \leq i \leq 4$, denotes the Λ_{r-1}-module which we obtain when we replace the direct summand S_r of Z_i by S_{r-1} and the direct summand P_{2r} by $P_{2,r-1}$. For $r-1 = 2$, let $P_{2,r-1} = 0$.

Note that this assumption is satisfied for $r-1 = 2$. We embedded $\overrightarrow{\mathcal{K}(\Lambda_2)}$ on a surface $\mathcal{S}_2$ of genus 2 and the subquivers

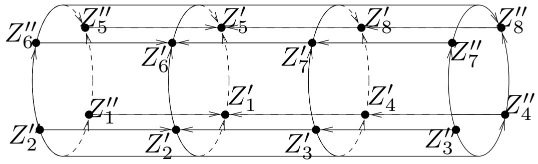
$$
\begin{array}{ccc} Z_5' = Y_5^2 & \longleftarrow & Z_1' = Y_6^2 \\ \uparrow & & \uparrow \\ Z_6' = Y_9^2 & \longleftarrow & Z_2' = Y_{10}^2 \end{array} \quad \text{and} \quad \begin{array}{ccc} Z_4' = Y_7^2 & \longrightarrow & Z_8' = Y_8^2 \\ \uparrow & & \uparrow \\ Z_3' = Y_{11}^2 & \longrightarrow & Z_7' = Y_{12}^2 \end{array}
$$

bound squares on $\mathcal{S}_2$.

The quiver $\overrightarrow{\mathcal{K}(\Lambda_{r-1})}$ is the full convex subquiver of $\overrightarrow{\mathcal{K}(\Lambda_r)}$ with vertices the tilting modules over Λ_r of the form ${}_{\Lambda_r}T \oplus {}_{\Lambda_r}P_r$, where ${}_{\Lambda_r}T$ is a tilting module over Λ_{r-1}. Note that there is an arrow ${}_{\Lambda_r}Z_i'' = {}_{\Lambda_r}Z_i' \oplus {}_{\Lambda_r}P_r \to {}_{\Lambda_r}Z_i$ in $\overrightarrow{\mathcal{K}(\Lambda_r)}$.

Let us assume first that we are in the situation (a). We cut out the interiors of the squares $\overrightarrow{Q}_1$ and $\overrightarrow{Q}_2$ and insert a handle

On this handle we embed $\overrightarrow{\mathcal{K}(\Lambda_r)} \setminus \overrightarrow{\mathcal{K}(\Lambda_{r-1})}$ and the arrows joining Z''_i and Z'_i:



This yields an embedding of $\overrightarrow{\mathcal{K}(\Lambda_r)}$ on a surface $\mathcal{S}_r$ of genus r and the squares

$$\begin{tikzcd} Z'_6 & Z'_7 \arrow[l] \\ Z'_2 \arrow[u] & Z'_3 \arrow[l] \arrow[u] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Z'_5 \arrow[r] & Z'_8 \\ Z'_1 \arrow[u] \arrow[r] & Z'_4 \arrow[u] \end{tikzcd}$$

bound squares on $\mathcal{S}_r$.

We proceed analogously in case (b), and it follows that $\gamma(\mathcal{K}(\Lambda_r)) \leq r$.

3.2. A lower bound for $\gamma(\mathcal{K}(\Lambda_r))$

If $r = 1$, then $\gamma(\mathcal{K}(\Lambda_1)) = 1$ as it was shown in 1.2. Hence we may assume that $r > 1$.

Consider $\overrightarrow{\mathcal{K}(\Lambda_r)} \setminus \overrightarrow{\mathcal{K}(\Lambda_1)}$. We embedded this quiver on a surface of genus $r - 1$, hence $\gamma(\mathcal{K}(\Lambda_r) \setminus \mathcal{K}(\Lambda_1)) \leq r - 1$. The graph $\mathcal{K}(\Lambda_2) \setminus \mathcal{K}(\Lambda_1)$ has 16 vertices and 32 edges. For all $2 \leq i \leq r$, the graphs $\mathcal{K}(\Lambda_i) \setminus \mathcal{K}(\Lambda_{i-1})$ have 8 vertices and 12 edges. Moreover, there are 8 edges joining vertices in $\mathcal{K}(\Lambda_i) \setminus \mathcal{K}(\Lambda_{i-1})$ with vertices in $\mathcal{K}(\Lambda_{i+1}) \setminus \mathcal{K}(\Lambda_i)$, $2 < i < r$. Hence $\mathcal{K}(\Lambda_r) \setminus \mathcal{K}(\Lambda_1)$ has $p = 16 + 8(r-2)$ vertices and $q = 32 + 20(r-2)$ edges. Since $\mathcal{K}(\Lambda_r) \setminus \mathcal{K}(\Lambda_1)$ has no triangles we may use the formula in 1.2 which gives $\gamma(\mathcal{K}(\Lambda_r) \setminus \mathcal{K}(\Lambda_1) \geq \frac{1}{4}q - \frac{1}{2}(p-2) = r - 1$, hence $\gamma(\mathcal{K}(\Lambda_r) \setminus \mathcal{K}(\Lambda_1)) = r - 1$.

We saw above that there are 4 arrows $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ joining vertices in $\overrightarrow{\mathcal{K}(\Lambda_1)}$ with vertices in $\overrightarrow{\mathcal{K}(\Lambda_2)} \setminus \overrightarrow{\mathcal{K}(\Lambda_1)}$. Let $\overrightarrow{\mathcal{K}(\Lambda_r)}'$ be the subquiver of $\overrightarrow{\mathcal{K}(\Lambda_r)}$ which we obtain by deleting three of these arrows, say $\alpha_2, \alpha_3, \alpha_4$. Then $\gamma(\mathcal{K}(\Lambda_r)) \geq \gamma(\mathcal{K}(\Lambda_r)')$. The blocks of $\mathcal{K}(\Lambda_r)'$, i.e. the maximal connected subgraphs of $\mathcal{K}(\Lambda_r)'$ which are connected, non trivial and have no cutpoints are $\mathcal{K}(\Lambda_1)$, $\circ \overset{\alpha_1}{—} \circ$ and $\mathcal{K}(\Lambda_r) \setminus \mathcal{K}(\Lambda_1)$. Since the genus of a graph is the sum of the genera of its blocks [2], we obtain that

$$\gamma(\mathcal{K}(\Lambda_r)) \geq \gamma(\mathcal{K}(\Lambda_r)') = \gamma(\mathcal{K}(\Lambda_1)) + \gamma(\circ — \circ) + \gamma(\mathcal{K}(\Lambda_r) \setminus \mathcal{K}(\Lambda_1)) = 1 + 0 + r - 1.$$

Hence $\gamma(\mathcal{K}(\Lambda_r)) = r$. To finish the proof of the theorem we have to show that there is an algebra Λ_0 with $\gamma(\mathcal{K}(\Lambda_0)) = 0$. If Λ_0 is the ground field, then $\mathcal{K}(\Lambda_0)$ consists of a single vertex, hence it has genus 0.

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