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# Higher-Order Preconnections in Synthetic Differential Geometry of Jet Bundles

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**Abstract.** In our previous papers (Nishimura [2001 and 2003]) we dealt with jet bundles from a synthetic perch by regarding a 1-jet as something like a pinpointed (nonlinear) connection (called a *preconnection*) and then looking on higher-order jets as repeated 1-jets. In this paper we generalize our notion of preconnection to higher orders, which enables us to develop a non-repetitive but still synthetic approach to jet bundles. Both our repetitive and non-repetitive approaches are coordinate-free and applicable to microlinear spaces in general. In our non-repetitive approach we can establish a theorem claiming that the $(n+1)$-th jet space is an affine bundle over the $n$-th jet space, while we have not been able to do so in our previous repetitive approach. We will show how to translate repeated 1-jets into higher-order preconnections. Finally we will demonstrate that our repetitive and non-repetitive approaches to jet bundles tally, as far as formal manifolds are concerned.

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## Introduction

In our previous papers (Nishimura [2001 and 2003]) we have approached the theory of jet bundles from a synthetic coign of vantage by regarding a 1-jet as a decomposition of the tangent space to the space at the point at issue (cf. Saunders [1989, Theorem 4.3.2]) and then looking on higher-order jets as repeated 1-jets (cf. Saunders [1989, §5.2 and §5.3]). In Nishimura [2001] a 1-jet put down in such a way was called a *preconnection*, which should have been called, exactly speaking, a 1-*preconnection*. In §1 of this paper we generalize our previous notion of 1-preconnection to higher-orders to get the notion of $n$-*preconnection* for any natural number $n$, reminiscent of higher-order generalizations of linear connection discussed by Lavendhomme [1996, p.107] and Lavendhomme and Nishimura [1998, Definition 2]. The immediate meed of our present approach to jet bundles is that we can establish a synthetic variant of Theorem 6.2.9 of Saunders [1989] claiming that the canonical projection from the $(n+1)$-th jet space to the $n$-th one is an affine bundle.

The remaining two sections are concerned with the comparison between our new approach to jet bundles by higher-order preconnections and our previous one by iterated 1-preconnections discussed in Nishimura [2001 and 2003]. In Section 2 we will explain how to translate the latter approach into the former, but we are not sure whether the translation gives a bijection in this general context. However, if we confine our scope to formal manifolds, the above translation indeed gives a bijection, which is the topic of Section 3.

Our standard reference of synthetic differential geometry is Lavendhomme [1996], but some material which is not easily available in his book or which had better be presented in this paper anyway is exhibited in §0 as preliminaries. Our standard reference of jet bundles is Saunders [1989], §5.2 and §5.3 of which have been constantly inspiring.

## 0. Preliminaries

### 0.1. Microcubes

Let $\mathbb{R}$ be the extended set of real numbers with cornucopia of nilpotent infinitesimals, which is expected to acquiesce in the so-called general Kock axiom (cf. Lavendhomme [1996, §2.1]). We denote by $D$ the totality of elements of $\mathbb{R}$ whose squares vanish. Given a microlinear space $M$ and an infinitesimal space $\mathbb{E}$, a mapping $\gamma$ from $\mathbb{E}$ to $M$ is called an $\mathbb{E}$-*microcube* on M. $D^n$-microcubes are usually called $n$-*microcubes*. In particular, 1-microcubes are called *tangent vectors*, and 2-microcubes are referred to as *microsquares*. We denote by $\mathbf{T}^{\mathbb{E}}(M)$ the totality of $\mathbb{E}$-microcubes on M. Given $x \in M$, we denote by $\mathbf{T}_x^{\mathbb{E}}(M)$ the totality of $\mathbb{E}$-microcubes $\gamma$ on $M$ with $\gamma(0,\dots,0) = \mathrm{x}$. $\mathbf{T}^{D^n}(M)$ and $\mathbf{T}_x^{D^n}(M)$ are usually denoted by $\mathbf{T}^n(M)$ and $\mathbf{T}_x^n(M)$ respectively. Given $\gamma \in \mathbf{T}^n(M)$ and a natural number $k$ with $k \leq n$, we can put down $\gamma$ as a tangent vector $\mathbf{t}_\gamma^k$ to $\mathbf{T}^{n-1}(M)$ mapping $d \in D$ to $\gamma_d^k \in \mathbf{T}^{n-1}(M)$, where

$$\gamma_d^k(d_1,\dots,d_{n-1}) = \gamma(d_1,\dots,d_{k-1},d,d_k,\dots,d_{n-1}) \tag{0.1.1}$$

for any $d_1,\dots,d_{n-1} \in D$. Given $\alpha \in \mathbb{R}$, we define $\alpha \underset{k}{\cdot} \gamma_k$ to be $\alpha \mathbf{t}_\gamma^k$. Given $\gamma_+, \gamma_- \in \mathbf{T}^n(M)$ with $\mathbf{t}_{\gamma_+}^k(0) = \mathbf{t}_{\gamma_-}^k(0)$, $\gamma_+ \underset{k}{-} \gamma_-$ is defined to be $\mathbf{t}_{\gamma_+}^k - \mathbf{t}_{\gamma_-}^k$. Given $\gamma_1,\dots,\gamma_m \in \mathbf{T}^n(M)$ with $\mathbf{t}_{\gamma_1}^k(0) = \dots = \mathbf{t}_{\gamma_m}^k(0)$, $\mathbf{t}_{\gamma_1}^k + \dots + \mathbf{t}_{\gamma_m}^k$ is denoted by $\gamma_1 \underset{k}{+} \dots \underset{k}{+} \gamma_m$ or $\Sigma_{k^{i=1}}^m \gamma_i$. Given $\gamma \in \mathbf{T}^n(M)$ and a mapping $f : M \to M'$, we will often denote $f \circ \gamma \in \mathbf{T}^n(M')$ by $f_*(\gamma)$.

We denote by $\mathfrak{S}_n$ the symmetric group of the set $\{1,\dots,n\}$, which is well known to be generated by $n-1$ transpositions $< i, i+1 >$ exchanging $i$ and $i+1 (1 \leq i \leq n-1)$ while keeping the other elements fixed. A cycle $\sigma$ of length $k$ is usually denoted by $<$

$j, \sigma(j), \sigma^2(j), \ldots, \sigma^{k-1}(j) >$, where $j$ is not fixed by $\sigma$. Given $\sigma \in \mathfrak{S}_n$ and $\gamma \in \mathbf{T}^n(M)$, we define $\Sigma_\sigma(\gamma) \in \mathbf{T}^n(M)$ to be

$$\Sigma_\sigma(\gamma)(d_1, \ldots, d_n) = \gamma(d_{\sigma(1)}, \ldots, d_{\sigma(n)}) \tag{0.1.2}$$

for any $(d_1, \ldots, d_n) \in D^n$. Given $\alpha \in \mathbb{R}$ and $\gamma \in \mathbf{T}^n(M)$, we define $\alpha \underset{i}{\cdot} \gamma \in \mathbf{T}_i^n(M)$ $(1 \leq i \leq n)$ to be

$$(\alpha \underset{i}{\cdot} \gamma)(d_i, \ldots, d_n) = \gamma(d_1, \ldots, d_{i-1}, \alpha d_i, d_{i+1}, \ldots, d_n) \tag{0.1.3}$$

for any $(d_1, \ldots, d_n) \in D^n$.

Some subspaces of $D^n$ will play an important role. We denote by $D(n)$ the set $\{(d_1, \ldots, d_n) \in D^n \mid d_i d_j = 0 \text{ for any } 1 \leq i, j \leq n\}$. We denote by $D(n;n)$ the set $\{(d_1, \ldots, d_n) \in D^n \mid d_1 \ldots d_n = 0\}$. Note that $D(2) = D(2;2)$.

Between $\mathbf{T}^n(M)$ and $\mathbf{T}^{n+1}(M)$ there are $2n+2$ canonical mappings:

$$\mathbf{T}^{n+1}(M) \underset{\mathbf{s}_i}{\overset{\mathbf{d}_i}{\rightleftarrows}} \mathbf{T}^n(M) \qquad (1 \leq i \leq n+1)$$

For any $\gamma \in \mathbf{T}^n(M)$, we define $\mathbf{s}_i(\gamma) \in \mathbf{T}^{n+1}(M)$ to be

$$\mathbf{s}_i(\gamma)(d_1, \ldots, d_{n+1}) = \gamma(d_1, \ldots, d_{i-1}, d_{i+1}, \ldots, d_{n+1}) \tag{0.1.4}$$

for any $(d_1, \ldots, d_{n+1}) \in D^{n+1}$. For any $\gamma \in \mathbf{T}^{n+1}(M)$, we define $\mathbf{d}_i(\gamma) \in \mathbf{T}^n(M)$ to be

$$\mathbf{d}_i(\gamma)(d_1, \ldots, d_n) = \gamma(d_1, \ldots, d_{i-1}, 0, d_i, \ldots, d_n) \tag{0.1.5}$$

for any $(d_1, \ldots, d_n) \in D^n$. These operators satisfy the so-called simplicial identities (cf. Goerss and Jardine [1999, p.4]).

Now we have

**Proposition 0.1.1.** *For any $\gamma_+, \gamma_- \in \mathbf{T}^n(M)$, $\gamma_+|_{D(n;n)} = \gamma_-|_{D(n;n)}$ iff $\mathbf{d}_i(\gamma_+) = \mathbf{d}_i(\gamma_-)$ for all $1 \leq i \leq n$.*

*Proof.* By the quasi-colimit diagram of Proposition 1 of Lavendhomme and Nishimura [1998].

### 0.2. Bundles

A mapping $\pi : E \to M$ of microlinear spaces is called a *bundle over* $M$, in which $E$ is called the *total space* of $\pi$, $M$ is called the *base space* of $\pi$, and $E_x = \pi^{-1}(x)$ is called the *fiber* over $x \in M$. Given $y \in E$, we denote by $\mathbf{V}_y^n(\pi)$ the totality of $n$-microcubes $\gamma$ on $E$ such that $\pi \circ \gamma$ is a constant function and $\gamma(0, \ldots, 0) = y$. We denote by $\mathbf{V}^n(\pi)$ the set-theoretic union of $\mathbf{V}_y^n(\pi)'s$ for all $y \in E$. A bundle $\pi : E \to M$ is called a *vector bundle* provided that $E_x$ is a Euclidean $\mathbb{R}$-*module* for every $x \in M$. The canonical projections $\tau_M : \mathbf{T}^1(M) \to M$ and $v_\pi : \mathbf{V}^1(\pi) \to E$ are vector bundles. A bundle $\pi : E \to M$ is called an *affine bundle over* a vector bundle $\pi' : E' \to M$ provided that $E_x$ is an affine space over the $\mathbb{R}$-module $E'_x$ for every $x \in M$. Given two bundles $\pi : E' \to M$ and $E' \to M$ over the same base space $M$, a mapping

$f : E \to E'$ is called a *morphism of bundles from* $\pi$ *to* $\pi'$ *over* $M$ if it induces the identity mapping on $M$. Given two bundles $\pi : E \to M$ and $\iota : M' \to M$ over the same base space $M$, the mapping $\iota^*(\pi)$ assigning $a \in E$ to each $(y, a) \in M' \underset{M}{\times} E = \{(y, a) \in M' \times E | \iota(y) = \pi(a)\}$ is called the bundle obtained by *pulling back* the bundle $\pi : E \to M$ along $\iota$.

### 0.3. Strong differences

Kock and Lavendhomme [1984] have provided the synthetic rendering of the notion of *strong difference* for microsquares, a good exposition of which can be seen in Lavendhomme [1996, §3.4]. Given two microsquares $\gamma_+$ and $\gamma_-$ on $M$, their strong difference $\gamma_+ \dot{-} \gamma_-$ is defined exactly when $\gamma_+|_{D(2)} = \gamma_-|_{D(2)}$, and it is a tangent vector to $M$ with $(\gamma_+ \dot{-} \gamma_-)(0) = \gamma_+(0, 0) = \gamma_-(0, 0)$. Given $t \in \mathbf{T}^1(M)$ and $\gamma \in \mathbf{T}^2(\gamma)$ with $t(0) = \gamma(0, 0)$, the strong addition $t \dot{+} \gamma$ is defined to be a microsquare on $M$ with $(t \dot{+} \gamma)|_{D(2)} = \gamma|_{D(2)}$. With respect to these operations Kock and Lavendhomme [1984] have shown that

**Theorem 0.3.1.** *The canonical projection* $\mathbf{T}^2(M) \to \mathbf{T}^{D(2)}(M)$ *is an affine bundle over the vector bundle* $\mathbf{T}^1(M) \underset{M}{\times} \mathbf{T}^{D(2)}(M) \to \mathbf{T}^{D(2)}(M)$ *assigning* $\gamma$ *to each* $(t, \gamma) \in \mathbf{T}^1(M) \underset{M}{\times} \mathbf{T}^{D(2)}(M) = \{(t, \gamma) \in \mathbf{T}^1(M) \times \mathbf{T}^{D(2)}(M) | t(0) = \gamma(0, 0)\}$. □

These considerations can be generalized easily to *n-microcubes* for any natural number $n$. More specifically, given two $n$-microsquares $\gamma_+$ and $\gamma_-$ on $M$, their strong difference $\gamma_+ \dot{-} \gamma_-$ is defined exactly when $\gamma_+|_{D(n;n)} = \gamma_-|_{D(n;n)}$, and it is a tangent vector to $M$ with $(\gamma_+ \dot{-} \gamma_-)(0) = \gamma_+(0, \dots, 0) = \gamma_-(0, \dots, 0)$. Given $t \in \mathbf{T}^1(M)$ and $\gamma \in \mathbf{T}^n(\gamma)$ with $t(0) = \gamma(0, \dots, 0)$, the strong addition $t \dot{+} \gamma$ is defined to be an $n$-microcube on $M$ with $(t \dot{+} \gamma)|_{D(n;n)} = \gamma|_{D(n;n)}$. So as to define $\dot{-}$ and $\dot{+}$, we need the following two lemmas. Their proofs are akin to their counterparts of microsquares (cf. Lavendhomme [1996, pp.92-93]).

**Lemma 0.3.2.** (cf. Nishimura [1997. Lemma 5.1] and Lavendhomme and Nishimura [1998, Proposition 3]). *The diagram*

$$\begin{array}{ccc} D(n;n) & \xrightarrow{i} & D^n \\ i \downarrow & & \downarrow \Psi \\ D^n & \xrightarrow{\Phi} & D^n \vee D \end{array}$$

*is a quasi-colimit diagram, where* $i : D(n; n) \to D^n$ *is the canonical injection,* $D^n \vee D = \{(d_1, \dots, d_n, e) \in D^{n+1} \mid d_1 e = \dots = d_n e = 0\}$, $\Phi(d_1, \dots, d_n) = (d_1, \dots, d_n, 0)$ *and* $\Psi(d_1, \dots, d_n) = (d_1, \dots, d_n, d_1 \dots d_n)$.

Given two $n$-microsquares $\gamma_+$ and $\gamma_-$ on $M$ with $\gamma_+|_{D(n;n)} = \gamma_-|_{D(n;n)}$, there exists a unique function $f : D^n \vee D \to M$ with $f \circ \Psi = \gamma_+$ and $f \circ \Phi = \gamma_-$. We define $(\gamma_+ \dot{-} \gamma_-)(d) = f(0, 0, d)$ for any $d \in D$. From the very definition of $\dot{-}$ we have

**Proposition 0.3.3.** *Let* $f : M \to M'$. *Given* $\gamma_+, \gamma_- \in \mathbf{T}^n(M)$ *with* $\gamma_+|_{D(n;n)} = \gamma_-|_{D(n;n)}$, *we have* $f_*(\gamma_+)|_{D(n;n)} = f_*(\gamma_-)|_{D(n;n)}$ *and*

$$f_*(\gamma_+ \dot{-} \gamma_-) = f_*(\gamma_+) \dot{-} f_*(\gamma_-). \tag{0.3.1}$$

**Lemma 0.3.4.** *The diagram*

$$\begin{array}{ccc} 1 & \xrightarrow{i} & D^n \\ i\downarrow & & \downarrow\Xi \\ D^n & \xrightarrow{\Phi} & D^n\vee D \end{array}$$

*is a quasi-colimit diagram, where $i : 1 \to D^n$ and $i : 1 \to D$ are the canonical injections and $\Xi(d) = (0,\dots,0,d)$.* □

Given $t \in \mathbf{T}^1(M)$ and $\gamma \in \mathbf{T}^n(\gamma)$ with $t(0) = \gamma(0,\dots,0)$, there exists a unique function $f : D^n \vee D \to M$ with $f \circ \Phi = \gamma$ and $f \circ \Xi = t$. We define $(t\dot{+}\gamma)(d_1,\dots,d_n) = f(d_1,\dots,d_n,d_1\dots d_n)$ for any $(d_1,\dots,d_n) \in D^n$. From the very definition of $\dot{+}$ we have

**Proposition 0.3.5.** *Let $f : M \to M'$. Given $t \in \mathbf{T}^1(M)$ and $\gamma \in \mathbf{T}^n(\gamma)$ with $t(0) = \gamma(0,\dots,0)$, we have $f_*(t)(0) = f_*(\gamma)(0,\dots,0)$ and*

$$f_*(t\dot{+}\gamma) = f_*(t)\dot{+}f_*(\gamma) \tag{0.3.2}$$

□

We can proceed as in the case of microsquares to get

**Theorem 0.3.6.** *The canonical projection $\mathbf{T}^n(M) \to \mathbf{T}^{D(n;n)}(M)$ is an affine bundle over the vector bundle $\mathbf{T}^1(M) \underset{M}{\times} \mathbf{T}^{D(n;n)}(M) \to \mathbf{T}^{D(n;n)}(M)$ assigning $\gamma$ to each $(t,\gamma) \in \mathbf{T}^1(M) \underset{M}{\times} \mathbf{T}^{D(n;n)}(M) = \{(t,\gamma) \in \mathbf{T}^1(M) \times \mathbf{T}^{D(n;n)}(M) | t(0) = \gamma(0,0)\}$.*

We have the following $n$-dimensional counterparts of Propositions 5, 6 and 7 of Lavendhomme [1996, §3.4].

**Proposition 0.3.7.** *For any $\alpha \in \mathbb{R}$, any $\gamma_+,\gamma_-,\gamma \in \mathbf{T}^n(M)$ and any $t \in \mathbf{T}^1(M)$ with $\gamma_+|_{D(n;n)} = \gamma_-|_{D(n;n)}$ and $t(0) = \gamma(0,\dots,0)$, we have*

$$\alpha(\gamma_+\dot{-}\gamma_-) = (\alpha \underset{i}{\cdot} \gamma_+)\dot{-}(\alpha \underset{i}{\cdot} \gamma_-). \tag{0.3.3}$$

$$\alpha \underset{i}{\cdot} (t_i\dot{+}\gamma) = \alpha t\dot{+}\alpha \underset{i}{\cdot} \gamma_i \tag{0.3.4}$$

□

**Proposition 0.3.8.** *For any $\sigma \in \sigma_n$, any $\gamma_+,\gamma_-,\ \gamma \in \mathbf{T}^n(M)$, and any $t \in \mathbf{T}^1(M)$ with $\gamma_+|_{D(n;n)} = \gamma_-|_{D(n;n)}$ and $t(0) = \gamma(0,\dots,0)$, we have*

$$\Sigma_\sigma(\gamma_+)\dot{-}\Sigma_\sigma(\gamma_-) = \gamma_+\dot{-}\gamma_- \tag{0.3.5}$$

$$\Sigma_\sigma(t\dot{+}\gamma) = t\dot{+}\Sigma_\sigma(\gamma) \tag{0.3.6}$$

□

**Proposition 0.3.9.** *For $\gamma_+,\gamma_-,\gamma \in \mathbf{T}^n(M)$ with $\gamma_+|_{D(n;n)} = \gamma_-|_{D(n;n)}$ we have*

$$\gamma_+\dot{-}\gamma_- = (\cdots(\gamma_+ \underset{1}{-} \gamma_-) \underset{2}{-} \mathbf{s}_1 \circ \mathbf{d}_1(\gamma_+)) \underset{3}{-} \mathbf{s}_1^2 \circ \mathbf{d}_1^2(\gamma_+)) \cdots \underset{n}{-} \mathbf{s}_1^{n-1} \circ \mathbf{d}_1^{n-1}(\gamma_+) \tag{0.3.7}$$

□

### 0.4. Symmetric forms

Given a vector bundle $\pi : E \to M$ and a bundle $\xi : P \to M$, a *symmetric $n$-form at $x \in P$ along $\xi$ with values in $\pi$* is a mapping $\omega : \mathbf{T}_x^n(P) \to E_{\xi(x)}$ such that for any $\gamma \in \mathbf{T}^n(P)$, any $\gamma' \in \mathbf{T}^{n-1}(P)$, any $\alpha \in \mathbb{R}$ and any $\sigma \in \mathfrak{S}_n$ we have

$$\omega(\alpha \underset{i}{\cdot} \gamma) = \alpha\omega(\gamma) \qquad (1 \leq i \leq n) \tag{0.4.1}$$

$$\omega(\Sigma_\sigma(\gamma)) = \omega(\gamma) \tag{0.4.2}$$

$$\omega((d_1, \dots, d_n) \in D^n \longmapsto \gamma'(d_1, \dots, d_{n-2}, d_{n-1}d_n)) = 0 \tag{0.4.3}$$

We denote by $\mathbf{S}_x^n(\xi;\pi)$ the totality of symmetric $n$-forms at $x$ along $\xi$ with values in $\pi$. We denote by $\mathbf{S}^n(\xi;\pi)$ the set-theoretic union of $\mathbf{S}_x^n(\xi;\pi)'s$ for all $x \in P$. If $P = M$ and $\xi : P \to M$ is the identity mapping, then $\mathbf{S}_x^n(\xi;\pi)$ and $\mathbf{S}^n(\xi;\pi)$ are usually denoted by $\mathbf{S}_x^n(M;\pi)$ and $\mathbf{S}^n(M;\pi)$ respectively.

**Proposition 0.4.1.** *Let $\omega \in \mathbf{S}^{n+1}(\xi;\pi)$. Then we have*

$$\omega(\mathbf{s}_i(\gamma)) = 0 \qquad (1 \leq i \leq n+1) \tag{0.4.4}$$

*for any $\gamma \in \mathbf{T}^n(P)$.*

*Proof.* For any $\alpha \in \mathbb{R}$, we have

$$\omega(\mathbf{s}_i(\gamma)) = \omega(\alpha \underset{i}{\cdot} \mathbf{s}_i(\gamma)) = \alpha\omega(\mathbf{s}_i(\gamma)) \tag{0.4.5}$$

Let $\alpha = 0$, we have the desired conclusion. □

### 0.5. Convention

Two bundles $\pi : E \to M$ and $\pi' : E' \to M$ over the same microlinear space $M$ shall be chosen once and for all.

## 1. Preconnections

Let $n$ be a natural number. An *$n$-pseudoconnection over the bundle $\pi : E \to M$ at $x \in E$* is a mapping $\nabla_x : \mathbf{T}_{\pi(x)}^n(M) \to \mathbf{T}_x^n(E)$ such that for any $\gamma \in \mathbf{T}_{\pi(x)}^n(M)$, any $\alpha \in \mathbb{R}$ and any $\sigma \in \mathfrak{S}_n$, we have the following:

$$\pi \circ \nabla_x(\gamma) = \gamma \tag{1.1}$$

$$\nabla_x(\alpha \underset{i}{\cdot} \gamma) = \alpha \underset{i}{\cdot} \nabla_x(\gamma) \qquad (1 \leq i \leq n) \tag{1.2}$$

$$\nabla_x(\Sigma_\sigma(\gamma)) = \Sigma_\sigma(\nabla_x(\gamma)) \tag{1.3}$$

We denote by $\hat{\mathbb{J}}_x^n(\pi)$ the totality of $n$-pseudoconnections $\nabla_x$ over the bundle $\pi : E \to M$ at $x \in E$. We denote by $\hat{\mathbb{J}}^n(\pi)$ the set-theoretic union of $\hat{\mathbb{J}}_x^n(\pi)'s$ for all $x \in E$. In particular, $\hat{\mathbb{J}}^0(\pi) = E$ by convention.

Let $\nabla_x$ be an $(n+1)$-pseudoconnection over the bundle $\pi : E \to M$ at $x \in E$. Let $\gamma \in \mathbf{T}^n_{\pi(x)}(M)$ and $(d_1, \dots, d_{n+1}) \in D^{n+1}$. Then we have

**Lemma 1.1.** *$\nabla_x(\mathbf{s}_{n+1}(\gamma))(d_1, \dots, d_n, d_{n+1})$ is independent of $d_{n+1}$, so that we can put down $\nabla_x(\mathbf{s}_{n+1}(\gamma))$ at $\mathbf{T}^n_x(E)$.*

*Proof.* The proof is similar to that of Proposition 0.4.1. For any $\alpha \in \mathbb{R}$ we have

$$\begin{aligned}(\nabla_x(\mathbf{s}_{n+1}(\gamma)))(d_1, \dots, d_n, \alpha d_{n+1}) &= (\alpha \underset{n+1}{\cdot} \nabla_x(\mathbf{s}_{n+1}(\gamma)))(d_1, \dots, d_n, d_{n+1}) \\ &= (\nabla_x(\alpha \underset{n+1}{\cdot} (\mathbf{s}_{n+1}(\gamma))))(d_1, \dots, d_n, d_{n+1}) \\ &= (\nabla_x(\mathbf{s}_{n+1}(\gamma)))(d_1, \dots, d_n, d_{n+1})\end{aligned} \tag{1.4}$$

Letting $\alpha = 0$ in (1.4), we have

$$(\nabla_x(\mathbf{s}_{n+1}(\gamma)))(d_1, \dots, d_n, 0) = (\nabla_x(\mathbf{s}_{n+1}(\gamma)))(d_1, \dots, d_n, d_{n+1}), \tag{1.5}$$

which shows that $\nabla_x(\mathbf{s}_{n+1}(\gamma))(d_1, \dots, d_n, d_{n+1})$ is independent of $d_{n+1}$. □

Now it is easy to see that

**Proposition 1.2.** *The assignment $\gamma \in \mathbf{T}^n_{\pi(x)}(M) \longmapsto \nabla_x(\mathbf{s}_{n+1}(\gamma)) \in \mathbf{T}^n_x(E)$ is an $n$-pseudoconnection over the bundle $\pi : E \to M$ at $x$.* □

By Proposition 1.2 we have the canonical projections $\underline{\hat{\pi}}_{n+1,n} : \hat{\mathbb{J}}^{n+1}(\pi) \to \hat{\mathbb{J}}^n(\pi)$. By assigning $\pi(x) \in M$ to each the canonical projections $\underline{\hat{\pi}}_n : \hat{\mathbb{J}}^n(\pi) \to M$. Note that $\underline{\hat{\pi}}_n \circ \underline{\hat{\pi}}_{n+1,n} = \underline{\hat{\pi}}_{n+1}$. For any natural numbers $n$, $m$ with $m \leq n$, we define $\underline{\hat{\pi}}_{n,m} : \hat{\mathbb{J}}^n(\pi) \to \hat{\mathbb{J}}^m(\pi)$ to be $\underline{\hat{\pi}}_{m+1,m} \circ \dots \circ \underline{\hat{\pi}}_{n,n-1}$.

Now we are going to show that

**Proposition 1.3.** *Let $\nabla_x \in \hat{\mathbb{J}}^{n+1}(\pi)$. Then the following diagrams are commutative:*

$$\begin{tikzcd} \mathbf{T}^{n+1}(M) \arrow[r, "\nabla_x"] & \mathbf{T}^{n+1}(E) \\ \mathbf{T}^n(M) \arrow[u, "\mathbf{s}_i"] \arrow[r, "\underline{\hat{\pi}}_{n+1,n}(\nabla_x)"'] & \mathbf{T}^n(E) \arrow[u, "\mathbf{s}_i"'] \end{tikzcd}$$

$$\begin{tikzcd} \mathbf{T}^{n+1}(M) \arrow[r, "\nabla_x"] \arrow[d, "\mathbf{d}_i"'] & \mathbf{T}^{n+1}(E) \arrow[d, "\mathbf{d}_i"] \\ \mathbf{T}^n(M) \arrow[r, "\underline{\hat{\pi}}_{n+1,n}(\nabla_x)"'] & \mathbf{T}^n(E) \end{tikzcd}$$

*Proof.* By the very definition of $\underline{\hat{\pi}}_{n+1,n}$ we have

$$\mathbf{s}_{n+1}(\underline{\hat{\pi}}_{n+1}(\nabla_x)(\gamma)) = \nabla_x(\mathbf{s}_{n+1}(\gamma)) \tag{1.6}$$

for any $\gamma \in \mathbf{T}^n_{\pi(x)}(M)$. For $i \neq n+1$, we have

$$\begin{aligned}\mathbf{s}_i(\underline{\hat{\pi}}_{n+1,n}(\nabla_x)(\gamma)) &= \Sigma_{<i+1,i+2,\dots,n,n+1>}(\Sigma_{<i,n+1>}(\mathbf{s}_{n+1}(\underline{\hat{\pi}}_{n+1,n}(\nabla_x)(\gamma)))) \\ &= \Sigma_{<i+1,i+2,\dots,n,n+1>}(\Sigma_{<i,n+1>}(\nabla_x(\mathbf{s}_{n+1}(\gamma)))) \quad [(1.6)] \\ &= \Sigma_{<i+1,i+2,\dots,n,n+1>}(\nabla_x(\Sigma_{<i,n+1>}(\mathbf{s}_{n+1}(\gamma)))) \quad [(1.3)] \\ &= \nabla_x(\Sigma_{<i+1,i+2,\dots,n,n+1>}(\Sigma_{<i,n+1>}(\mathbf{s}_{n+1}(\gamma)))) \quad [(1.3)] \\ &= \nabla_x(\mathbf{s}_i(\gamma))\end{aligned} \tag{1.7}$$

Now we are going to show that

$$\mathbf{d}_i(\nabla_x(\gamma)) = (\underline{\hat{\pi}}_{n+1,n}(\nabla_x))(\mathbf{d}_i(\gamma)) \tag{1.8}$$

for any $\gamma \in \mathbf{T}^{n+1}_{\pi(x)}(M)$. First we deal with the case of $i = n+1$. For any $(d_1, \dots, d_{n+1}) \in D^{n+1}$ we have

$$\begin{aligned}(\mathbf{d}_{n+1}(\nabla_x(\gamma)))(d_1,\dots,d_n) &= (\nabla_x(\gamma))(d_1,\dots,d_n,0) \\ &= (\nabla_x(\gamma))(d_1,\dots,d_n,0d_{n+1}) \\ &= (0 \underset{n+1}{\cdot} \nabla_x(\gamma))(d_1,\dots,d_n,d_{n+1}) \\ &= (\nabla_x(0 \underset{n+1}{\cdot} \gamma))(d_1,\dots,d_n,d_{n+1}) \\ &= (\nabla_x(\mathbf{s}_{n+1}(\mathbf{d}_{n+1}(\gamma))))(d_1,\dots,d_n,d_{n+1}) \\ &= (\underline{\hat{\pi}}_{n+1,n}(\nabla_x))(\mathbf{d}_{n+1}(\gamma))(d_1,\dots,d_n)\end{aligned} \tag{1.9}$$

For $i \neq n+1$ we have

$$\begin{aligned}\mathbf{d}_i(\nabla_x(\gamma)) &= \Sigma_{<n,n-1,\dots,i+1,i>}(\mathbf{d}_{n+1}(\Sigma_{<i,n+1>}(\nabla_x(\gamma)))) \\ &= \Sigma_{<n,n-1,\dots,i+1,i>}(\mathbf{d}_{n+1}(\nabla_x(\Sigma_{<i,n+1>}(\gamma)))) \\ &\quad [(1.3)] \\ &= \Sigma_{<n,n-1,\dots,i+1,i>}(\underline{\hat{\pi}}_{n+1,n}(\nabla_x)(\mathbf{d}_{n+1}(\Sigma_{<i,n+1>}(\gamma)))) \\ &\quad [(1.9)] \\ &= \underline{\hat{\pi}}_{n+1,n}(\nabla_x)(\Sigma_{<n,n-1,\dots,i+1,i>}(\mathbf{d}_{n+1}(\Sigma_{<i,n+1>}(\gamma)))) \\ &\quad [(1.3)] \\ &= \underline{\hat{\pi}}_{n+1,n}(\nabla_x)(\mathbf{d}_i(\gamma))\end{aligned} \tag{1.10}$$

□

**Corollary 1.4.** *Let $\nabla^+_x$, $\nabla^-_x \in \hat{\mathbb{J}}^{n+1}(\pi)$ with $\underline{\hat{\pi}}_{n+1,n}(\nabla^+_x) = \underline{\pi}_{n+1,n}(\nabla^-_x)$. Then*

*$\nabla^+_x(\gamma)|_{D(n+1;n+1)} = \nabla^-_x(\gamma)|_{D(n+1;n+1)}$ for any $\gamma \in \mathbf{T}^{n+1}_{\pi(x)}(M)$.*

*Proof.* By Lemma 0.1.1 and Proposition 1.3. □

The notion of an *n-preconnection* is defined inductively on $n$. The notion of a 1-*preconnection* shall be identical with that of a 1-pseudoconnection. Now we proceed inductively. An $(n+1)$-pseudoconnection $\nabla_x : \mathbf{T}^{n+1}_{\pi(x)}(M) \to \mathbf{T}^{n+1}_x(E)$ over the bundle $\pi : E \to M$ at $x \in E$ is called an $(n+1)$-*preconnection over* the bundle $\pi : E \to M$ *at* $x$ if it acquiesces in the following two conditions

$$\hat{\underline{\pi}}_{n+1,n}(\nabla_x) \text{ is an } n\text{−preconnection.} \tag{1.11}$$

For any $\gamma \in \mathbf{T}^n_{\pi(x)}(M)$, we have

$$\begin{aligned} &\nabla_x((d_1,\dots,d_{n+1}) \in D^{n+1} \longmapsto \gamma(d_1,\dots,d_{n-1},d_n d_{n+1})) \\ &= (d_1,\dots,d_{n+1}) \in D^{n+1} \longmapsto \hat{\underline{\pi}}_{n+1,n}(\nabla_x)(\gamma)(d_1,\dots,d_{n-1},d_n d_{n+1}). \end{aligned} \tag{1.12}$$

We denote by $\mathbb{J}^n_x(\pi)$ the totality of $n$-preconnections $\nabla_x$ over the bundle $\pi : E \to M$ at $x \in E$. We denote by $\mathbb{J}^n(\pi)$ the set-theoretic union of $\mathbb{J}^n_x(\pi)'s$ for all $x \in E$. In particular, $\mathbb{J}^0(\pi) = \hat{\mathbb{J}}^0(\pi) = E$ by convention and $\mathbb{J}^1(\pi) = \hat{\mathbb{J}}^1(\pi)$ by definition. By the very definition of $n$-preconnection, the projections $\hat{\underline{\pi}}_{n+1,n} : \hat{\mathbb{J}}^{n+1}(\pi) \to \hat{\mathbb{J}}^n(\pi)$ are naturally restricted to mappings $\underline{\pi}_{n+1,n} : \mathbb{J}^{n+1}(\pi) \to \mathbb{J}^n(\pi)$. Similarly for $\underline{\pi}_n : \mathbb{J}^n(\pi) \to M$ and $\underline{\pi}_{n,m} : \mathbb{J}^n(\pi) \to \mathbb{J}^m(\pi)$ with $m \leq n$.

**Proposition 1.5.** *Let $m,n$ be natural numbers with $m \leq n$. Let $k_1,\dots,k_m$ be positive integers with $k_1+\cdots+k_m = n$. For any $\nabla_x \in \mathbb{J}^n(\pi)$, any $\gamma \in \mathbf{T}^m_{\pi(x)}(M)$ and any $\sigma \in \mathfrak{S}_n$ we have*

$$\begin{aligned} &\nabla_x((d_1,\dots,d_n) \in D^n \longmapsto \gamma(d_{\sigma(1)}\dots d_{\sigma(k_1)}, \\ &\qquad d_{\sigma(k_1+1)}\dots d_{\sigma(k_1+k_2)},\dots,d_{\sigma(k_1+\cdots+k_{m-1}+1)}\cdots{}_{\sigma(n)})) \\ &= (d_1,\dots,d_n) \in D^n \longmapsto \underline{\pi}_{n,m}(\nabla_x)(\gamma)(d_{\sigma(1)}\dots d_{\sigma(k_1)}, d_{\sigma(k_1+1)}\dots \\ &\qquad d_{\sigma(k_1+k_2)},\dots,d_{\sigma(k_1+\cdots+k_{m-1}+1)}\dots d_{\sigma(n)}) \end{aligned} \tag{1.13}$$

*Proof.* This follows simply from repeated use of (1.3) and (1.12). □

The following proposition will be used in the proof of Proposition 3.6.

**Proposition 1.6.** *Let $\nabla_x \in \mathbb{J}^n(\pi)$, $t \in \mathbf{T}^1_{\pi(x)}(M)$ and $\gamma$, $\gamma_+$, $\gamma_- \in \mathbf{T}^n_{\pi(x)}(M)$ with $\gamma_+|_{D(n;n)} = \gamma_-|_{D(n;n)}$. Then we have*

$$\nabla_x(\gamma_+)\dot{-}\nabla_x(\gamma_-) = \underline{\pi}_{n,1}(\nabla_x)(\gamma_+\dot{-}\gamma_-) \tag{1.14}$$

$$\underline{\pi}_{n,1}(\nabla_x)(t)\dot{+}\nabla_x(\gamma) = \nabla_x(t\dot{+}\gamma). \tag{1.15}$$

*Proof.* It is an easy exercise of affine geometry to show that (1.14) and (1.15) are equivalent. Here we deal only with (1.14) in case of $n = 2$. For any $d_1, d_2 \in D$, we have

$$
\begin{aligned}
(\nabla_x(\gamma_+)\dot{-}\nabla_x(\gamma_-))(d_1d_2) =&((\nabla_x(\gamma_+) \underset{1}{-} \nabla_x(\gamma_-)) \underset{2}{-} (\mathbf{s}_1 \circ \mathbf{d}_1)(\nabla_{\mathbf{x}}(\gamma_+)))(d_1, d_2) \\
&\text{[By Proposition 0.3.9]} \\
=&\nabla_x((\gamma_+ \underset{1}{-} \gamma_-) \underset{2}{-} (\mathbf{s}_1 \circ \mathbf{d}_1)(\gamma_+))(d_1, d_2) \\
&\text{[By (1.2) and Proposition 1.3]} \\
=&\nabla_x(((e_1, e_2) \in D^2 \longmapsto (\gamma_+\dot{-}\gamma_-)(e_1e_2)))(d_1, d_2) \\
&\text{[By Proposition 0.3.9 again]} \\
=&\underline{\pi}_{2,1}(\nabla_x)(\gamma_+\dot{-}\gamma-)(d_1d_2) \\
&\text{[By Proposition 1.5]},
\end{aligned} \tag{1.16}
$$

so that (1.14) in case of $n = 2$ obtains. □

**Proposition 1.7.** *Let $\nabla_x^+$, $\nabla_x^- \in \mathbb{J}_x^{n+1}(\pi)$ with $\hat{\underline{\pi}}_{n+1,n}(\nabla_x^+) = \hat{\underline{\pi}}_{n+1,n}(\nabla_x^-)$. Then the assignment $\gamma \in \mathbf{T}_{\pi(x)}^{n+1}(M) \longmapsto \nabla_x^+(\gamma)\dot{-}\nabla_x^-(\gamma)$ belongs to $\mathbf{S}_{\pi(x)}^{n+1}(M; v_\pi)$.*

*Proof.* Since

$$
\begin{aligned}
\pi_*(\nabla_x^+(\gamma)\dot{-}\nabla_x^-(\gamma)) &= \pi_*(\nabla_x^+(\gamma))\dot{-}\pi_*(\nabla_x^-(\gamma)) \quad \text{[By Proposition 0.3.3]} \\
&= 0 \quad [(1.1)],
\end{aligned} \tag{1.17}
$$

$\nabla_x^+(\gamma)\dot{-}\nabla_x^-(\gamma)$ belongs in $\mathbf{V}_x^1(\pi)$. For any $\alpha \in \mathbb{R}$ and any natural number $i$ with $1 \leq 1 \leq n+1$, we have

$$
\begin{aligned}
\nabla_x^+(\alpha \underset{i}{\cdot} \gamma)_i\dot{-}\nabla_x^-(\alpha \underset{i}{\cdot} \gamma) &= \alpha \underset{i}{\cdot} \nabla_x^+(\gamma)\dot{-}\alpha \underset{i}{\cdot} \nabla_x^-(\gamma) \quad [(1.2)] \\
&= \alpha(\nabla_x^+(\gamma)\dot{-}\nabla_x^-(\gamma)) \qquad [(0.3.3)],
\end{aligned} \tag{1.18}
$$

which implies that the assignment $\gamma \in \mathbf{T}_{\pi(x)}^{n+1}(M) \longmapsto \nabla_x^+(\gamma)\dot{-}\nabla_x^-(\gamma)$ abides by (0.4.1). For any $\sigma \in \mathfrak{S}_{n+1}$ we have

$$
\begin{aligned}
\nabla_x^+(\Sigma_\sigma(\gamma))\dot{-}\nabla_x^-(\Sigma_\sigma(\gamma)) &= \Sigma_\sigma(\nabla_x^+(\gamma))\dot{-}\Sigma_\sigma(\nabla_x^-(\gamma)) \quad [(1.3)] \\
&= \Sigma_\sigma(\nabla_x^+(\gamma)\dot{-}\nabla_x^-(\gamma)) \qquad [(0.3.5)],
\end{aligned} \tag{1.19}
$$

which implies that the assignment $\gamma \in \mathbf{T}_{\pi(x)}^{n+1}(M) \longmapsto \nabla_x^+(\gamma)\dot{-}\nabla_x^-(\gamma)$ abides by (0.4.2). It remains to show that the assignment $\gamma \in \mathbf{T}_{\pi(x)}^{n+1}(M) \longmapsto \nabla_x^+(\gamma)\dot{-}\nabla_x^-(\gamma)$ abides by (0.4.3), which follows directly from (1.12) and the assumption that $\hat{\underline{\pi}}_{n+1,n}(\nabla_x^+) = \hat{\underline{\pi}}_{n+1,n}(\nabla_x^-)$. □

**Proposition 1.8.** *Let $\nabla_x \in \mathbb{J}_x^{n+1}(\pi)$ and $\omega \in \mathbf{S}_{\pi(x)}^{n+1}(M; v_\pi)$. Then the assignment $\gamma \in \mathbf{T}_{\pi(x)}^{n+1}(M) \longmapsto \omega(\gamma)\dot{+}\nabla_x(\gamma)$ belongs to $\mathbb{J}_x^{n+1}(\pi)$.*

*Proof.* Since

$$
\begin{aligned}
\pi_*(\omega(\gamma)\dot{+}\nabla_x(\gamma)) &= \pi_*(\omega(\gamma))\dot{+}\pi_*(\nabla_x(\gamma)) \quad [(0.3.2)] \\
&= \gamma \quad [(1.1)],
\end{aligned} \tag{1.20}
$$

the assignment $\gamma \in \mathbf{T}^{n+1}_{\pi(x)}(M) \longmapsto \omega(\gamma)\dot{+}\nabla_x(\gamma)$ stands to (1.1). For any $\alpha \in \mathbb{R}$ and any natural number $i$ with $1 \leq i \leq n+1$, we have

$$\begin{aligned}\omega(\alpha \underset{i}{\cdot} \gamma)_i\dot{+}\nabla_x(\alpha \underset{i}{\cdot} \gamma) &= \alpha\omega(\gamma)\dot{+}\alpha \underset{i}{\cdot} \nabla_x(\gamma) \quad [(0.4.1) \text{ and } (1.2)] \\ &= \alpha \underset{i}{\cdot} (\omega(\gamma)_i\dot{+}\nabla_x(\gamma)) \quad [(0.3.4)],\end{aligned} \tag{1.21}$$

so that the assignment $\gamma \in \mathbf{T}^{n+1}_{\pi(x)}(M) \longmapsto \omega(\gamma)\dot{+}\nabla_x(\gamma)$ stands to (1.2). For any $\sigma \in \mathfrak{S}_{n+1}$ we have

$$\begin{aligned}\omega(\Sigma_\sigma(\gamma))\dot{+}\nabla_x(\Sigma_\sigma(\gamma)) &= \omega(\gamma)\dot{+}\Sigma_\sigma(\nabla_x(\gamma)) \quad [(0.4.2)\text{and } (1.2)] \\ &= \Sigma_\sigma(\omega(\gamma)\dot{+}\nabla_x(\gamma)) \quad [(0.3.6)],\end{aligned} \tag{1.22}$$

so that the assignment $\gamma \in \mathbf{T}^{n+1}_{\pi(x)}(M) \longmapsto \omega(\gamma)\dot{+}\nabla_x(\gamma)$ stands to (1.3). That the assignment $\gamma \in \mathbf{T}^{n+1}_{\pi(x)}(M) \longmapsto \omega(\gamma)\dot{+}\nabla_x(\gamma)$ stands to (1.11) follows from the simple fact that the image of the assignment under $\underline{\hat{\pi}}_{n+1,n}$ coincides with $\underline{\hat{\pi}}_{n+1,n}(\nabla_x)$, which is consequent upon Proposition 0.4.1. It remains to show that the assignment $\gamma \in \mathbf{T}^{n+1}_{\pi(x)}(M) \longmapsto \omega(\gamma)\dot{+}\nabla_x(\gamma)$ abides by (1.12), which follows directly from (0.4.3) and (1.12). □

For any $\nabla_x^+, \nabla_x^- \in \mathbb{J}^{n+1}(\pi)$ with $\underline{\hat{\pi}}_{n+1,n}(\nabla_x^+) = \underline{\hat{\pi}}_{n+1,n}(\nabla_x^-)$, we define $\nabla_x^+\dot{-}\nabla_x^- \in \mathbf{S}^{n+1}_{\pi(x)}(M; v_\pi)$ to be

$$(\nabla_x^+\dot{-}\nabla_x^-)(\gamma) = \nabla_x^+(\gamma)\dot{-}\nabla_x^-(\gamma) \tag{1.23}$$

for any $\gamma \in \mathbf{T}^{n+1}_{\pi(x)}(M)$. This is well defined by dint of Lemma 1.4 and Propositions 0.3.5 and 0.3.6. For any $\omega \in \mathbf{S}^{n+1}_{\pi(x)}(M; v_\pi)$ and any $\nabla_x \in \mathbb{J}^{n+1}(\pi)$ we define $\omega\dot{+}\nabla_x \in \mathbb{J}^{n+1}_x(\pi)$ to be

$$(\omega\dot{+}\nabla_x)(\gamma) = \omega(\gamma)\dot{+}\nabla_x(\gamma) \tag{1.24}$$

for any $\gamma \in \mathbf{T}^{n+1}_{\pi(x)}(M)$. This is well defined by dint of Propositions 0.3.5 and 0.3.6

With these two operations defined in (1.23) and (1.24) it is easy to see that

**Theorem 1.9** (cf. Saunders [1989, Theorem 6.2.9]). *The bundle* $\underline{\pi}_{n+1,n} : \mathbb{J}^{n+1}(\pi) \to \mathbb{J}^n(\pi)$ *is an affine bundle over the vector bundle* $\mathbb{J}^n(\pi) \underset{M}{\times} \mathbf{S}^{n+1}(M; v_\pi) \to \mathbb{J}^n(\pi)$. □

An *$n$-connection* $\nabla$ *over* $\pi$ is simply an assignment of an $n$-preconnection $\nabla_x$ over $\pi$ at $x$ to each point $x$ of $E$, in which we will often write $\nabla(\gamma, x)$ in place of $\nabla_x(\gamma)$. 1-preconnections over $\pi$ (at $x \in E$) in this paper were called simply preconnections over $\pi$ (at $x \in E$) in Nishimura [2001].

Let $f$ be a morphism of bundles over $M$ from $\pi$ to $\pi'$. We say that an $n$-preconnection $\nabla_x$ over $\pi$ at a point $x$ of $E$ *is $f$-related to* an $n$-preconnection $\nabla_y$ over $\pi'$ at a point $y = f(x)$ of $E'$ provided that

$$f \circ \nabla_x(\gamma) = \nabla_y(\gamma) \tag{1.25}$$

for any $\gamma \in \mathbf{T}^n_a(M)$ with a $= \pi(x) = \pi'(y)$.

Now we recall the construction of $\mathbf{J}^n(\pi)'s$ in Nishimura [2003]. By convention we let $\mathbf{J}^0(\pi) = \mathbb{J}^0(\pi) = E$ with $\pi_{0,0} = \underline{\pi}_{0,0} = id_E$ and $\pi_0 = \underline{\pi}_0 = \pi$. We let $\mathbf{J}^1(\pi) = \mathbb{J}^1(\pi)$ with $\pi_{1,0} = \underline{\pi}_{1,0}$ and $\pi_1 = \underline{\pi}_1$. Now we are going to define $\mathbf{J}^{n+1}(\pi)$ together with the canonical mapping $\pi_{n+1,n} : \mathbf{J}^{n+1}(\pi) \to \mathbf{J}^n(\pi)$ by induction on $n \geq 1$. These are intended for holonomic jet bundles (cf. Saunders [1989, Chapter 5]). We define $\mathbf{J}^{n+1}(\pi)$ to be the subspace of $\mathbf{J}^1(\pi_n)$ consisting of $\nabla_x' s$ with $x = \nabla_y \in \mathbf{J}^n(\pi)$ pursuant to the following two conditions:

$$\nabla_x \text{ is } \pi_{n,n-1}\text{−related to } \nabla_y. \tag{1.26}$$

$$\begin{aligned} &\text{Let } d_1, d_2 \in D \text{ and } \gamma \text{ a microsquare on } M \text{ with} \\ &\gamma(0,0) = \pi_n(x). \text{ Let it be that} \\ &z = \nabla_y(\gamma(\cdot,0))(d_1) \qquad (1.27.1) \\ &w = \nabla_y(\gamma(0,\cdot))(d_2) \qquad (1.27.2) \\ &\nabla_z = \nabla_x(\gamma(\cdot,0))(d_1) \qquad (1.27.3) \\ &\nabla_w = \nabla_x(\gamma(0,\cdot))(d_2) \qquad (1.27.4) \end{aligned} \tag{1.27}$$

Then we have

$$\nabla_z(\gamma(d_1,\cdot))(d_2) = \nabla_w(\gamma(\cdot,d_2))(d_1) \tag{1.27.5}$$

We define $\pi_{n+1,n}$ to be the restriction of $(\pi_n)_{1,0} : \mathbf{J}^1(\mathbf{J}^n(\pi)) \to \mathbf{J}^n(\pi)$ to $\mathbf{J}^{n+1}(\pi)$. We let $\pi_{n+1} = \pi_n \circ \pi_{n+1,n}$

## 2. Translation of repeated 1-jets into higher-order Preconnections

Mappings $\varphi_n : \mathbf{J}^n(\pi) \to \mathbb{J}^n(\pi)(n = 0,\, 1)$ shall be the identity mappings. We are going to define $\varphi_n : \mathbf{J}^n(\pi) \to \mathbb{J}^n(\pi)$ for any natural number $n$ by induction on $n$. Let $x_n = \nabla_{x_{n-1}} \in \mathbf{J}^n(\pi)$ and $\nabla_{x_n} \in \mathbf{J}^{n+1}(\pi)$. We define $\varphi_{n+1}(\nabla_{x_n})$ as follows:

$$\varphi_{n+1}(\nabla_{x_n})(\gamma)(d_1,\dots,d_{n+1}) = \varphi_n(\nabla_{x_n}(\gamma(0,\dots,0,\cdot))(d_{n+1}))(\gamma(\cdot,\dots,\cdot,d_{n+1}))(d_1,\dots,d_n) \tag{2.1}$$

for any $\gamma \in \mathbf{T}^{n+1}_{\pi_n(x_n)}(M)$ and any $(d_1,\dots,d_{n+1}) \in D^{n+1}$. Then we have

**Lemma 2.1.** $\varphi_{n+1}(\nabla_{x_n}) \in \hat{\mathbb{J}}^{n+1}(\pi)$.

*Proof.* It suffices to show that for any $\gamma \in \mathbf{T}^{n+1}_{\pi_n(x_n)}(M)$, any $\alpha \in \mathbb{R}$ and any $\sigma \in \mathfrak{S}_{n+1}$ we have

$$\pi \circ \varphi_{n+1}(\nabla_{x_n})(\gamma) = \gamma \tag{2.2}$$

$$\varphi_{n+1}(\nabla_{x_n})(\alpha \underset{i}{\cdot} \gamma) = \alpha \underset{i}{\cdot} \varphi_{n+1}(\nabla_{x_n})(\gamma) \quad (1 \leq i \leq n+1) \tag{2.3}$$

$$\varphi_{n+1}(\nabla_{x_n})(\Sigma_\sigma(\gamma)) = \Sigma_\sigma(\varphi_{n+1}(\nabla_{x_n})(\gamma)) \tag{2.4}$$

We proceed by induction on $n$. First we deal with (2.2)

$$\begin{aligned}\pi \circ \varphi_{n+1}(\nabla_{x_n})(\gamma)(d_1, \dots, d_{n+1}) &= \pi(\varphi_{n+1}(\nabla_{x_n})(\gamma)(d_1, \dots, d_{n+1})) \\ &= \pi(\varphi_n(\nabla_{x_n}(\gamma(0, \dots, 0, \cdot))(d_{n+1}))(\gamma(\cdot, \dots, \cdot, d_{n+1}))(d_1, \dots, d_n)) \\ &\quad \text{[By the definition of } \varphi_{\mathrm{n+1}}] \\ &= \pi \circ \varphi_n(\nabla_{x_n}(\gamma(0, \dots, 0, \cdot))(d_{n+1}))(\gamma(\cdot, \dots, \cdot, d_{n+1}))(d_1, \dots, d_n) \\ &= \gamma(\cdot, \dots, \cdot, d_{n+1})(d_1, \dots, d_n) \\ &\quad \text{[By induction hypothesis]} \\ &= \gamma(d_1, \dots, d_{n+1})\end{aligned} \tag{2.5}$$

Next we deal with (2.3), the treatment of which is divided into two cases, namely, $i \leq n$ and $i = n + 1$. For the former case we have

$$\begin{aligned}&\varphi_{n+1}(\nabla_{x_n})(\alpha \underset{i}{\cdot} \gamma)(d_i, \dots, d_{n+1}) \\ &\quad = \varphi_n(\nabla_{x_n}(\alpha \underset{i}{\cdot} \gamma(0, \dots, 0, \cdot))(d_{n+1}))(\alpha \underset{i}{\cdot} \gamma(\cdot, \dots, \cdot, d_{n+1}))(d_1, \dots, d_n) \\ &\quad\quad \text{[By the definition of } \varphi_{n+1}] \\ &\quad = \alpha \underset{i}{\cdot} \varphi_i(\nabla_{x_n}(\gamma(0, \dots, 0, \cdot))(d_{n+1}))(\gamma(\cdot, \dots, \cdot, d_{n+1}))(d_1, \dots, d_n) \\ &\quad\quad \text{[By induction hypothesis]} \\ &\quad = \varphi_n(\nabla_{x_n}(\gamma(0, \dots, 0, \cdot))(d_{n+1}))(\gamma(\cdot, \dots, \cdot, d_{n+1}))(d_1, \dots, d_{i-1}, \alpha d_i, d_{i+1}, \cdots, d_n) \\ &\quad = \varphi_{n+1}(\nabla_{x_n})(\gamma)(d_1, \dots, d_{i-1}, \alpha d_i, d_{i+1}, \cdots, d_{n+1}) \\ &\quad = \alpha \underset{i}{\cdot} \varphi_i(\nabla_{x_n})(\gamma)(d_1, \dots, d_{n+1})\end{aligned} \tag{2.6}$$

For the latter case of our treatment of (2.3) we have

$$\begin{aligned}&\varphi_{n+1}(\nabla_{x_n})(\alpha \underset{n+1}{\cdot} \gamma)(d_1, \dots, d_{n+1}) \\ &\quad = \varphi_n(\nabla_{x_n}(\alpha \underset{n+1}{\cdot} \gamma(0, \dots, 0, \cdot))(d_{n+1}))(\alpha \underset{n+1}{\cdot} \gamma(\cdot, \dots, \cdot, d_{n+1}))(d_1, \dots, d_n) \\ &\quad\quad \text{[By the definition of } \varphi_{\mathrm{n+1}}] \\ &\quad = \varphi_n(\nabla_{x_n}(\gamma(0, \dots, 0, \cdot))(\alpha d_{n+1}))(\gamma(\cdot, \dots, \cdot, \alpha d_{n+1}))(d_1, \dots, d_n) \\ &\quad = \alpha \underset{n+1}{\cdot} \varphi_{n+1}(\nabla_{x_n})(\gamma)(d_1, \dots, d_{n+1})\end{aligned} \tag{2.7}$$

Finally we deal with (2.4), for which it suffices to handle $\sigma =< i, i+1 >$ $(1 \leq i \leq n)$. The treatment of the simple case of $i \leq n-1$ can safely be left to the reader. Here we deal with (2.4) in case of $\sigma =< n, n+1 >$. Let it be that

$$y_{n-1} = \nabla_{x_{n-1}}(\gamma(0, \dots, 0, \cdot))(d_{n+1}) \tag{2.8}$$

$$z_{n-1} = \nabla_{x_{n-1}}(\gamma(0, \dots, 0, \cdot, 0))(d_n) \tag{2.9}$$

$$\nabla_{y_{n-1}} = \nabla_{x_n}(\gamma(0, \dots, 0, \cdot))(d_{n+1}) \tag{2.10}$$

$$\nabla_{z_{n-1}} = \nabla_{x_n}(\gamma(0, \dots, 0, \cdot, 0))(d_n) \tag{2.11}$$

On the one hand we have

$$
\begin{aligned}
&\varphi_{n+1}(\nabla_{x_n})(\Sigma_{<n,n+1>}(\gamma))(d_1,\dots,d_{n+1})\\
&=\varphi_n(\nabla_{x_n}(\Sigma_{<n,n+1>}(\gamma)(0,\dots,0,\cdot))(d_{n+1}))(\Sigma_{<n,n+1>}(\gamma)(\cdot,\dots,\cdot,d_{n+1}))(d_1,\dots,d_n)\\
&\quad\text{[By the definition of } \varphi_{\mathrm{n+1}}]\\
&=\varphi_n(\nabla_{x_n}(\gamma(0,\dots,0,\cdot,0))(d_{n+1}))(\gamma(\cdot,\dots,\cdot,d_{n+1},\cdot))(d_1,\dots,d_n)\\
&=\varphi_{n-1}(\nabla_{z_{n-1}}(\gamma(0,\dots,0,d_{n+1},\cdot)(d_n))(\gamma(\cdot,\dots,\cdot,d_{n+1},d_n))(d_1,\dots,d_{n-1})\\
&\quad\text{[By the definition of } \varphi_{\mathrm{n}}]\\
&=\varphi_{n-1}(\nabla_{y_{n-1}}(\gamma(0,\dots,0,\cdot,d_n)(d_{n+1}))(\gamma(\cdot,\dots,\cdot,d_{n+1},d_n))(d_1,\dots,d_{n-1})\\
&\quad[(1.27.5)]
\end{aligned}
\tag{2.12}
$$

On the other hand we have

$$
\begin{aligned}
&\Sigma_{<n,n+1>}(\varphi_{n+1}(\nabla_{x_n})(\gamma))(d_1,\dots,d_{n+1})\\
&=\varphi_{n+1}(\nabla_{x_n})(\gamma)(d_1,\dots,d_{n-1},d_{n+1},d_n)\\
&=\varphi_n(\nabla_{x_n}(\gamma(0,\dots,0,\cdot))(d_n))(\gamma(\cdot,\dots,\cdot,d_n))(d_1,\dots,d_{n-1},d_{n+1})\\
&\quad\text{[By the definition of } \varphi_{\mathrm{n+1}}]\\
&=\varphi_{n-1}(\nabla_{y_{n-1}}(\gamma(0,\dots,0,\cdot,d_n)(d_{n+1}))(\gamma(\cdot,\dots,\cdot,d_{n+1},d_n))(d_1,\dots,d_{n-1})\\
&\quad\text{[By the definition of } \varphi_{\mathrm{n}}]
\end{aligned}
\tag{2.13}
$$

It follows from (2.12) and (2.13) that

$$\varphi_{n+1}(\nabla_{x_n})(\Sigma_{<n,n+1>}(\gamma))=\Sigma_{<n,n+1>}(\varphi_{n+1}(\nabla_{x_n})(\gamma)) \tag{2.14}$$

This completes the proof. □

**Lemma 2.2.** *The diagram*

$$
\begin{array}{ccc}
\mathbf{J}^{n+1}(\pi) & \xrightarrow{\quad\varphi_{n+1}\quad} & \hat{\mathbb{J}}^{n+1}(\pi)\\
\pi_{n+1,n}\ \downarrow & & \downarrow\ \underline{\hat{\pi}}_{n+1,n}\\
\mathbf{J}^{n}(\pi) & \xrightarrow[\quad\varphi_{n}\quad]{} & \hat{\mathbb{J}}^{n}(\pi)
\end{array}
\tag{2.15}
$$

*is commutative.*

*Proof.* Let $\nabla_{x_n}\in\mathbf{J}^{n+1}(\pi)$ and $x_n=\nabla_{x_{n-1}}\in\mathbf{J}^n(\pi)$. For any $\gamma\in\mathbf{T}^n_{\pi_{n-1}(x_{n-1})}(M)$ and any $(d_1,\dots,d_n)\in D^n$ we have

$$
\begin{aligned}
&((\underline{\pi}_{n+1,n}\circ\varphi_{n+1})(\nabla_{x_n}))(\gamma)(d_1,\dots,d_n)\\
&=(\varphi_{n+1}(\nabla_{x_n}))(\mathbf{s}_{n+1}(\gamma))(d_1,\dots,d_n,0)\\
&\quad\text{[By the definition of } \underline{\pi}_{\mathrm{n+1,n}}]\\
&=\varphi_n(\nabla_{x_n}(\mathbf{s}_{n+1}(\gamma)(0,\dots,0,\cdot))(0))(\mathbf{s}_{n+1}(\gamma)(\cdot,\dots,\cdot,0))(d_1,\dots,d_n)\\
&\quad\text{[By the definition of } \varphi_{\mathrm{n+1}}]\\
&=\varphi_n(\nabla_{x_{n-1}})(\gamma)(d_1,\dots,d_n),
\end{aligned}
\tag{2.16}
$$

which shows the commutativity of the diagram (2.15). □

Lemma 2.1 can be strengthened as follows:

**Lemma 2.3.** $\varphi_{n+1}(\nabla_{x_n}) \in \mathbb{J}^{n+1}(\pi)$.

*Proof.* With due regard to Lemmas 2.1 and 2.2, we have only to show that for any $\gamma \in \mathbf{T}^n_{\pi_n(x_n)}(M)$, we have

$$\begin{aligned}&\varphi_{n+1}(\nabla_{x_n})(((d_1,\dots,d_{n+1}) \in D^{n+1} \longmapsto \gamma(d_1,\dots,d_{n-1},d_nd_{n+1})))\\ &= (d_1,\dots,d_{n+1}) \in D^{n+1} \longmapsto \hat{\underline{\pi}}_{n+1,n}(\varphi_{n+1}(\nabla_{x_n}))(\gamma)(d_1,\dots,d_{n-1},d_nd_{n+1})\end{aligned} \tag{2.17}$$

We proceed by induction on $n$. For $n$ =0 there is nothing to prove. Let $\bar{\gamma}$ be the $(n+1)$-microcube $(d_1,\dots,d_{n+1}) \in D^{n+1} \longmapsto \gamma(d_1,\dots,d_{n-1},d_nd_{n+1})$. For any $d_1,\dots,d_{n+1} \in D$ we have

$$\begin{aligned}&\varphi_{n+1}(\nabla_{x_n})(\bar{\gamma})(d_1,,,,d_{n+1})\\ &= \varphi_n(\nabla_{x_n}(\bar{\gamma}(0,\dots,0,\cdot))(d_{n+1}))(\bar{\gamma}(\cdot,\dots,\cdot,d_{n+1}))(d_1,\dots,d_n)\\ &\quad \text{[By the definition of } \varphi_{\mathrm{n+1}}]\\ &= \varphi_n(\nabla_{x_{n-1}})(\bar{\gamma}(\cdot,\dots,\cdot,d_{n+1}))(d_1,\dots,d_n)\\ &= \varphi_n(\nabla_{x_{n-1}})(d_{n+1_n} \underset{n}{\cdot} \bar{\gamma})(d_1,\dots,d_n)\\ &= d_{n+1} \underset{n}{\cdot} (\varphi_n(\nabla_{x_{n-1}})(\bar{\gamma}))(d_1,\dots,d_n) \quad \text{[By Lemma 2.1]}\\ &= \varphi_n(\nabla_{x_{n-1}})(\bar{\gamma})(d_1,\dots,d_{n-1},d_nd_{n+1})\end{aligned} \tag{2.18}$$

Thus we have established the mappings $\varphi_n : \mathbf{J}^n(\pi) \to \mathbb{J}^n(\pi)$. □

## 3. Preconnections in formal bundles

In this section we will assume that the bundle $\pi : E \to M$ is a formal bundle of fiber dimension $q$ over the formal manifold of dimension $p$. For the exact definition of a formal bundle, the reader is referred to Nishimura [n.d.]. Since our considerations to follow are always infinitesimal, this means that we can assume without any loss of generality that $M = \mathbb{R}^p$, $E = \mathbb{R}^{p+q}$, and $\pi : \mathbb{R}^{p+q} \to \mathbb{R}^p$ is the canonical projection to the first $p$ axes. We will let $i$ with or without subscripts range over natural numbers between 1 and $p$ (including endpoints), while we will let $j$ with or without subscripts range over natural numbers between 1 and $q$ (including endpoints). For any natural number $n$, we denote by $\mathcal{J}^n(\pi)$ the totality of $(x^i, y^j, \alpha^j_i, \alpha^j_{i_1i_2}, \dots, \alpha^j_{i_1\dots i_n})'s$ of $p+q+pq+p^2q+\cdots+p^nq$ elements of $\mathbb{R}$ such that $\alpha^j_{i_1\dots i_k}{}'s$ are symmetric with respect to subscripts, i.e., $\alpha^j_{i_{\sigma(1)}\cdots i_{\sigma(k)}} = \alpha^j_{i_1\dots i_k}$ for any $\sigma \in \mathfrak{S}_k (2 \leq k \leq n)$. Therefore the number of independent components in $(x^i, y^j, \alpha^j_i, \alpha^j_{i_1i_2}, \dots, \alpha^j_{i_1\dots i_n}) \in \mathcal{J}^n(\pi)$ is $p + q\Sigma^n_{k=0}\binom{p+k-1}{p-1} = p + q\binom{p+n}{n}$. The canonical projection $(x^i, y^j, \alpha^j_i, \alpha^j_{i_1i_2}, \dots, \alpha^j_{i_1\dots i_n}, \alpha^j_{i_1\dots i_{n+1}}) \in \mathcal{J}^{n+1}(\pi) \longmapsto (x^i, y^j, \alpha^j_i, \alpha^j_{i_1i_2}, \dots, \alpha^j_{i_1\dots i_n}) \in \mathcal{J}^n(\pi)$ is denoted by $\underset{\sim}{\pi}_{n+1,n}$. We will use Einstein's summation convention to suppress $\Sigma$.

The principal objective in this section is to define mappings $\widetilde{\theta}_n : \mathcal{J}^n(\pi) \to \mathbb{J}^n(\pi)$ and $\underset{\sim}{\theta}_n : \mathbb{J}^n(\pi) \to \mathcal{J}^n(\pi)$, which are to be shown to be the inverse of each other. Let $\widetilde{\theta}_0$ be the identity mapping. We define $\widetilde{\theta}_1 : \mathcal{J}^1(\pi) \to \mathbb{J}^1(\pi)$ to be

$$\widetilde{\theta}_1((x^i, y^j, \alpha^j_i))(d \in D \longmapsto (x^i) + d(a^i)) = d \in D \longmapsto (x^i, y^j) + d(a^i, a^i\alpha^j_i) \tag{3.1}$$

We define $\widetilde{\theta}_2 : \mathcal{J}^2(\pi) \to \mathbb{J}^2(\pi)$ to be

$$\begin{aligned}&\widetilde{\theta}_2((x^i, y^j, \alpha^j_i, \alpha^j_{i_1 i_2}))((d_1, d_2) \in D^2 \\ &\longmapsto (x^i) + d_1(a^i_1) + d_2(a^i_2) + d_1 d_2(a^i_{12})) \\ &= (d_1, d_2) \in D^2 \\ &\longmapsto (x^i, y^j) + d_1(a^i_1, a^i_1\alpha^j_i) + d_2(a^i_2, a^i_2\alpha^j_i) + d_1 d_2(a^i_{12}, a^{i_1}_1 a^{i_2}_2 \alpha^j_{i_1 i_2} + a^i_{12}\alpha^j_i)\end{aligned} \tag{3.2}$$

Generally we define $\widetilde{\theta}_n : \mathcal{J}^n(\pi) \to \mathbb{J}^n(\pi)$ to be

$$\begin{aligned}&\widetilde{\theta}_n((x^i, y^j, \alpha^j_i, \alpha^j_{i_1 i_2}, \dots, \alpha^j_{i_1 i_2 \dots i_n}))((d_1, \dots, d_n) \in D^n \\ &\longmapsto (x^i) + \Sigma^n_{r=1}\Sigma_{1 \le k_1 < \dots < k_r \le n} d_{k_1} \dots d_{k_r}(a^i_{k_1 \dots k_r})) \\ &= (d_1, \dots, d_n) \in D^n \\ &\longmapsto (x^i, y^j) + \Sigma^n_{r=1}\Sigma_{1 \le k_1 < \dots < k_r \le n} d_{k_1} \dots d_{k_r}(a^i_{k_1 \dots k_r}, \Sigma a^{i_{\mathbf{J}_1}}_{\mathbf{J}_1} \dots a^{i_{\mathbf{J}_s}}_{\mathbf{J}_s} \alpha^j_{i_{\mathbf{J}_1} \dots i_{\mathbf{J}_s}})),\end{aligned} \tag{3.3}$$

where the last $\Sigma$ is taken over all partitions of the set $\{k_1, \dots, k_r\}$ into nonempty subsets $\{\mathbf{J}_1, \dots, \mathbf{J}_s\}$, and if $\mathbf{J} = \{k_1, \dots, k_t\}$ is a set of natural numbers with $k_1 < \dots < k_t$, then $a^{i_\mathbf{J}}_\mathbf{J}$ denotes $a^{i_{k_1 \cdots k_t}}_{k_1 \cdots k_t}$.

First of all we note that

**Proposition 3.1.** *For any* $(x^i, y^j, \alpha^j_i, \dots, \alpha^j_{i_1 \dots i_n}) \in \mathcal{J}^n(\pi)$*, we have* $\widetilde{\theta}_n((x^i, y^j, \alpha^j_i, \dots, \alpha^j_{i_1 \dots i_n})) \in \hat{\mathbb{J}}^n(\pi)$

*Proof.* For the sake of simplicity of notation we deal only with the case of $n = 2$, leaving the general case safely to the reader. It should be obvious that $\widetilde{\theta}_2((x^i, y^j, \alpha^j_i, \dots, \alpha^j_{i_1 \dots i_n}))$ satisfies conditions (1.1) and (1.2). To show that it abides also by (1.3), we note that given $\gamma \in \mathbf{T}^2(M)$ of the form $(d_1, d_2) \in D^2 \longmapsto (x^i) + d_1(a^i_1) + d_2(a^i_2) + d_1 d_2(a^i_{12})$, we have

$$\begin{aligned}&\widetilde{\theta}_2((x^i, y^j, \alpha^j_i, \alpha^j_{i_1 i_2}))(\Sigma_{<1,2>}(\gamma)) \\ &= (d_1, d_2) \in D^2 \\ &\longmapsto (x^i, y^j) + d_2(a^i_1, a^i_1\alpha^j_i) + d_1(a^i_2, a^i_2\alpha^j_i) + d_1 d_2(a^i_{12}, a^{i_1}_2 a^{i_2}_1 \alpha^j_{i_1 i_2} + a^i_{12}\alpha^j_i) \\ &= (d_1, d_2) \in D^2 \\ &\longmapsto (x^i, y^j) + d_2(a^i_1, a^i_1\alpha^j_i) + d_1(a^i_2, a^i_2\alpha^j_i) + d_1 d_2(a^i_{12}, a^{i_1}_2 a^{i_2}_1 \alpha^j_{i_2 i_1} + a^i_{12}\alpha^j_i) \\ &\qquad [\text{since } \alpha^j_{i_1 i_2} = \alpha^j_{i_2 i_1}] \\ &= \Sigma_{<1,2>}(\widetilde{\theta}_2((x^i, y^j, \alpha^j_i, \alpha^j_{i_1 i_2}))(\gamma))\end{aligned} \tag{3.4}$$

This completes the proof. □

**Proposition 3.2.** *The following diagram is commutative:*

$$
\begin{tikzcd}
\mathcal{J}^{n+1}(\pi) \arrow[r, "\widetilde{\theta}_{n+1}"] \arrow[d, "\underset{\sim}{\pi}_{n+1,n}"'] & \hat{\mathbb{J}}^{n+1}(\pi) \arrow[d, "\hat{\underset{\sim}{\pi}}_{n+1,n}"] \\
\mathcal{J}^{n}(\pi) \arrow[r, "\widetilde{\theta}_{n}"'] & \hat{\mathbb{J}}^{n}(\pi)
\end{tikzcd}
$$

*Proof.* For the sake of simplicity of notation we deal only with the case of $n = 1$, leaving the general case safely to the reader. Given $t \in \mathbf{T}^1(M)$ of the form $d \in D \longmapsto (x^i) + d(a^i)$, $\mathbf{s}_2(t)$ is seen to be of the form $(d_1, d_2) \in D^2 \longmapsto (x^i) + d_1(a^i)$, so that $\widetilde{\theta}_2((x^i, y^j, \alpha_i^j, \alpha_{i_1 i_2}^j))(\mathbf{s}_2(t))$ is of the form $(d_1, d_2) \in D^2 \longmapsto (x^i, y^j) + d_1(a^i, a^i \alpha_i^j)$, which means the commutativity of the above diagram. □

**Proposition 3.3.** *For any* $(x^i, y^j, \alpha_i^j, \dots, \alpha_{i_1 \dots i_n}^j) \in \mathcal{J}^n(\pi)$, *we have* $\widetilde{\theta}_n((x^i, y^j, \alpha_i^j, \dots, \alpha_{i_1 \dots i_n}^j)) \in \mathbb{J}^n(\pi)$

*Proof.* With due regard to Proposition 3.1 we have only to deal with conditions (1.11) and (1.12). For $n = 0$ and $n = 1$ there is nothing to prove. For $n = 2$ it is easy to see that

$$
\begin{aligned}
&\widetilde{\theta}_2((x^i, y^j, \alpha_i^j, \alpha_{i_1 i_2}^j))((d_1, d_2) \in D^2 \longmapsto (x^i) + d_1 d_2 (a^i)) \\
&= (d_1, d_2) \in D^2 \longmapsto (x^i, y^j) + d_1 d_2 (a^i, a^i \alpha_i^j) \\
&= (d_1, d_2) \in D^2 \longmapsto \widetilde{\theta}_1((x^i, y^j, \alpha_i^j))(d \in D \longmapsto (x^i) + d(a^i))(d_1 d_2),
\end{aligned}
\tag{3.5}
$$

from which (1.12) is easily seen to hold by dint of Proposition 3.2. The condition (1.11) holds trivially. We can continue by induction on $n$ by dint of Proposition 3.2. □

Now we are going to define mappings $\underset{\sim}{\theta}_n : \mathbb{J}^n(\pi) \to \mathcal{J}^n(\pi)$ by induction on $n$ such that the diagram

$$
\begin{tikzcd}
\mathbb{J}^{n+1}(\pi) \arrow[r, "\underset{\sim}{\theta}_{n+1}"] \arrow[d, "\underset{\sim}{\pi}_{n+1,n}"'] & \mathcal{J}^{n+1}(\pi) \arrow[d, "\underset{\sim}{\pi}_{n+1,n}"] \\
\mathbb{J}^{n}(\pi) \arrow[r, "\underset{\sim}{\theta}_{n}"'] & \mathcal{J}^{n}(\pi)
\end{tikzcd}
$$

is commutative. The mapping $\underset{\sim}{\theta}_0 : \mathbb{J}^0(\pi) \to \mathcal{J}^0(\pi)$ shall be the identity mapping. Assuming that $\underset{\sim}{\theta}_n : \mathbb{J}^n(\pi) \to \mathcal{J}^n(\pi)$ is defined, we are going to define $\underset{\sim}{\theta}_{n+1} : \mathbb{J}^{n+1}(\pi) \to \mathcal{J}^{n+1}(\pi)$, for which it suffices by the required commutativity of the above diagram only to give $\alpha_{i_1 \dots i_{n+1}}^j$'s for each $\nabla_x \in \mathbb{J}^{n+1}(\pi)$ with $x = (x^i, y^j)$. Let $\mathbf{e}_i$ denote $(0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^p$, where 1 is inserted at the $i$-th position while the other $p - 1$ elements are fixed zero. By the general Kock axiom (cf. Lavendhomme [1996, §2.1.3]), $\nabla_x((d_1, \dots, d_{n+1}) \in D^{n+1} \longmapsto (x^i) + d_1 \mathbf{e}_{i_1} + \dots + d_{n+1} \mathbf{e}_{i_{n+1}}))$ should be a polynomial of $d_1, \dots, d_{n+1}$, in which the coefficient of $d_1 \dots d_{n+1}$ should be of the form $(0, \dots, 0, \alpha_{i_1 \dots i_{n+1}}^1, \dots, \alpha_{i_1 \dots i_{n+1}}^q) \in \mathbb{R}^{p+q}$ for some $(\alpha_{i_1 \dots i_{n+1}}^1, \dots, \alpha_{i_1 \dots i_{n+1}}^q) \in \mathbb{R}^q$ and we choose them as our desired $\alpha_{i_1 \dots i_{n+1}}^j$'s. Now we have

**Proposition 3.4.** *For any $\nabla_{\mathbf{x}} \in \mathbb{J}^n(\pi)$, we have $\underset{\sim}{\theta}_n(\nabla_{\mathbf{x}}) \in \mathcal{J}^n(\pi)$.*

*Proof.* We have only to check the symmetric nature of $\alpha^j_{i_1 \ldots i_k}$'s with respect to subscripts, which follows easily from (1.3) by induction on $n$. □

It is easy to see that

**Proposition 3.5.** *The composition $\underset{\sim}{\theta}_n \circ \widetilde{\theta}_n$ is the identity mapping of $\mathcal{J}^n(\pi)$.*

*Proof.* Using the commutative diagram

$$\begin{array}{ccccc}
\mathcal{J}^{n+1}(\pi) & \xrightarrow{\widetilde{\theta}_{n+1}} & \mathbb{J}^{n+1}(\pi) & \xrightarrow{\underset{\sim}{\theta}_{n+1}} & \mathcal{J}^{n+1}(\pi) \\
\underset{\sim}{\pi}_{n+1,n} \downarrow & & \downarrow \ \underset{\sim}{\pi}_{n+1,n} & & \downarrow \ \underset{\sim}{\pi}_{n+1,n} \\
\mathcal{J}^n(\pi) & \xrightarrow[\widetilde{\theta}_n]{} & \mathbb{J}^n(\pi) & \xrightarrow[\underset{\sim}{\theta}_n]{} & \mathcal{J}^n(\pi)
\end{array}$$

we can easily establish the desired result by induction on $n$. □

**Proposition 3.6.** *The mapping $\underset{\sim}{\theta}_n : \mathbb{J}^n(\pi) \to \mathcal{J}^n(\pi)$ is one-to-one.*

*Proof.* For $n = 0, 1$, there is nothing to prove. For $n = 2$ we have

$$\begin{aligned}
&(d_1, d_2) \in D^2 \longmapsto (x^i) + d_1(a^i_1) + d_2(a^i_2) + d_1 d_2(a^i_{12}) \\
&= (d \in D \longmapsto (x^i) + d(a^i_{12})) \dot{+} ((d_1, d_2) \in D^2 \longmapsto (x^i) + d_1(a^i_1) + d_2(a^i_2)) \\
&= \underset{1}{\sum}{}_{i=1}^{p} a^i_{12}(d \in D \longmapsto (x^i) + \mathbf{e}_i)) \\
&\quad \dot{+} \underset{2}{\sum}{}_{i''=1}^{p} a^{i''}_2 \underset{2}{\cdot} ((d_1, d_2) \in D^2 \longmapsto (x^i) + d_1(a^i_1) + d_2 \mathbf{e}_{i''}) \\
&= \sum_{i=1}^{p} a^i_{12}(d \in D \longmapsto \mathbf{e}_i)) \\
&\quad \dot{+} \underset{2}{\sum}{}_{i''=1}^{p} a^{i''}_2 \underset{2}{\cdot} (\underset{1}{\sum}{}_{i'=1}^{p} a^i_1 \underset{1}{\cdot} ((d_1, d_2) \in D^2 \longmapsto (x^i) + d_1 \mathbf{e}_{i'} + d_2 \mathbf{e}_{i''}))
\end{aligned} \tag{3.6}$$

Therefore the desired statement follows from Proposition 1.6. We can continue to proceed by induction on $n$ by using Propositions 1.3, 1.5 and 1.6. □

These considerations finally yield the following main theorem of this section.

**Theorem 3.7.** *The mappings $\underset{\sim}{\theta}_n : \mathbb{J}^n(\pi) \to \mathcal{J}^n(\pi)$ and $\widetilde{\theta}_n : \mathcal{J}^n(\pi) \to \mathbb{J}^n(\pi)$ are the inverse of each other. In particular, both of them are bijective.*

*Proof.* This follows directly from Propositions 3.5 and 3.6. □

By combining the above theorem with a main result of our previous paper (Nishimura [2003, Theorem 4.8]), we have

**Theorem 3.8.** *Under the present assumption that the bundle $\pi : E \to M$ is a formal bundle of dimension $q$ over the formal manifold of dimension $p$, the translation $\varphi_n : \mathbf{J}^n(\pi) \to \mathbb{J}^n(\pi)$ is a bijective correspondence.* □

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