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Mappings of the Sets of Invariant Subspaces of Null Systems

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Abstract. Let $\mathcal{P}$ and $\mathcal{P}'$ be $(2k+1)$-dimensional Pappian projective spaces. Let also $f:\ \mathcal{P}\to\mathcal{P}^*$ and $f':\ \mathcal{P}'\to\mathcal{P}'^*$ be null systems. Denote by $\mathcal{G}_k(f)$ and $\mathcal{G}_k(f')$ the sets of all invariant k-dimensional subspaces of f and f', respectively. In the paper we show that if $k\geq 2$ then any mapping of $\mathcal{G}_k(f)$ to $\mathcal{G}_k(f')$ sending base subsets to base subsets is induced by a strong embedding of $\mathcal{P}$ to $\mathcal{P}'$.

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1. Introduction

Let $\mathcal{P}$ be an n-dimensional projective space. For each number $m=0,1,\dots,n-1$ we denote by $\mathcal{G}_m(\mathcal{P})$ the Grassmann space consisting of all m-dimensional subspaces of $\mathcal{P}$. Then $\mathcal{G}_0(\mathcal{P})=\mathcal{P}$. Note also that $\mathcal{G}_{n-1}(\mathcal{P})$ is an n-dimensional projective space; it is called *dual* to $\mathcal{P}$ and denoted by $\mathcal{P}^*$.

A mapping $f:\ \mathcal{P}\to\mathcal{P}^*$ is called a *polarity* if

$$q\in f(p)\Rightarrow p\in f(q)$$

for any two points p and q of $\mathcal{P}$. It is well known that any polarity is a collineation of $\mathcal{P}$ to $\mathcal{P}^*$.

A polarity $f:\ \mathcal{P}\to\mathcal{P}^*$ is said to be a *null system* if for each point p of $\mathcal{P}$ the subspace $f(p)$ contains p. Null systems of $\mathcal{P}$ exist only for the case when n is odd and the projective space $\mathcal{P}$ is Pappian, see [1], [2]. The last means that $\mathcal{P}$ is isomorphic to the projective space of 1-dimensional subspaces of some $(n+1)$-dimensional vector space over a field. For this case any null system of $\mathcal{P}$ is associated with some non-degenerate alternating form, see [2] or [18].

From this moment we will assume that our projective space $\mathcal{P}$ is Pappian and $n = 2k+1$.

Let $f : \mathcal{P} \to \mathcal{P}^*$ be a null-system. Since f is a collineation, for any m-dimensional subspace $S \subset \mathcal{P}$ the set $f(S)$ is an m-dimensional subspace of $\mathcal{P}^*$. Then the principle of duality of projective geometry (see, for example, [2]) shows that $f(S)$ can be considered as an $(n-m-1)$-dimensional subspace of $\mathcal{P}$. Thus for each number $m = 0, 1, \dots, n-1$ the mapping f induces some bijection

$$f_m : \mathcal{G}_m(\mathcal{P}) \to \mathcal{G}_{n-m-1}(\mathcal{P});$$

clearly, $f_0 = f$ and f_k is a bijective transformation of $\mathcal{G}_k(\mathcal{P})$. If $m \le k$ then we set

$$\mathcal{G}_m(f) := \{ S \in \mathcal{G}_m(\mathcal{P}) \mid S \subset f_m(S) \}.$$

In particular, $\mathcal{G}_0(f)$ coincides with $\mathcal{P}$ and $\mathcal{G}_k(f)$ is the set consisting of all k-dimensional subspaces $S \subset \mathcal{P}$ such that $f_k(S) = S$.

Recall that two m-dimensional subspaces S and U of $\mathcal{P}$ are called *adjacent* if the dimension of $S \cap U$ is equal to $m-1$ (this condition holds if and only if the subspace spanned by S and U is $(m+1)$-dimensional). It is trivial that any two 0-dimensional or $(n-1)$-dimensional subspaces are adjacent; for the general case this fails.

Adjacency preserving transformations of $\mathcal{G}_k(f)$ were studied by W.L. Chow [7] and W.-l. Huang [11], [12]. The classical Chow's theorem [7] states that any bijective transformation of $\mathcal{G}_k(f)$ preserving the adjacency relation in both directions is induced by a collineation of $\mathcal{P}$ to itself. W.-l. Huang [12] has shown that any surjective adjacency preserving transformation of $\mathcal{G}_k(f)$ is a bijection which preserves the adjacency relation in both directions; a more general result was given in Huang's subsequent paper [13] (it will be formulated in Section 4).

Let $\mathcal{P}'$ be another n-dimensional Pappian projective space and $f' : \mathcal{P}' \to \mathcal{P}'^*$ be a null system ($\mathcal{P}'^*$ is the projective space dual to $\mathcal{P}'$). In the present paper we consider mappings of $\mathcal{G}_k(f)$ to $\mathcal{G}_k(f')$ which send base subsets of $\mathcal{G}_k(f)$ to base subsets of $\mathcal{G}_k(f')$ (the definition will be given in the next section) and show that these mappings are induced by strong embeddings of $\mathcal{P}$ to $\mathcal{P}'$.

2. Base subsets of $\mathcal{G}_k(f)$

First of all we recall the concept of base subsets of Grassmann spaces (see [15], [16] and [17]). Let $I := \{1, \dots, n+1\}$ and $\mathcal{B} = \{p_i\}_{i \in I}$ be a base for the projective space $\mathcal{P}$. For each natural number $m = 1, \dots, n-1$ the finite set $\mathcal{B}_m$ consisting of all m-dimensional subspaces

$$\overline{\{p_{i_1}, \dots, p_{i_{m+1}}\}}$$

is called the *base subset* of $\mathcal{G}_m(\mathcal{P})$ *associated* with $\mathcal{B}$ (for any set $X \subset \mathcal{P}$ we denote by $\overline{X}$ the subspace of $\mathcal{P}$ spanned by X).

We say that $\mathcal{B}$ is an f-*base* if for all $i \in I$

$$f(p_i) = \overline{\mathcal{B} - \{p_{\sigma(i)}\}}$$

where

$$\sigma(i) = \begin{cases} i+k+1 & \text{if } 1 \le i \le k+1 \\ i-k-1 & \text{if } k+2 \le i \le 2k+2. \end{cases}$$

For this case

$$\mathcal{B}_{fm} := \mathcal{B}_m \cap \mathcal{G}_m(f), \quad m = 1, \dots, k$$

is said to be the *base subset* of $\mathcal{G}_m(f)$ associated with the f-base $\mathcal{B}$.

Lemma 1. *The following statements hold true:*

(1) *The set $\mathcal{B}_{fm}$ consists of all subspaces $U \in \mathcal{B}_m$ such that*

$$p_i \in U \Longrightarrow p_{\sigma(i)} \notin U \qquad \forall\, i \in I.$$

(2) *For any subspace $U \in \mathcal{B}_{fk}$ and any $i \in I$ we have*

$$p_i \in U \quad or \quad p_{\sigma(i)} \in U.$$

Proof. These statements are direct consequences of the definition. □

Proposition 1. *Any base subset of $\mathcal{G}_k(f)$ contains*

$$\sum_{m=0}^{k+1} \binom{k+1}{m}$$

elements.

Proof. For any $m = 0, 1, \dots, k+1$ we put $\mathcal{B}^m_{fk}$ for the set of all subspaces $U \in \mathcal{B}_{fk}$ containing exactly m points p_i such that $i \le k+1$. The second statement of Lemma 1 shows that for each set

$$\{i_1, \dots, i_m\} \subset \{1, \dots, k+1\}$$

there is unique subspace belonging to $\mathcal{B}^m_{fk}$ and containing $p_{i_1}, \dots p_{i_m}$. Hence

$$|\mathcal{B}^m_{fk}| = \binom{k+1}{m}$$

and

$$|\mathcal{B}_{fk}| = \sum_{m=0}^{k+1} |\mathcal{B}^m_{fk}| = \sum_{m=0}^{k+1} \binom{k+1}{m}.$$

□

3. Mappings of $\mathcal{G}_m(f)$ to $\mathcal{G}_m(f')$ induced by strong embeddings

An injective mapping $g: \mathcal{P} \to \mathcal{P}'$ is called an *embedding* if it is collinearity and non-collinearity preserving (g sends triples of collinear points and non-collinear points to collinear points and non-collinear points, respectively). Any surjective embedding is a collineation. An embedding is said to be *strong* if it transfers independent sets to independent sets (recall that a set $X \subset \mathcal{P}$ is independent if the subspace $\overline{X}$ is not spanned by a proper subset of X).

Since our projective spaces have the same dimension, any strong embedding of $\mathcal{P}$ to $\mathcal{P}'$ maps bases to bases.

For each number $m = 0, \dots, n-1$ any strong embedding $g: \mathcal{P} \to \mathcal{P}'$ induces the injection

$$g_m: \mathcal{G}_m(\mathcal{P}) \to \mathcal{G}_m(\mathcal{P}')$$

which sends an m-dimensional subspace $S \subset \mathcal{P}$ to the subspace $\overline{g(S)}$. For the case when $m = n-1$ it is a strong embedding of $\mathcal{P}^*$ to $\mathcal{P}'^*$; we will denote this embedding by g^*. If

$$f'g = g^*f \tag{1}$$

then for any $m \leq k$ the g_m-image of $\mathcal{G}_m(f)$ is contained in $\mathcal{G}_m(f')$, in other words, g induces an injection of $\mathcal{G}_m(f)$ to $\mathcal{G}_m(f')$.

Proposition 2. *Let $g: \mathcal{P} \to \mathcal{P}'$ be a strong embedding satisfying* (1) *and such that the mapping*

$$g_m: \mathcal{G}_m(f) \to \mathcal{G}_m(f') \tag{2}$$

is bijective for some natural number $m \leq k$. Then g is a collineation.

Proof. We show that the mapping

$$g_{m-1}: \mathcal{G}_{m-1}(f) \to \mathcal{G}_{m-1}(f') \tag{3}$$

is bijective.

For any subspace S' belonging to $\mathcal{G}_{m-1}(f')$ there exist subspaces $U'_1, U'_2 \in \mathcal{G}_m(f')$ such that $S' = U'_1 \cap U'_2$. Then U'_1 and U'_2 are adjacent and the subspace spanned by them is $(m+1)$-dimensional. By our hypothesis, the mapping (2) is bijective and the equalities

$$\overline{g(U_1)} = U'_1 \text{ and } \overline{g(U_2)} = U'_2$$

hold for some subspaces U_1 and U_2 belonging to $\mathcal{G}_m(f)$. An immediate verification shows that

$$g(\overline{U_1 \cup U_2}) \subset \overline{g(U_1) \cup g(U_2)} \subset \overline{U'_1 \cup U'_2}.$$

Since $\overline{U'_1 \cup U'_2}$ is $(m+1)$-dimensional, the dimension of $\overline{U_1 \cup U_2}$ is not greater than $k+1$; this dimension is equal to $k+1$ (U_1 and U_2 are distinct k-dimensional subspaces). In other words, U_1 and U_2 are adjacent and

$$S := U_1 \cap U_2$$

belongs to $\mathcal{G}_{m-1}(f)$. Then

$$\overline{g(S)} = \overline{g(U_1) \cap g(U_2)} \subset \overline{g(U_1)} \cap \overline{g(U_2)} = U'_1 \cap U'_2 = S'.$$

The subspaces $\overline{g(S)}$ and S' are both $(k-1)$-dimensional, hence $\overline{g(S)} = S'$. We have established that (3) is surjective; but this mapping is injective and we get the required.

By induction, we can prove that the embedding g is surjective. This means that g is a collineation. $\square$

4. Result

Theorem 1. (W.-l. Huang [13]) *Let g be an adjacency preserving mapping of $\mathcal{G}_k(f)$ to $\mathcal{G}_k(f')$ and suppose that for each $S \in \mathcal{G}_k(f)$ there exists $U \in \mathcal{G}_k(f)$ such that*

$$g(S) \cap g(U) = \emptyset.$$

Then g is induced by a strong embedding of $\mathcal{P}$ to $\mathcal{P}'$.

In the present paper the following statement will be proved.

Theorem 2. *Let $k \geq 2$ and g be a mapping of $\mathcal{G}_k(f)$ to $\mathcal{G}_k(f')$ sending base subsets to base subsets. Then g is induced by a strong embedding of $\mathcal{P}$ to $\mathcal{P}'$.*

Theorem 2 and Proposition 2 give the following.

Corollary 1. *Let $k \geq 2$ and g be a surjection of $\mathcal{G}_k(f)$ to $\mathcal{G}_k(f')$ sending base subsets to base subsets. Then g is induced by a collineation of $\mathcal{P}$ to $\mathcal{P}'$.*

Remark 1. Adjacency preserving mappings of Grassmann spaces were studied by many authors, see [6], [7], [11], [9], [10], [14]. These results are closely related with the discipline known as *characterizations of geometrical mappings under mild hypotheses*, see [3].

Remark 2. Mappings of Grassmann spaces transferring base subsets to base subsets were considered in author's papers [15], [16], and [17].

Remark 3. Let $\mathcal{G}$ be the Grassmann space of m-dimensional subspaces of some $(2m+1)$-dimensional projective space. A. Blunck and H. Havlicek [5] have characterized the adjacency relation on $\mathcal{G}$ in terms of non-intersecting subspaces; this result was exploited to study transformations of $\mathcal{G}$ sending non-intersecting subspaces to non-intersecting subspaces.

5. Proof of Theorem 2

5.1.

Let S be a subspace belonging to $\mathcal{G}_k(f)$. Consider a base subset $\mathcal{B}_{fk}$ containing S (it is trivial that this base set exists). By our hypothesis, $g(\mathcal{B}_{fk})$ is a base subset of $\mathcal{G}_k(f')$ and there exists $U' \in g(\mathcal{B}_{fk})$ such that

$$g(S) \cap U' = \emptyset.$$

Since $U' = g(U)$ for some $U \in \mathcal{B}_{fk}$, the mapping g satisfies the second condition of Theorem 1. Thus we need to prove that g is adjacency preserving.

Now we want to show that g is injective. We will exploit the following statement which is a simple consequence of more general results related with Tits buildings (see [4], [8], [18] or [19]).

Lemma 2. *For any two elements of $\mathcal{G}_k(f)$ there exists a base subset containing them.*

Let S and U be distinct elements of $\mathcal{G}_k(f)$ and $\mathcal{B}_{fk}$ be a base subset of $\mathcal{G}_k(f)$ containing them. If $g(S) = g(U)$ then the cardinal number of $g(\mathcal{B}_{fk})$ is less than the cardinal number of $\mathcal{B}_{fk}$. Then $g(\mathcal{B}_{fk})$ is not a base subset of $\mathcal{G}_k(f')$; this contradicts to our hypothesis. Therefore, f is injective.

5.2.

Let $\mathcal{B} = \{p_i\}_{i\in I}$ be an f-base for $\mathcal{P}$ and $\mathcal{B}_{fk}$ be the base subset of $\mathcal{G}_k(f)$ associated with $\mathcal{B}$. We say that $\mathcal{R} \subset \mathcal{B}_{fk}$ is an *exact* subset of $\mathcal{B}_{fk}$ if $\mathcal{B}_{fk}$ is unique base subset of $\mathcal{G}_k(f)$ containing $\mathcal{R}$; otherwise, the subset $\mathcal{R}$ is said to be *inexact*.

1) *By our hypothesis, $g(\mathcal{B}_{fk})$ is a base subset of $\mathcal{G}_k(f')$. The mapping g transfers inexact subsets of $\mathcal{B}_{fk}$ to inexact subsets of $g(\mathcal{B}_{fk})$.*

Proof. If $\mathcal{R}$ is an inexact subset of $\mathcal{B}_{fk}$ then there is another base subset of $\mathcal{G}_k(f)$ containing $\mathcal{R}$. Since g is injective, there exist at least two distinct base subsets of $\mathcal{G}_k(f')$ containing $g(\mathcal{R})$; hence $g(\mathcal{R})$ is inexact. □

For any set $\mathcal{R} \subset \mathcal{B}_{fk}$ and any number $i \in I$ denote by $S_i(\mathcal{R})$ the intersection of all subspaces $U \in \mathcal{R}$ containing p_i. Clearly, $\mathcal{R}$ is exact if each $S_i(\mathcal{R})$ is a one-point set. Now we show that the inverse statement holds true.

2) *A subset $\mathcal{R}$ of $\mathcal{B}_{fk}$ is exact if and only if*

$$S_i(\mathcal{R}) = \{p_i\} \qquad \forall\, i \in I.$$

Proof. If $S_i(\mathcal{R}) \neq \{p_i\}$ for some number i then one of the following possibilities is realized:

(A) $S_i(\mathcal{R})$ is empty,

(B) $S_i(\mathcal{R})$ contains a point p_j, $j \neq i$.

We show that for each of these cases there exists an f-base $\mathcal{B}'_{fk}$ different from $\mathcal{B}_{fk}$ and such that the base subset of $\mathcal{G}_k(f)$ associated with $\mathcal{B}'_{fk}$ contains $\mathcal{R}$; this means that $\mathcal{R}$ is inexact.

Case (A): Let p'_i be a point of the line $p_ip_{\sigma(i)}$ (spanned by the points p_i and $p_{\sigma(i)}$) such that $p'_i \neq p_i, p_{\sigma(i)}$. Set

$$\mathcal{B}' := \ (\mathcal{B} - \{p_i\}) \cup \{p'_i\}.$$

Then

$$f(p'_i) = \overline{(\mathcal{B} - \{p_i, p_{\sigma(i)}\}) \cup \{p'_i\}} = \overline{\mathcal{B}' - \{p_{\sigma(i)}\}}$$

and

$$f(p_{\sigma(i)}) = \overline{\mathcal{B} - \{p_i\}} = \overline{\mathcal{B}' - \{p'_i\}};$$

for any $j \neq i, \sigma(i)$ we have

$$f(p_j) = \overline{\mathcal{B} - \{p_{\sigma(j)}\}} = \overline{(\mathcal{B} - \{p_i, p_{\sigma(j)}\}) \cup \{p'_i\}} = \overline{\mathcal{B}' - \{p_{\sigma(j)}\}}.$$

Therefore, $\mathcal{B}'$ is an f-base. Each subspace $S \in \mathcal{R}$ is spanned by points of the set

$$\mathcal{B} - \{p_i\} = \mathcal{B}' - \{p'_i\}$$

and $\mathcal{R}$ is contained in the base subset of $\mathcal{G}_k(f)$ associated with $\mathcal{B}'$.

Case (B): Lemma 1 shows that $j \neq \sigma(i)$. Besides $p_{\sigma(i)}$ belongs to $S_{\sigma(j)}(\mathcal{R})$; indeed, if some subspace $U \in \mathcal{R}$ does not contain $p_{\sigma(i)}$ then $p_i \in U$ (Lemma 1) and the condition (B) guarantees that p_j is a point of U, hence $p_{\sigma(j)} \notin U$. Now take two points

$$p'_i \in p_ip_j \ \text{ and } \ p'_{\sigma(j)} \in p_{\sigma(i)}p_{\sigma(j)}$$

such that

$$p'_i \neq p_i, p_j \ \text{ and } \ p'_{\sigma(j)} \neq p_{\sigma(i)}, p_{\sigma(j)}$$

and set

$$\mathcal{B}' := \ (\mathcal{B} - \{p_i, p_{\sigma(j)}\}) \cup \{p'_i, p'_{\sigma(j)}\}.$$

Then

$$f(p'_i) = \overline{(\mathcal{B} - \{p_i, p_{\sigma(i)}\}) \cup \{p'_i\}} =$$
$$\overline{(\mathcal{B} - \{p_i, p_{\sigma(i)}, p_{\sigma(j)}\}) \cup \{p'_i, p'_{\sigma(j)}\}} = \overline{\mathcal{B}' - \{p_{\sigma(i)}\}}\,,$$
$$f(p_{\sigma(i)}) = \overline{\mathcal{B} - \{p_i\}} = \overline{(\mathcal{B} - \{p_i, p_{\sigma(j)}\}) \cup \{p'_{\sigma(j)}\}} = \overline{\mathcal{B}' - \{p'_i\}}$$

and

$$f(p_j) = \overline{\mathcal{B} - \{p_{\sigma(j)}\}} = \overline{(\mathcal{B} - \{p_i, p_{\sigma(j)}\}) \cup \{p'_i\}} = \overline{\mathcal{B}' - \{p'_{\sigma(j)}\}}\,,$$
$$f(p'_{\sigma(j)}) = \overline{(\mathcal{B} - \{p_j, p_{\sigma(j)}\}) \cup \{p'_{\sigma(j)}\}} =$$
$$\overline{(\mathcal{B} - \{p_i, p_j, p_{\sigma(j)}\}) \cup \{p'_i, p'_{\sigma(j)}\}} = \overline{\mathcal{B}' - \{p_j\}};$$

if $m \neq i, j, \sigma(i), \sigma(j)$ then

$$f(p_m) = \overline{\mathcal{B} - \{p_{\sigma(m)}\}} = \overline{(\mathcal{B} - \{p_i, p_{\sigma(j)}, p_{\sigma(m)}\}) \cup \{p'_i, p'_{\sigma(j)}\}} = \overline{\mathcal{B}' - \{p_{\sigma(m)}\}}.$$

We have established that $\mathcal{B}'$ is an f-base. Lemma 1 and the condition (B) show that each subspace $S \in \mathcal{R}$ contains one of the lines p_ip_j or $p_{\sigma(i)}p_{\sigma(j)}$; i.e. S is spanned by one of these lines and points of the set

$$\mathcal{B} - \{p_i, p_j, p_{\sigma(i)}, p_{\sigma(j)}\} = \mathcal{B}' - \{p'_i, p_j, p_{\sigma(i)}, p'_{\sigma(j)}\}$$

This implies that $\mathcal{R}$ is contained in the base subset of $\mathcal{G}_k(f)$ associated with $\mathcal{B}'$. □

Let $0 \leq m \leq k$ and U be an m-dimensional subspace spanned by points of the base $\mathcal{B}$ (in other words, U is an element of the base subset of $\mathcal{G}_m(\mathcal{P})$ associated with $\mathcal{B}$ or a point of $\mathcal{B}$ if $m = 0$). Put $\mathcal{B}_{fk}(U)$ for the set of all subspaces belonging to $\mathcal{B}_{fk}$ and containing U. This set is empty for the case when $U \notin \mathcal{B}_{fm}$. If U is an element of $\mathcal{B}_{fm}$ then $\mathcal{B}_{fk}(U)$ is not empty; the cardinal number of this set will be denoted by t_m (it does not depend on the choice of $U \in \mathcal{B}_{fm}$).

3) *If $i < j \leq k$ then $t_i > t_j$.*

Proof. Let us consider two subspaces $T \in \mathcal{B}_{fi}$ and $U \in \mathcal{B}_{fj}$ such that $T \subset U$. It is trivial that $\mathcal{B}_{fk}(U)$ is a proper subspace of $\mathcal{B}_{fk}(T)$. This implies the required inequality. □

Now consider two distinct points p_i and p_j such that $\sigma(i) \neq j$. The line p_ip_j belongs to $\mathcal{B}_{f1}$ and the set

$$\mathcal{B}_{fk}(p_ip_j) \cup \mathcal{B}_{fk}(p_{\sigma(i)}) \tag{4}$$

is inexact (if some subspace S belongs to (4) and contains p_i then $p_j \in S$). Since $\mathcal{B}_{fk}(p_ip_j)$ and $\mathcal{B}_{fk}(p_{\sigma(i)})$ are non-intersecting sets, the cardinal number of (4) is equal to $t_0 + t_1$.

4) *If $\mathcal{R}$ is an inexact subset of $\mathcal{B}_{fk}$ containing t_0+t_1 elements then there exist two distinct numbers i and j such that $\sigma(i) \neq j$ and*

$$\mathcal{R} = \mathcal{B}_{fk}(p_i p_j) \cup \mathcal{B}_{fk}(p_{\sigma(i)}).$$

Proof. Since $\mathcal{R}$ is inexact, $S_i(\mathcal{R}) \neq \{p_i\}$ for some number i. If $S_i(\mathcal{R})$ is not empty then we take any point p_j, $j \neq i$ belonging to $S_i(\mathcal{R})$; by Lemma 1, $j \neq \sigma(i)$. For the case when $S_i(\mathcal{R})$ is empty we can take an arbitrary point p_j such that $j \neq i, \sigma(i)$. Then for any subspace $U \in \mathcal{R}$ one of the following possibilities is realized:

(A) $p_i \in U$ then U belongs to $\mathcal{B}_{fk}(p_i p_j)$,

(B) $p_i \notin U$ then $p_{\sigma(i)} \in U$ and U belongs to $\mathcal{B}_{fk}(p_{\sigma(i)})$.

Hence

$$\mathcal{R} \subset \mathcal{B}_{fk}(p_i p_j) \cup \mathcal{B}_{fk}(p_{\sigma(i)}).$$

These sets have the same cardinal number and the inclusion can be replaced by the equality. □

5.3.

We say that $\mathcal{R} \subset \mathcal{B}_{fk}$ is a *c-subset* of $\mathcal{B}_{fk}$ if its complement $\mathcal{B}_{fk} - \mathcal{R}$ is an inexact subset containing $t_0 + t_1$ elements.

5) *The mapping g transfers c-subsets of $\mathcal{B}_{fk}$ to c-subsets of $g(\mathcal{B}_{fk})$.*

Proof. Since g is an injection, it is a direct consequence of the definition of c-subsets and the statement 1). □

6) *For any set $\mathcal{R} \subset \mathcal{B}_{fk}$ the following conditions are equivalent:*

(A) *$\mathcal{R}$ is a c-subset,*

(B) *there exists a line $L \in \mathcal{B}_{f1}$ such that $\mathcal{R} = \mathcal{B}_{fk}(L)$.*

Proof. $(A) \Rightarrow (B)$. Assume that $\mathcal{R}$ is a c-subset of $\mathcal{B}_{fk}$. Then

$$\mathcal{R} = \mathcal{B}_{fk} - (\mathcal{B}_{fk}(p_i p_j) \cup \mathcal{B}_{fk}(p_{\sigma(i)})) \tag{5}$$

for some numbers i and j such that $j \neq \sigma(i)$. We show that the line

$$L := \ p_i p_{\sigma(j)}$$

has the required property.

Let $S \in \mathcal{R}$. By (5), $p_{\sigma(i)}$ does not belong to S. Thus $p_i \in S$. The equation (5) implies also that the line $p_i p_j$ is not contained in S. Since p_i is a point of S, $p_j \notin S$ and $p_{\sigma(j)}$ belongs to S. Therefore, $S \in \mathcal{B}_{fk}(L)$.

Consider a subspace S belonging to $\mathcal{B}_{fk}(L)$. Since $p_i \in S$, S does not belong to $\mathcal{B}_{fk}(p_{\sigma(i)})$. The condition $p_{\sigma(j)} \in S$ guarantees that $p_j \notin S$ and the line $p_i p_j$ is not contained in S. Then S does not belong to $\mathcal{B}_{fk}(p_i p_j)$. By (5), $S \in \mathcal{R}$.

$(B) \Rightarrow (A)$. Let $L \in \mathcal{B}_{f1}$. Then $L = p_i p_j$ and $j \neq \sigma(i)$. If some subspace $S \in \mathcal{B}_{fk}$ is not contained in the set $\mathcal{B}_{fk}(L)$ then one of the following cases is realized:

— $p_i \notin S$ then $p_{\sigma(i)} \in S$ and S belongs to $\mathcal{B}_{fk}(p_{\sigma(i)})$,

— $p_i \in S$ and $p_j \notin S$ then $p_{\sigma(j)} \in S$ and S is an element of $\mathcal{B}_{fk}(p_i p_{\sigma(j)})$.

Therefore, $\mathcal{B}_{fk} - \mathcal{B}_{fk}(L)$ is contained in

$$\mathcal{B}_{fk}(p_{\sigma(i)}) \cup \mathcal{B}_{fk}(p_i p_{\sigma(j)}).$$

The arguments given above show that these sets are coincident. □

Let $\mathcal{R}$ and $\mathcal{R}'$ be distinct c-subsets of $\mathcal{B}_{fk}$. Then

$$\mathcal{R} = \mathcal{B}_{fk}(L) \text{ and } \mathcal{R}' = \mathcal{B}_{fk}(L'),$$

where L and L' are distinct elements of $\mathcal{B}_{f1}$. Denote by S the subspace spanned by L and L'; the dimension of S is equal to 2 or 3.

Consider the case when $k = 2$. If $S \in \mathcal{B}_{f2}$ then

$$\mathcal{R} \cap \mathcal{R}' = \{S\};$$

for this case we will say that our c-subsets form an (A)-*pair*. If S does not belong to $\mathcal{B}_{f2}$ then $\mathcal{R} \cap \mathcal{R}'$ is empty.

If $k \geq 3$ then there are the following possibilities for the subspace S:

(A) S belongs to $\mathcal{B}_{f2}$ then
$$\mathcal{R} \cap \mathcal{R}' = \mathcal{B}_{fk}(S)$$
contains t_2 elements,

(B) S belongs to $\mathcal{B}_{f3}$ then
$$\mathcal{R} \cap \mathcal{R}' = \mathcal{B}_{fk}(S)$$
contains t_3 elements,

(C) S does not belong to $\mathcal{B}_{f2}$ and $\mathcal{B}_{f3}$, for this case the set $\mathcal{R} \cap \mathcal{R}'$ is empty.

We say that our c-subsets form an (A)-*pair* or a (B)-*pair* if the corresponding case is realized.

7) *The mapping g transfers any (A)-pair of c-subsets to an (A)-pair of c-subsets. If $k \geq 3$ then g maps (B)-pairs to (B)-pairs.*

Proof. Let $\mathcal{R}$ and $\mathcal{R}'$ be distinct c-subsets of $\mathcal{B}_{fk}$. By (5), $g(\mathcal{R})$ and $g(\mathcal{R}')$ are c-subsets of $g(\mathcal{B}_{fk})$. Since f is injective, $\mathcal{R} \cap \mathcal{R}'$ and $g(\mathcal{R}) \cap g(\mathcal{R}')$ have the same cardinal numbers and the arguments given before (7) imply the required. □

8) *Let S and S' be distinct elements of $\mathcal{B}_{fk}$. Then the following statements are fulfilled:*

(i) *For the case when $k = 2$ the subspaces S and S' are adjacent if and only if there exists a c-subset of $\mathcal{B}_{fk}$ containing them.*

(ii) *For the case when $k = 2m > 2$ our subspaces are adjacent if and only if there exists a sequence $\mathcal{R}_1, \dots, \mathcal{R}_m$ of c-subsets of $\mathcal{B}_{fk}$ such that any two $\mathcal{R}_i$ and $\mathcal{R}_j$ form a (B)-pair if $i \neq j$ and each $\mathcal{R}_i$ contains S and S'.*

(iii) *Let* $k = 2m + 1 \geq 3$. *Then* S *and* S' *are adjacent if and only if there exists a sequence* $\mathcal{R}_1, \dots, \mathcal{R}_{m+1}$ *of c-subsets of* $\mathcal{B}_{fk}$ *such that each* $\mathcal{R}_i$ *contains* S *and* S' *and the following conditions hold true:*

— $\mathcal{R}_i$ *and* $\mathcal{R}_j$ *form a* (B)*-pair if* $i \neq j$ *and* i, j *are both less than* $m+1$,

— *if* $i < m$ *then* $\mathcal{R}_i$ *and* $\mathcal{R}_{m+1}$ *is a* (B)*-pair,*

— $\mathcal{R}_m$ *and* $\mathcal{R}_{m+1}$ *form an* (A)*-pair.*

Proof. The statement (i) is trivial. For the case (ii) or (iii) the existence of a sequence of c-subsets satisfying the corresponding conditions implies that the subspace $S \cap S'$ is $(k-1)$-dimensional (i.e. S and S' are adjacent).

Now assume that S and S' are adjacent. Then the dimension of $S \cap S'$ is equal to $k-1$. Clearly, we can restrict ourself to the case when $S \cap S'$ is spanned by $p_1, \dots, p_k$. If $k = 2m$ then the lines

$$L_i := p_{2i-1}p_{2i}$$

$i = 1, \dots, m$ define a sequence of c-subsets satisfying the required conditions. For the case when $k = 2m+1$ we set

$$L_i := p_{2i-1}p_{2i} \text{ if } i = 1, \dots, m \text{ and } L_{m+1} := p_{k-1}p_k.$$

It is easy to see that the c-subsets associated with these lines are as required. □

The statements (7) and (8) show that the restriction of g to each base subset of $\mathcal{G}_k(f)$ is adjacency preserving. Since for any two elements of $\mathcal{G}_k(f)$ there exists a base subset containing them (Lemma 2), the mapping g preserves the adjacency.

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