Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 45 (2004), No. 2, 557-568.

# Bounds of the Affine Breadth Eccentricity of Convex Bodies via Semi-infinite Optimization

*Dedicated to the memory of Bernulf Weißbach*

**Friedrich Juhnke**

*Otto-von-Guericke-Universität, Fakultät für Mathematik*
*PSF 4120, 39016 Magdeburg, Germany*

**Abstract.** In this contribution we give a semi-infinite optimization approach to investigate the affine breadth eccentricity of convex bodies. An optimization-technique-based description of the minimal ellipsoid (Loewner-ellipsoid) of a convex body is used to derive an upper bound of the affine eccentricity in a very natural way. An additional special (integer programming) optimization problem shows that the obtained upper bound is the best possible one.

MSC 2000: 52A20, 52A40, 90C34
Keywords: Affine breadth eccentricity, minimal ellipsoid

## 1. Introduction

Let $\mathbb{K} := \{K \subseteq \mathbb{R}^d \mid \text{K convex, compact, int } K \neq \emptyset\}$ be the class of all convex bodies of $\mathbb{R}^d$. In $\mathbb{K}$ we have the geometrical quantities *width* $\triangle(K)$ and *diameter* $D(K)$ as the minimal or maximal distance respectively between two parallel supporting hyperplanes of $K \in \mathbb{K}$. Furtheron the *inradius* $\varrho(K) := \sup\{r | B(x,r) \subseteq K\}$ and the *circumradius* $R(K) := \inf\{r | B(x,r) \supseteq K\}$ are the radii of a largest ball contained in $K$ or the smallest containing $K$ respectively, $B(x,r) := x + rB$, $B := \{u \in \mathbb{R}^d \,|\, u^T u \leq 1\}$ denotes the $d$-dimensional Euclidean unit ball.

According to Steinhagen and Jung the ratios $\Delta(K)/\varrho(K)$ and $R(K)/D(K)$ are bounded above in $\mathbb{K}$, the sharp upper bounds depend on the dimension $d$ only, see [5], [7].

The quotient $\dfrac{D(C)}{\triangle(C)}$, the *breadth eccentricity* of $C$, is unbounded in $\mathbb{K}$. Leichtweiss [6] showed that the *affine breadth eccentricity*

$$\beta(C) := \inf_{\varphi\in\Phi} \frac{D(\varphi(C))}{\Delta(\varphi(C))} = \inf_{K\in\mathbb{K}_C} \frac{D(K)}{\triangle(K)}$$

is bounded above in $\mathbb{K}$ by the sharp upper bound $\sqrt{d}$; $\Phi$ denotes the group of all regular affine mappings of $\mathbb{R}^d$ into itself and $\mathbb{K}_C := \{K \in \mathbb{K} \,|\, K = \varphi(C),\, \varphi \in \Phi\}$ is the *affine class* corresponding to $C$. The affine breadth eccentricity is closely related to the *affine radial eccentricity* $\alpha(C) = \inf_{K\in\mathbb{K}_C} \dfrac{R(K)}{\varrho(K)}$. It turns out that $\beta(C) = \alpha(C^*)$, where $C^* := \dfrac{1}{2}(C+(-C))$ denotes the *central symmetrization* of $C$. In what follows we will use an optimization approach for determining the smallest upper bound (already given by Leichtweiss) of the affine breadth eccentricity.

## 2. Minimal ellipsoids and upper bounds of affine eccentricity

We consider ellipsoids

$$E = E(Q,x) := \left\{ t \in \mathbb{R}^d \;\middle|\; (t-x)^T Q(t-x) \le 1 \right\}$$

with center $x = (x_1, \dots, x_d)^T \in \mathbb{R}^d$ and $Q = (q_{ik}) \in S_+^{d\times d}$ := set of all real symmetric positive definite $(d,d)$-matrices. Let

$$k(u) := \sup_{t\in C} u^T t\,, \qquad u \in \mathbb{R}^d$$

and

$$h(u) := \sup_{t\in E(Q,x)} u^T t = u^T x + \sqrt{u^T X u}, \qquad u \in \mathbb{R}^d$$

be the corresponding support functions of $C$ and $E$ respectively, where $X = (x_{ik}) := Q^{-1} \in S_+^{d\times d}$. Let $0 \ne u \in \mathbb{R}^d$ be a given direction, the corresponding supporting hyperplane $\{t \in \mathbb{R}^d \,|\, u^T t = h(u)\}$ of the ellipsoid $E(Q,x)$ has the supporting point

$$y := x + \frac{Xu}{\sqrt{u^T X u}} \in \partial E(Q,x). \tag{2.1}$$

The volume of $E$ is given by

$$V(E) = V(Q,x) = \frac{\omega_d}{\sqrt{\det Q}} = \omega_d \sqrt{\det X}$$

where $\omega_d := \dfrac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}+1\right)}$ denotes the volume of the unit ball $B$.

Let $C \in \mathbb{K}$. Due to $\operatorname{int} C \ne \emptyset$ we can assume $0 \in \operatorname{int} C$. Among all ellipsoids covering $C$ we look at first for that one with minimal volume (Loewner-ellipsoid, minimal ellipsoid). Using the equivalence

$$C \subseteq E(Q,x) \Longleftrightarrow \forall u \in \mathbb{R}^d : k(u) \le h(u),$$

we obtain the semi-infinite optimization problem

$$\min\left\{F(X,x)=\det\, X \;\middle|\; (X,x)\in UM\right\} \tag{2.2}$$

$$UM:=\left\{(X,x)\in S_+^{d\times d}\times\mathbb{R}^d \;\middle|\; \forall u\in\partial B: u^Tx+\sqrt{u^TXu}\geq k(u)\right\} \tag{2.3}$$

with the $\frac{d(d+3)}{2}$ variables $x_i$ and $x_{ik}$, $i,k=1,\ldots,d$, $i\geq k$. The pair $(X,x)\in UM$ will also be regarded as a point $\xi:=(x_{11},\ldots,x_{1d},x_{22},\ldots,x_{2d},x_{33},\ldots,x_{dd},x_1,\ldots,x_d)\in\mathbb{R}^m$, $m=\dfrac{d(d+3)}{2}$. Due to the compactness of $C$ and $0\in\operatorname{int}C$ there exist radii $r,R>0$ with $rB\subseteq C\subseteq RB$, such that $0<r\leq k(u)\leq R$ holds for all $u\in\partial B$. Hence the feasible region (2.3) can be reduced to such points $(X,x)$, for which $r^d\omega_d\leq V(E(X^{-1},x))\leq R^d\omega_d$ holds, i.e. to

$$\overline{UM}:=\left\{(X,x)\;\middle|\;\begin{array}{lrcl}\forall u\in\partial B: & u^Tx+\sqrt{u^TXu} & \geq & k(u)\\ & r^{2d}\leq\det\, X & \leq & R^{2d}\\ & X\in S_+^{d\times d}, x\in\mathbb{R}^d & & \end{array}\right\}.$$

Since $\overline{UM}$ is a compact subset of $\mathbb{R}^m$, (2.2) has always an optimal solution $(X^0,x^0)$, which gives the minimal ellipsoid (Loewner-ellipsoid) $E(X^{0^{-1}},x^0)$. Due to John [1] there exist directions $u^1,\ldots,u^r\in\partial B$, $0\leq r\leq m=\frac{d(d+3)}{2}$, which correspond to active restrictions (in the point $(X^0,x^0)$) of (2.3), and multipliers $\lambda^T=(\lambda_0,\ldots,\lambda_r)\neq(0,\ldots,0)$, $\lambda_0\geq 0$, $\lambda_1>0,\ldots,\lambda_r>0$ such that the Lagrange-function

$$L(X,x,\lambda):=\lambda_0\det\, X+\sum_{j=1}^r\lambda_j\left[k(u^j)-u^{j^T}x-\sqrt{u^{j^T}Xu^j}\right],$$

built with these restrictions, is stationary in $(X^0,x^0)$.

In what follows, $u^1,\ldots,u^r$ are called *characteristic directions* and the corresponding supporting points $y^1,\ldots,y^r$ (cf. (2.1)) *characteristic supporting points.*

With $\dfrac{\partial L}{\partial x_i}=-\sum\limits_{j=1}^r\lambda_ju_i^j$ and

$$\frac{\partial L}{\partial x_{ik}}=\begin{cases}\lambda_0X_{ii} - \frac{1}{2}\sum\limits_{j=1}^r\lambda_j\dfrac{(u_i^j)^2}{\sqrt{u^{j^T}Xu^j}} & \text{for } i=k\\ 2\lambda_0X_{ik} - \sum\limits_{j=1}^r\lambda_j\dfrac{u_i^ju_k^j}{\sqrt{u^{j^T}Xu^j}} & \text{for } j\neq k\end{cases}$$

($X_{ik}=$ adjoint of $x_{ik}$ in $X$) we get necessary optimality conditions, which turn out to be also sufficient ones. The uniqueness of the minimal ellipsoid can be deduced from these conditions, and it can be shown that the minimal ellipsoid of $C$ is already the minimal ellipsoid of a certain polytope contained in $C$ (with at most $\frac{d(d+3)}{2}$ vertices). Summarized we have (cf. [3], [4])

**Theorem 1.**

(i) *$E(X^{0^{-1}}, x^0)$ is minimal ellipsoid of a convex body $C$ if and only if $E(X^{0^{-1}}, x^0) \supseteq C$ and the following conditions hold: There exist directions $u^1, \dots, u^r \in \partial B$ and scalars $\lambda_0, \dots, \lambda_r \in \mathbb{R}$ with*

$$d + 1 \le r \le \frac{d(d+3)}{2}, \quad \lambda_0 > 0, \dots, \lambda_r > 0,$$

*such that*

$$u^{j^T} x^0 + \sqrt{u^{j^T} X^0 u^j} = k(u^j), \;\; j = 1, \dots, r \tag{2.4}$$

$$\sum_{j=1}^{r} \lambda_j u^j = 0 \tag{2.5}$$

$$(2\lambda_0 \det X^0) I = \sum_{j=1}^{r} \lambda_j \frac{X^0 u^j u^{j^T}}{\sqrt{u^{j^T} X^0 u^j}}. \tag{2.6}$$

(ii) *The minimal ellipsoid of $C$ is uniquely determined.*

(iii) *The minimal ellipsoid of $C$ is also minimal ellipsoid of the polytope* conv $\{y^1, \dots, y^r\}$,

$$y^j := x^0 + \frac{X^0 u^j}{\sqrt{u^{j^T} X^0 u^j}}, \qquad j = 1, \dots, r$$

*($I$ denotes the $(d, d)$-identity matrix).*

Forming the trace in (2.6) we get

$$2d\lambda_0 \det X^0 = \sum_{j=1}^{r} \lambda_j \sqrt{u^{j^T} X^0 u^j}. \tag{2.7}$$

This shows once more $\lambda_0 > 0$, i.e. John's optimality conditions of part (i) are of Karush-Kuhn-Tucker type.

A regular affine mapping has no essential influence to the optimization problem (2.2), since all volume ratios remain invariant. Furtheron the minimal ellipsoid $E_0$ of $C$ is affine-invariantly connected with $C$ (due to its uniqueness): $\varphi(E_0)$ is minimal ellipsoid of $\varphi(C)$ for all regular affine transformations $\varphi \in \Phi$. Therefore we may assume the minimal ellipsoid $E(X^{0^{-1}}, x^0)$ to be the unit ball after a suitable affine transformation, i.e. $X^0 = I$, $x^0 = 0$ and $u^j = y^j$, $j = 1, \dots, r$. In this case, the optimality conditions (2.4) and (2.6) simplify to

$$k(u^j) = 1, \; j = 1, \dots, r$$

$$2\lambda_0 I = \sum_{j=1}^{r} \lambda_j u^j u^{j^T} \tag{2.8}$$

and (2.7) turns into

$$2d\lambda_0 = \sum_{j=1}^{r} \lambda_j. \tag{2.9}$$

**Lemma 1.** *Let $C \in \mathbb{K}$ be a convex body and $E_0$ the corresponding minimal ellipsoid. Let $\varphi_0 \in \Phi$ be a regular affine transformation, which maps $E_0$ onto the unit ball $B$. Then*

$$\frac{D(\varphi_0(C))}{\triangle(\varphi_0(C))} \leq \sqrt{d}.$$

*Proof.* The minimal ellipsoid of $\varphi_0(C)$ is $B$. The corresponding characteristic directions $u^1, \dots, u^r \in \partial B$ are equal to the characteristic supporting points $y^1, \dots, y^r$. Let

$$g(u) := \max_{j=1,\dots,r} u^T y^j$$

be the support function of the polytope conv $\{y^1, \dots, y^r\}$, such that

$$(g(u) - u^T y^j)(g(-u) + u^T y^j) \geq 0, \ \ j = 1, \dots, r.$$

With the multipliers $\lambda_1, \dots, \lambda_r$ according to Theorem 1, this leads to

$$\sum_{j=1}^{r} \lambda_j \, g(u)g(-u) + (g(u) - g(-u))u^T \sum_{j=1}^{r} \lambda_j y^j - u^T \left( \sum_{j=1}^{r} \lambda_j y^j {y^j}^T \right) u \geq 0,$$

which (using the optimality conditions (2.5), (2.8) and (2.9)) results in

$$2d\,\lambda_0\, g(u)g(-u) - u^T(2\lambda_0 I)\, u \geq 0,$$

and therefore

$$\forall u \in \partial B: \quad g(u)g(-u) \geq \frac{1}{d}\,.$$

Since $0 \leq g(u) \leq 1 \ \forall u \in \partial B$, we get

$$\frac{g(u) + g(-u)}{2} \geq \sqrt{g(u)g(-u)} \geq \frac{1}{\sqrt{d}}$$

for all $u \in \partial B$. The inclusions $\{y^1, \dots, y^r\} \subseteq \varphi_0(C) \subseteq B$ lead to

$$\triangle(\varphi_0(C)) \geq \triangle(\text{conv}\left\{y^1, \dots, y^r\right\}) = \inf_{u \in \partial B} (g(u) + g(-u)) \geq \frac{2}{\sqrt{d}}$$

and

$$D(\varphi_0(C)) \leq D(B) = 2,$$

out of which the claim follows. □

Every affine class $\mathbb{K}_C$ contains according to Lemma 1 a convex body, whose breadth eccentricity is at the most $\sqrt{d}$, so the inequality

$$\forall C \in \mathbb{K} : 1 \leq \beta(C) \leq \sqrt{d}$$

given by Leichtweiss [6] is verified, only using optimization techniques. We show now, also with means of optimization theory, that the upper bound $\sqrt{d}$ is the best (smallest) possible one in the class $\mathbb{K}$.

To prove this property, a particular body $C_0 \subseteq \mathbb{K}$ with

$$\forall\, \varphi \in \Phi : \frac{D(\varphi(C_0))}{\Delta(\varphi(C_0))} \geq \sqrt{d}$$

is to present. We will show that for instance every cross polytope of $\mathbb{R}^d$ is such a body. For this purpose let $S_0 := \{e^1, \ldots, e^d, -e^1, \ldots, -e^d\}$ be the vertex set of a regular cross polytope $P_0 := \operatorname{conv} S_0$ (with edge length $\sqrt{2}$); $e^i := (\delta_{i1}, \ldots, \delta_{id})^T \in \mathbb{R}^d$ denotes the $i$-th canonical basis unit vector of $\mathbb{R}^d$, $(e^1 \cdots e^d)_{d,d} = I$.

An arbitrary affine transformation $\varphi \in \Phi$ has the form $\varphi(x) = A^T x + b$ with a certain regular matrix $A^T = (a^1 \cdots a^d)_{d,d}$. Since diameter and width are invariant under translations, we may restrict ourselves to $b = 0$. Then

$$\varphi(S_0) = \{a^1, \ldots, a^d,\ -a^1, \ldots, -a^d\}$$

is the vertex set of the (general) cross polytope

$$P := \varphi(P_0) = \varphi(\operatorname{conv} S_0) = \operatorname{conv} \varphi(S_0).$$

The linearly independent vectors $a^1, \ldots, a^d \in \mathbb{R}^d$ are the "generating vectors" of $P$. Inversely each cross polytope of $\mathbb{R}^d$ can be represented as affine image of the regular cross polytope $P_0$. Hence the affine breadth eccentricity of the regular cross polytope is given by

$$\beta(P_0) = \inf_{\varphi \in \Phi} \frac{D(\varphi(P_0))}{\Delta(\varphi(P_0))} = \inf_{P \in \mathbb{KP}} \frac{D(P)}{\Delta(P)}.$$

$\mathbb{KP}$ denotes the set of all cross polytopes of $\mathbb{R}^d$. (Furthermore we realize: $\forall\, P \in \mathbb{KP} : \beta(P) = \beta(P_0)$, i.e. $\beta$ is constant on $\mathbb{KP}$.)

The width $\Delta$ of a general cross polytope can be obtained from the solution of a certain discrete optimization problem according to

**Lemma 2.** *The cross polytope $P = \operatorname{conv}\{a^1, \ldots, a^d, -a^1, \ldots, -a^d\}$ of $\mathbb{R}^d$ with the linearly independent generating vectors $a^1, \ldots, a^d \in \mathbb{R}^d$ has the width*

$$\Delta(P) = \frac{2}{\max\limits_{\varepsilon} ||A^{-1}\varepsilon||}$$

*and the diameter*

$$D(P) = 2 \max_{i=1,\dots,d} ||a^i||,$$

*where* $A^T := (a^1 \cdots a^d)_{d,d}$ *and* $\varepsilon^T := (\varepsilon_1, \dots, \varepsilon_d)$ *with* $\varepsilon_1 \in \{-1, 1\}$, $i = 1, \dots, d$.

*Proof.* Let $S := \{a^1, \dots, a^d, -a^1, \dots, -a^d\}$ be the set of vertices of the cross polytope $P$, then

$$\begin{aligned} \Delta(P) &= \Delta(\operatorname{conv} S) = \Delta(S) \\ &= \inf_{u \in \partial B} \{\max_{t \in S} u^T t - \min_{t \in S} u^T t\} \\ &= \inf_{u \in \partial B} \{\max_{i=1,\dots,d} (\pm u^T a^i) - \min_{i=1,\dots,d} (\pm u^T a^i)\} \\ &= 2 \inf_{u \in \partial B} (\max_{i=1,\dots,d} |u^T a^i|) \end{aligned} \tag{2.10}$$

holds. With

$$z(u) := \max_{i=1,\dots,d} |u^T a^i| , \tag{2.11}$$

$\frac{1}{2}\Delta(P)$ is the optimal value of the optimization problem

$$\min \left\{ z(u) \;\middle|\; ||u|| = 1 \right\}, \tag{2.12}$$

which is equivalent to

$$\min \left\{ \varrho \;\middle|\; -\varrho \le a^{i^T} u \le \varrho,\ i = 1, \dots, d,\ ||u|| = 1 \right\}.$$

(Geometrically (2.12) means: A hyperplane $H$ through the origin with normal vector $u$ is to determine in such a way, that the greatest distance between the points $a^1, \dots, a^d$ and the hyperplane $H$ gets minimal.) With $v := u/\varrho$ we get

$$\max \left\{ ||v|| \;\middle|\; v \in M \right\}, \tag{2.13}$$

$$M := \left\{ v \in \mathbb{R}^d \;\middle|\; \begin{bmatrix} A \\ -A \end{bmatrix} v \le \begin{bmatrix} e \\ e \end{bmatrix} \right\},$$

$e := (1, \dots, 1)^T \in \mathbb{R}^d$. The polyhedral constraint set $M$ is compact, because if $M$ would be unbounded, the corresponding homogeneous constraint system

$$\begin{bmatrix} A \\ -A \end{bmatrix} v \le 0$$

would have a nontrivial solution $v$ (recession direction of $M$) with $Av = 0$. That is impossible due to the linear independency of the generating vectors $a^1, \dots, a^d$. The points of $M$ build a parallelotope with center 0, whose $d$ pairs of $(d-1)$-dimensional faces have the normal vectors $a^1, \dots, a^d$ and whose distances to the origin are $1/||a^1||, \dots, 1/||a^d||$. (2.13) requires to determine a point in $M$ with greatest distance to the origin 0.

On the compact convex set $M$ the convex objective function $||v||$ attains its maximum in a vertex of $M$. The point $v \in M$ is a vertex of $M$ iff there are $d$ linearly independent constraints in the set of all in $v$ active constraints. This occurs if and only if

$$a^{i^T} v = \varepsilon_i, \ i = 1, \dots, d$$

holds with $\varepsilon_1, \dots, \varepsilon_d \in \{-1, 1\}$, i.e. $Av = \varepsilon, \ \varepsilon = (\varepsilon_1, \dots, \varepsilon_d)^T$. There are exactly $2^d$ such vectors $\varepsilon$. These lead to the $2^d$ different vertices of $M$, which correspond to the $2^d$ different $(d-1)$-dimensional faces of the cross polytope. Hence (2.13) is equivalent to the discrete optimization problem

$$\max \left\{ ||v|| \ \middle| \ Av = \varepsilon, \ \varepsilon = (\varepsilon_1, \dots, \varepsilon_d), \ \varepsilon_1, \dots, \varepsilon_d \in \{-1, 1\} \right\}$$

with the optimal value $||v||_{max} = \max\limits_{\varepsilon} ||A^{-1}\varepsilon||$, such that

$$\Delta(P) = 2\varrho_{min} = \frac{2}{||v||_{max}} = \frac{2}{\max\limits_{\varepsilon} ||A^{-1}\varepsilon||}, \qquad \square$$

Due to (2.10) and (2.11) the diameter of the cross polytope $P$ becomes $D(P) = 2 \sup\limits_{u \in \partial B} z(u)$. With $\max\limits_{i=1,\dots,d} ||a^i|| =: ||a^k||$ we get

$$\forall \ u \in \partial B: \ z(u) = \max_{i=1,\dots,d} |u^T a^i| \le ||a^k||$$

and $z(u^0) = ||a^k||$ for $u^0 = a^k/||a^k||$, such that

$$\sup_{u \in \partial B} z(u) = ||a^k||$$

and

$$D(P) = 2 \sup_{u \in \partial B} z(u) = 2 \max_{i=1,\dots,d} ||a^i||, \qquad \square$$

**Lemma 3.** *Let $P \subseteq \mathbb{R}^d$ be an arbitrary cross polytope. Then*

$$\frac{D(P)}{\Delta(P)} \begin{cases} > \sqrt{d}, & \text{if } P \text{ not regular} \\ = \sqrt{d}, & \text{if } P \text{ regular.} \end{cases}$$

*Proof.* Due to Lemma 2

$$\frac{D(P)}{\Delta(P)} = \max_{\varepsilon} ||A^{-1}\varepsilon|| \max_{i=1,\dots,d} ||a^i||$$

holds for $P = \operatorname{conv}\{a^1, \dots, a^d, -a^1, \dots, -a^d\}$, in which $A = (a^1 \cdots a^d)^T_{d,d}$ is the matrix of the generating vectors of $P$ and $\varepsilon = (\varepsilon_1, \dots, \varepsilon_d)^T,\ \varepsilon_i \in \{-1, 1\},\ i = 1, \dots, d$. The generating matrix $\varrho A$ belongs to the cross polytope $P_\varrho := \operatorname{conv}\{\varrho a^1, \dots, \varrho a^d, -\varrho a^1, \dots, -\varrho a^d\}$ and

$$\frac{D(P_\varrho)}{\Delta(P_\varrho)} = \frac{D(P)}{\Delta(P)}$$

holds, so that $\max\limits_{i=1,\dots,d} ||a^i|| = 1$ can be assumed. Let $A^{-1} =: (b^1 \cdots b^d)$, we have

$$\forall\, j \in \{1, \dots, d\} :\ 1 = a^{j^T} b^j \leq ||a^j||\, ||b^j|| \leq ||b^j|| \tag{2.14}$$

and

$$\begin{aligned} \frac{D^2(P)}{\Delta^2(P)} &= \max_\varepsilon ||b^1\varepsilon_1 + \cdots + b^d\varepsilon_d||^2 \\ &= \max_\varepsilon \left( ||b^1||^2 + \cdots + ||b^d||^2 + 2 \sum_{\substack{i,k=1 \\ i<k}}^{d} \varepsilon_i\varepsilon_k b^{i^T} b^k \right) \\ &= ||b^1||^2 + \cdots + ||b^d||^2 + 2\gamma_d \end{aligned} \tag{2.15}$$

with $\gamma_d := \max\limits_\varepsilon\ s_d, \quad s_d := \sum\limits_{\substack{i,k=1 \\ i<k}}^{d} \varepsilon_i\varepsilon_k b^{i^T} b^k.$ Then

$$\begin{aligned} \gamma_1 &= 0, \\ \gamma_2 &= \max_{\varepsilon_1,\varepsilon_2} \varepsilon_1\varepsilon_2 b^{1^T} b^2 = |b^{1^T} b^2| \geq 0, \\ \gamma_d &= \max_{\varepsilon_1,\dots,\varepsilon_d} (s_{d-1} + \varepsilon_d(\varepsilon_1 b^{1^T} b^d + \cdots + \varepsilon_{d-1} b^{d-1^T} b^d)) \end{aligned}$$

holds. Choosing $(\varepsilon_1, \dots, \varepsilon_{d-1}) = (\varepsilon_1^0, \dots, \varepsilon_{d-1}^0)$ in such a way, that $s_{d-1}$ becomes maximal, we obtain

$$\begin{aligned} \gamma_d &\geq \gamma_{d-1} + \max_{\varepsilon_d} \varepsilon_d(\varepsilon_1^0 b^{1^T} b^d + \cdots + \varepsilon_{d-1}^0 b^{d-1^T} b^d) \\ &= \gamma_{d-1} + |\varepsilon_1^0 b^{1^T} b^d + \cdots + \varepsilon_{d-1}^0 b^{d-1^T} b^d| \\ &\geq \gamma_{d-1}, \end{aligned}$$

all in all

$$\gamma_d \geq \gamma_{d-1} \geq \cdots \geq \gamma_2 \geq \gamma_1 = 0.$$

Furthermore

$$\gamma_d \begin{cases} = 0, & \text{if } b^{i^T} b^k = 0 \text{ for } i, k = 1, \dots, d,\ i \neq k, \\ > 0 & \text{otherwise} \end{cases}$$

holds, because if $b^{p^T} b^q \neq 0$ for a certain pair of indices $p, q \in \{1, \dots, d\}$, then

$$\gamma_d \geq \max_{\varepsilon_p,\varepsilon_q} \varepsilon_p\varepsilon_q b^{p^T} b^q = |b^{p^T} b^q| > 0$$

would hold. Together with (2.14) and (2.15) this leads to

$$\frac{D(P)}{\Delta(P)} \geq \sqrt{d}, \tag{2.16}$$

and equality holds in (2.16) if and only if the vectors $b^1, \dots, b^d$ build an orthonormal system, i.e. the matrix $A^{-1}$ is orthogonal. Hence the generating vectors $a^1, \dots, a^d$ of the cross polytope build an orthonormal system, i.e. $P$ is a regular cross polytope. This proves the claim. □

The Lemmata 1 and 3 show, that $\sqrt{d}$ is the best possible upper bound of $\beta$ for all convex bodies, so we have proven the

**Theorem 2.** *In the class $\mathbb{K}$ of the convex bodies $K$ of $\mathbb{R}^d$ the affine breadth eccentricity*

$$\beta(K) := \inf_{\varphi \in \Phi} \frac{D(\varphi(K))}{\Delta(\varphi(K))}$$

*has the upper bound $\sqrt{d}$, which is the best possible one. It will be attained for instance in the affine class of cross polytopes of $\mathbb{R}^d$.*

Analogously to the proofs of Lemmata 2 and 3 it can be verified, that in addition to the cross polytopes the parallelotopes of $\mathbb{R}^d$ (which are "polar" to the cross polytopes) also are convex bodies with maximal affine breadth eccentricity $\sqrt{d}$.

## 3. Application: Minimal ellipsoid of a general cross polytope

Let $E_0 = E(C^0, x^0)$ be the minimal ellipsoid of the cross polytope

$$P = \mathrm{conv}\{a^1, \dots, a^d, -a^1, \dots, -a^d\} \subseteq \mathbb{R}^d \tag{3.1}$$

with the linearly independent generating vectors $a^1, \dots, a^d \in \mathbb{R}^d$. The centers of $P$ and $E_0$ coincide (as it is true for every central-symmetric convex body), i.e. $x^0 = 0$. Let $\varphi = \varphi_P$ be an affine transformation of $\mathbb{R}^d$ into itself with $\varphi_P(E_0) = B(0,1)$. According to Lemma 1, we get

$$\frac{D(\varphi_P(K))}{\Delta(\varphi_P(K))} \leq \sqrt{d}$$

for the cross polytope $\varphi_P(P)$. Due to Lemma 3 equality holds and hence $\varphi_P(P)$ is a regular cross polytope. After a suitable orthogonal transformation it can be written as

$$\varphi_P(P) = \mathrm{conv}\{e^1, \dots, e^d, -e^1, \dots, -e^d\}$$

with $\varphi_P(a^i) = e^i,\ i = 1, \dots, d$. Hence the mapping $\varphi_P$ must have the form

$$\varphi_P(t) = A^{-1^T} t,$$

where $A = (a^1 \cdots a^d)^T_{d,d}$. The condition $\varphi_P(E_0) = B(0,1)$ has the consequence $AQ^0A^T = I$, thus the explicit representation of the minimal ellipsoid $E(Q^0, x^0)$ of the cross polytope (3.1) is given by

$$Q^0 = (A^T A)^{-1},\ x^0 = 0.$$

The generating vectors $a^1, \dots, a^d$ build a system of pairwise conjugate half axes of the minimal ellipsoid, because

$$a^{i^T} Q^0 a^k = \begin{cases} 0, & \text{if } \ i \neq k \\ 1, & \text{if } \ i = k \end{cases} \ , i, k = 1, \dots, d. \tag{3.2}$$

**Remark.** If the vectors $a^1, \dots, a^d \in \mathbb{R}^d$ are given, the matrix $Q^0$ is uniquely determined by property (3.2): At first (3.2) implies the linear independency of $a^1, \dots, a^d$. For every

$$x = \sum_{k=1}^{d} \xi_k a^k \in \mathbb{R}^d$$

the relation

$$(\sum_{i=1}^{d} a^i a^{i^T})\, Q^0 x = \sum_{\substack{i,k=1 \\ i<k}}^{d} \xi_k a^i a^{i^T} Q^0 a^k = x$$

holds, hence $Q^{0^{-1}} = \sum\limits_{i=1}^{d} a^i a^{i^T} = A^T A$.

So we have as a by-product the following geometric property: Let $\{a^1, \dots, a^d\}$ be an arbitrary system of conjugate half axes of an ellipsoid $E = \{t \in \mathbb{R}^d \mid t^T Q t \leq 1\} \subseteq \mathbb{R}^d$. Then $E$ is minimal ellipsoid of the cross polytope $\mathrm{conv}\{a^1, \dots, a^d, -a^1, \dots, -a^d\}$, which is generated by these conjugate half axes. By the way, additionally $E$ is also the maximal ellipsoid of the parallelotope (*polar* convex body) given as intersection of the supporting halfspaces of $E$ in the boundary points $a^1, \dots, a^d, -a^1, \dots, -a^d$, cf. [4].

## References


[1] John, F.: *Extremum problems with inequalities as subsidiary conditions.* Studies Essays, Courant anniversary volume, Intersciene Publ. Inc., New York 1948, 187–204. Zbl 0034.10503

[2] Juhnke, F.: *Das Umkugelproblem und lineare semiinfinite Optimierung.* Beitr. Algebra Geom. **28** (1989), 147–156. Zbl 0689.52003

[3] Juhnke, F.: *Beiträge zur optimierungstheoretischen Behandlung volumenextremaler Ellipsoidüberdeckungen und -einbettungen.* Dissertation B, TU Magdeburg 1990.

[4] Juhnke, F.: *Polarity of Embedded and Circumscribed Ellipsoids.* Beitr. Algebra Geom. **36** (1995), 17–24. Zbl 0819.52007

[5] Jung, H. W. E.: *Über die kleinste Kugel, die eine räumliche Figur einschließt*. J. für Math. **123** (1901), 241–257. JFM 32.0296.05

[6] Leichtweiss, K.: *Über die affine Exzentrizität konvexer Körper*. Arch. Math. **10** (1959), 187–199. Zbl 0089.38303

[7] Steinhagen, P.: *Über die größte Kugel in einer konvexen Punktmenge*. Hamb. Abh. **1** (1921), 15–26. JFM 48.0837.03

Received November 9, 2003