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Asymptotic Mean Values of Gaussian Polytopes

Dedicated to the memory of Bernulf Weißbach

Daniel Hug **Götz Olaf Munsonius** **Matthias Reitzner**

Mathematisches Institut, Albert-Ludwigs-Universität
Eckerstr. 1, D-79104 Freiburg i. Br., Germany
e-mail: daniel.hug@math.uni-freiburg.de olaf@munsonius.de

Institut für Analysis und Technische Mathematik, Technische Universität Wien
Wiedner Hauptstrasse 8–10, A-1040 Vienna, Austria
e-mail: Matthias.Reitzner+e1142@tuwien.ac.at

Abstract. We consider geometric functionals of the convex hull of normally distributed random points in Euclidean space $\mathbb{R}^d$. In particular, we determine the asymptotic behaviour of the expected value of such functionals and of related geometric probabilities, as the number of points increases.

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1. Introduction

Let $X_1, X_2, \dots$ be independent identically distributed random points in Euclidean space $\mathbb{R}^d$. Geometric functionals such as volume, surface area, mean width, or the number of k-faces, of the convex hull of such random points have been studied repeatedly in the literature. A recent survey is provided in [12]. If the random points are chosen from a given compact convex set $K \subset \mathbb{R}^d$ with non-empty interior, it is natural to consider the uniform distribution on K. More generally, the distribution function may have a density with respect to Lebesgue measure. In case the domain is $\mathbb{R}^d$, the normal distribution is a canonical choice. Another method of generating $n+1$ random points in $\mathbb{R}^d$ goes back to a suggestion by Goodman and Pollack. Let R denote a random rotation of $\mathbb{R}^n$, i.e. a stochastic choice from the orthogonal group $O(n)$ under normalized Haar measure, let $\Pi_d : \mathbb{R}^n \to \mathbb{R}^d$ be the projection to the first d components ($d < n$), put $\Pi := \Pi_d \circ R$, and let $v_1, \dots, v_{n+1}$ be the vertices of a regular simplex T^n in $\mathbb{R}^n$. Then $\Pi(v_1), \dots, \Pi(v_{n+1})$ are $n+1$ random points in $\mathbb{R}^d$ in the *Goodman-Pollack model.* Clearly, as long as one considers rotation invariant functionals of such random points, one can project to a random linear subspace, instead of first rotating randomly and then projecting to a fixed subspace. For further information on this 'Grassmann approach' and related work of Vershik and Sporyshev [15], we refer to [3].

In connection with the Goodman-Pollack model, Affentranger and Schneider [3] especially found an expression for the expected value $\mathbb{E}f_k(\Pi T^n)$ of the number of k-faces of the random polytope ΠT^n, for $0 \le k < d < n$, in terms of external and internal angles of T^n and its faces. In addition, they showed that asymptotically

$$\mathbb{E}f_k(\Pi T^n) \sim \frac{2^d}{\sqrt{d}} \binom{d}{k+1} \beta(T^k, T^{d-1}) \, (\pi \log n)^{\frac{d-1}{2}} \tag{1.1}$$

as $n \to \infty$, where $\beta(T^k, T^{d-1})$ is the internal angle of a regular $(d-1)$-simplex at one of its k-dimensional faces. It was also observed by these authors that the value $\mathbb{E}f_{d-1}(\Pi T^n)$ coincides with the expected number of facets of the convex hull of $n+1$ independent and normally distributed random points in $\mathbb{R}^d$. An explanation for this relationship was subsequently found by Baryshnikov and Vitale [4]. To describe an important consequence of their result, and for later use, we call an i.i.d. sequence of (standard) Gaussian random points in $\mathbb{R}^d$ a *Gaussian sample* in $\mathbb{R}^d$. Let $X_1, \dots, X_{n+1}$ be a Gaussian sample in $\mathbb{R}^d$. Its convex hull will be called a *Gaussian polytope* in $\mathbb{R}^d$ and denoted by $[X_1, \dots, X_{n+1}]$. Let φ be an affine invariant (measurable) functional on the convex polytopes. Then it is shown in [4] that

$$\varphi(\Pi T^n) \stackrel{d}{=} \varphi([X_1, \dots, X_{n+1}]), \tag{1.2}$$

where $\stackrel{d}{=}$ means equality in distribution. Thus, if $X_1, \dots, X_n$ is a standard Gaussian sample and f_k denotes the number of k-faces, combining (1.1) and (1.2), we get

$$\mathbb{E}f_k([X_1, \dots, X_n]) \sim \frac{2^d}{\sqrt{d}} \binom{d}{k+1} \beta(T^k, T^{d-1}) \, (\pi \log n)^{\frac{d-1}{2}} \tag{1.3}$$

as $n \to \infty$ (see [4, Theorem 3]).

A direct derivation of this asymptotic expansion has been given by Raynaud [11] in the special case when $k = d-1$. A main objective of the present work is to provide a direct derivation of (1.3) for all $k \in \{0, \dots, d-1\}$. Incidentally, the present approach leads to a new expression for the internal angles of a regular simplex. A basic idea of the geometric part of our method is to characterize a k-face of a polytope by considering the projection of the vertices of the polytope to the orthogonal complement of that face. Another geometric tool, which we will apply repeatedly, is the classical affine Blaschke-Petkantschin formula. Thus, exploiting the fact that the projection to a subspace of a normally distributed point is again normally distributed, we can rewrite the expected value in terms of geometric probabilities of the form

$$\mathbb{P}(Y \notin [Y_1, \dots, Y_{n-k-1}]), \tag{1.4}$$

where $Y, Y_1, \dots, Y_{n-k-1}$ are independent normally distributed random points in $\mathbb{R}^{d-k}$ (with different variances). Note that (1.4) is the probability that a normally distributed random point is contained in a Gaussian polytope in $\mathbb{R}^{d-k}$. In a second step, we then derive the asymptotic behaviour of such geometric probabilities.

A major advantage of the present more direct treatment of normally distributed random points is that it can be applied to functionals which are not necessarily affine invariant. More explicitly, we are able to obtain results for a class of rotation invariant functionals that has been introduced by Wieacker [17], and has further been studied by Affentranger and Wieacker [2] and Affentranger [1]. Particular cases of such functionals are the total k-dimensional volume $V_k(\mathrm{skel}_k(P))$ of the k-faces of a polytope P, and the number of k-faces $f_k(P)$. For these we obtain as a consequence a more general result:

Theorem 1.1. *Let $X_1, \dots, X_n$ be i.i.d. random points in $\mathbb{R}^d$ with common standard normal distribution. Then*

$$\mathbb{E}V_k(\mathrm{skel}_k([X_1, \dots, X_n])) \sim c_{(k,d)}\ (\log n)^{\frac{d-1}{2}} \tag{1.5}$$

and

$$\mathbb{E}f_k([X_1, \dots, X_n]) \sim \bar{c}_{(k,d)}\ (\log n)^{\frac{d-1}{2}} \tag{1.6}$$

as $n \to \infty$, where $c_{(k,d)}$ and $\bar{c}_{(k,d)}$ are constants depending only on k and d.

The constants $c_{(k,d)}$ and $\bar{c}_{(k,d)}$ are given in Section 4. These results complement asymptotic expansions for the mean values of quermassintegrals of Gaussian polytopes, which were given by Affentranger [1]. Important further contributions to convex hulls of normally distributed random points are due to Hueter [7], who proved a Central Limit Theorem for $V_d([X_1, \dots, X_n])$ and $f_0([X_1, \dots, X_n])$.

We also investigate functionals of the (centrally) *symmetric convex hull* $[\pm X_1, \dots, \pm X_n]$, where again $X_1, \dots, X_n$ is a (standard) Gaussian sample in $\mathbb{R}^d$. It follows from [4] that the symmetric convex hull of a Gaussian sample can be obtained by randomly rotating and projecting to $\mathbb{R}^d$ a regular crosspolytope in $\mathbb{R}^n$. This fact was used by Böröczky and Henk [5], who thus found the surprising result that the asymptotic expansion does not change if T^n in (1.1) is replaced by a regular n-dimensional crosspolytope. Apart from admitting the treatment of more general functionals also in the symmetric situation, our method leads to an alternative and more direct explanation for this phenomenon.

2. Auxiliary results

In this section, we will fix our notation and provide some auxiliary results.

We will work in Euclidean spaces $\mathbb{R}^n$ of varying dimensions n. The norm in these spaces will always be denoted by $\|\cdot\|$. For points $x_1, \dots, x_m \in \mathbb{R}^n$, the convex hull of these points is denoted by $[x_1, \dots, x_m]$. If $P \subset \mathbb{R}^n$ is a (convex) polytope, then we write $\mathcal{F}_k(P)$ for the set of its k-dimensional faces and $f_k(P)$ for the number of these k-faces, where $k \in \{0, \dots, n\}$. The k-dimensional volume of the convex hull of $k+1$ points $x_0, \dots, x_k$ is denoted by $\Delta_k(x_0, \dots, x_k)$. Finally, the k-dimensional Lebesgue measure in a k-dimensional flat $E \subset \mathbb{R}^n$ is denoted by λ_E, or simply by λ_k, if the affine subspace E is clear from the context.

The affine Blaschke-Petkantschin formula will be an important tool in our analysis. Let $\mathcal{E}_k^n$ be the space of k-flats in $\mathbb{R}^n$, and let $\mathcal{L}_k^n$ be the space of k-dimensional linear subspaces of $\mathbb{R}^n$, $k \in \{0, \dots, n\}$. Both spaces are endowed with the usual topologies. The rotation invariant Haar probability measure on $\mathcal{L}_k^n$ is denoted by ν_k (the dimension n will always be clear from the context). Moreover, a motion invariant Haar measure on $\mathcal{E}_k^n$ is defined by

$$\mu_k := \int_{\mathcal{L}_k^n} \int_{L^\perp} \mathbf{1}\{L + y \in \cdot\}\, \lambda_{L^\perp}(dy)\, \nu_k(dL),$$

where $L^\perp$ is the orthogonal complement of $L \in \mathcal{L}_k^n$ in $\mathbb{R}^n$. Then, for $n \geq 1$, $q \in \{0, \dots, n\}$ and any non-negative measurable function $f : (\mathbb{R}^n)^{q+1} \to \mathbb{R}$, the *affine Blaschke-Petkantschin formula* (see [13, § 6.1]) states that

$$\begin{aligned} &\int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} f(x_0, \dots, x_q)\, \lambda_n(dx_0) \dots \lambda_n(dx_q) \qquad (2.1)\\ &= c_{nq} (q!)^{n-q} \int_{\mathcal{E}_q^n} \int_E \cdots \int_E f(x_0, \dots, x_q) \Delta_q(x_0, \dots, x_q)^{n-q}\, \lambda_E(dx_0) \dots \lambda_E(dx_q)\, \mu_q(dE), \end{aligned}$$

where

$$c_{nq} := \frac{\omega_{n-q+1} \cdots \omega_n}{\omega_1 \cdots \omega_q}$$

and $\omega_r := 2\pi^{\frac{r}{2}} / \Gamma\left(\frac{r}{2}\right)$, $r > 0$; for $r \in \mathbb{N}$, ω_r is the volume of the $(r-1)$-dimensional unit sphere.

In addition to the Blaschke-Petkantschin formula, we will require more specific preparations related to the multidimensional normal distribution. As usual, we fix an underlying probability space $(\Omega, \mathfrak{A}, \mathbb{P})$. A random point X in $\mathbb{R}^n$, defined on Ω, is said to be normally distributed with positive definite $n \times n$-covariance matrix Σ (and mean 0) if $X(\mathbb{P})$ has the density

$$f_\Sigma(x) = \left((2\pi)^n \det \Sigma\right)^{-\frac{1}{2}} \exp\left(-\frac{1}{2} x^T \Sigma^{-1} x\right), \qquad x \in \mathbb{R}^n;$$

then we write $X \stackrel{d}{=} N(0, \Sigma)$. For simplicity, we will exclusively consider the case $\Sigma = \sigma \cdot I_n$, $\sigma > 0$. The distribution function of the one-dimensional normal distribution $N(0, \frac{1}{2})$ is given by

$$\phi(z) := \frac{1}{\sqrt{\pi}} \int_{-\infty}^{z} e^{-t^2}\, dt, \qquad z \in \mathbb{R}.$$

The Landau symbols o and O, which will be used several times in the following, are defined as usual. Moreover, writing $f(x) \sim g(x)$ for real-valued functions f, g, defined on a suitable subset of $\mathbb{R}$, we mean that $f(x)/g(x) \to 1$ as $x \to \infty$. The natural logarithm will be denoted by log. Finally, all constants which are used subsequently, depend only on the parameters that are indicated.

Lemma 2.1. *For $\alpha, \beta > 0$ and $s \in \mathbb{R}$,*

$$\int_{1}^{\infty} \phi(z)^{\beta-\alpha} z^s \exp\left(-\alpha z^2\right)\, dz = \Gamma(\alpha) 2^{\alpha-1} \pi^{\frac{\alpha}{2}} \beta^{-\alpha} (\log \beta)^{\frac{\alpha+s-1}{2}} (1 + o(1))$$

and

$$\int_{1}^{\infty} (2\phi(z) - 1)^{\beta-\alpha} z^s \exp\left(-\alpha z^2\right)\, dz = \Gamma(\alpha) 2^{-1} \pi^{\frac{\alpha}{2}} \beta^{-\alpha} (\log \beta)^{\frac{\alpha+s-1}{2}} (1 + o(1))$$

as $\beta \to \infty$.

Proof. A complete proof can be given, for instance, by refining and extending an argument of Affentranger (see [1, Appendix II]) or by generalizing an alternative approach indicated in [8]. □

The asymptotic expansion provided in Lemma 2.1 will be used several times. A first application is given in the proof of the next result, which will be needed in Section 4. There the following expressions arise naturally. For $a \geq 0$, $p, q, r \in \mathbb{R}$ with $p > q > r > 0$ and $\gamma \in \mathbb{R}$, we define

$$I_a(p, q, r; \gamma) := \int_{1}^{\infty}\int_{1}^{\infty} \phi(z)^{p-q} z s^{a+q-r-1} \left(\gamma^2 + z^2\right)^{a/2} \exp\left(-r s^2(\gamma^2 + z^2) - (q - r) z^2\right)\, ds\, dz.$$

This quantity will be compared with

$$J_a(p, q, r; \gamma) := \frac{1}{2r} \int_{1}^{\infty} \phi(z)^{p-q} z^{a-1} \exp\left(-q z^2 - r\gamma^2\right)\, dz$$

as $p \to \infty$.

Lemma 2.2. *Let $a \geq 0$, and let $p, q, r \in \mathbb{R}$ satisfy $q > r > 0$. Then, uniformly in $\gamma \in \mathbb{R}$,*

$$|I_a(p, q, r; \gamma) - J_a(p, q, r; \gamma)| = O\left(p^{-q} (\log p)^{\frac{q+a-3}{2}}\right)$$

as $p \to \infty$.

Proof. By Fubini's theorem,

$$\begin{aligned} I_a(p,q,r;\gamma) &= \int_1^\infty \phi(z)^{p-q} z \sqrt{\gamma^2+z^2}^{\,a} \exp\left(-(q-r)z^2\right) \\ &\qquad \times \int_1^\infty s^{a+q-r-1} \exp\left(-r\left(\gamma^2+z^2\right)s^2\right)\, ds\, dz. \end{aligned} \tag{2.2}$$

Repeated partial integration yields that

$$\begin{aligned} &\left| \int_1^\infty s^{a+q-r-1} \exp\left(-r(\gamma^2+z^2)s^2\right)\, ds - \frac{1}{2r(\gamma^2+z^2)} \exp\left(-r(\gamma^2+z^2)\right) \right| \\ &\qquad \le \frac{c_1(a,q,r)}{(\gamma^2+z^2)^2} \exp\left(-r(\gamma^2+z^2)\right), \end{aligned} \tag{2.3}$$

where $c_1(a,q,r)$ is a constant. Hence, (2.2) and (2.3) imply that

$$\begin{aligned} &\left| I_a(p,q,r;\gamma) - \frac{1}{2r} \int_1^\infty \phi(z)^{p-q} z \sqrt{\gamma^2+z^2}^{\,a-2} \exp\left(-qz^2 - r\gamma^2\right)\, dz \right| \\ &\qquad \le c_1(a,q,r) \int_1^\infty \phi(z)^{p-q} z \sqrt{\gamma^2+z^2}^{\,a-4} \exp\left(-qz^2 - r\gamma^2\right)\, dz. \end{aligned}$$

Then, for $z \ge 1$, $r > 0$, $a \ge 0$ and $\gamma \in \mathbb{R}$, we use the estimates

$$\left| z\sqrt{\gamma^2+z^2}^{\,a-2} - z^{a-1} \right| \exp\left(-r\gamma^2\right) \le c_2(a,r) z^{a-2}$$

and

$$z\sqrt{\gamma^2+z^2}^{\,a-4} \exp\left(-r\gamma^2\right) \le c_2(a,r) z^{a-3},$$

with a constant $c_2(a,r)$, to infer that

$$|I_a(p,q,r;\gamma) - J_a(p,q,r;\gamma)| \le c_3(a,q,r) \int_1^\infty \phi(z)^{p-q} z^{a-2} \exp\left(-qz^2\right)\, dz,$$

where $c_3(a,q,r)$ is a constant. Now an application of Lemma 2.1 completes the proof. $\square$

We remark that a similar result holds, with essentially the same proof, if in the definition of I_a and J_a the function ϕ is replaced by $2\phi - 1$.

3. Transition to probabilities

Throughout this paper, $X_1, \ldots, X_n$ will be independent random points with $X_i \stackrel{d}{=} N\left(0, \frac{1}{2} I_d\right)$. For $n \geq d+1$, $k \in \{0, \ldots, d-1\}$ and $I \subset \{1, \ldots, n\}$ with $|I| = k+1$, we define

$$h_I(x_1, \ldots, x_n) := \mathbf{1}\{[x_i : i \in I] \in \mathcal{F}_k([x_1, \ldots, x_n])\},$$

$x_1, \ldots, x_n \in \mathbb{R}^d$, and put $I_0 := \{1, \ldots, k+1\}$. By symmetry, we then obtain for the mean number of k-faces of the $\mathbb{P}$-almost surely simplicial Gaussian polytope $[X_1, \ldots, X_n]$ that

$$\begin{aligned} E f_k([X_1, \ldots, X_n]) &= \sum_{|I|=k+1} \int h_I(X_1, \ldots, X_n)\, d\mathbb{P} \\ &= \binom{n}{k+1} \int h_{I_0}(X_1, \ldots, X_n)\, d\mathbb{P}. \end{aligned} \tag{3.1}$$

In order to transform this mean value into a basic geometric probability, we define for $d, k \in \mathbb{N}$ and $q \geq 0$ the constant

$$M(d, k, q) := \pi^{-\frac{d}{2}(k+1)} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \Delta_k(x_0, \ldots, x_k)^q \exp\left(-\sum_{i=0}^{k} \|x_i\|^2\right) \lambda_d(dx_0) \ldots \lambda_d(dx_k).$$

In the case when $q \in \mathbb{N}$, $M(d, k, q)$ is the q-th moment of the random k-dimensional volume of a random k-simplex $[X_0, \ldots, X_k]$ with $k+1$ independent and normally distributed vertices $X_i \stackrel{d}{=} N\left(0, \frac{1}{2} I_d\right)$, $i = 0, \ldots, k$. The following lemma will be applied in the special case when $d = k$.

Lemma 3.1. *For $d, k \in \mathbb{N}$, $k \leq d$ and $q \geq 0$,*

$$M(d, k, q) = \pi^{\frac{k}{2} q} \frac{c_{dk}}{c_{(q+d)k}} \left(\frac{\sqrt{k+1}}{k!}\right)^q.$$

Proof. For $q \in \mathbb{N}_0$ this can be shown as in the proof of Satz 6.3.1 in [13]. The general case follows by using the connection with the Wishart distribution (cf. [10], [9, p. 437, (4.5.3)] and [6, pp. 303, 315]; see also [14]). □

Theorem 3.2. *Let $X_1, \ldots, X_n$ be $n \geq d+1$ independent random points in $\mathbb{R}^d$ with $X_i \stackrel{d}{=} N\left(0, \frac{1}{2} I_d\right)$. Then, for $k \in \{0, \ldots, d-1\}$,*

$$\mathbb{E} f_k([X_1, \ldots, X_n]) = \binom{n}{k+1} \mathbb{P}(Y \notin [Y_1, \ldots, Y_{n-k-1}]),$$

where $Y, Y_1, \ldots, Y_{n-k-1}$ are independent random points in $\mathbb{R}^{d-k}$ with $Y \stackrel{d}{=} N\left(0, \frac{1}{2(k+1)} I_{d-k}\right)$ and $Y_i \stackrel{d}{=} N\left(0, \frac{1}{2} I_{d-k}\right)$ for $i = 1, \ldots, n-k-1$.

Proof. Using (3.1), the Blaschke-Petkantschin formula (2.1) and the definition of μ_k, we obtain

$$
\begin{aligned}
&\mathbb{E}f_k([X_1,\dots,X_n])\\
&= \binom{n}{k+1}\pi^{-\frac{d}{2}n}\int\limits_{\mathbb{R}^d}\cdots\int\limits_{\mathbb{R}^d} h_{I_0}(x_1,\dots,x_n)\exp\left(-\sum_{i=1}^{n}\|x_i\|^2\right)\lambda_d(dx_1)\dots\lambda_d(dx_n)\\
&= c_4(n,k,d)\int\limits_{\mathbb{R}^d}\cdots\int\limits_{\mathbb{R}^d}\int\limits_{\mathcal{L}_k^d}\int\limits_{L^\perp}\int\limits_{L}\cdots\int\limits_{L} h_{I_0}(z_1+y,\dots,z_{k+1}+y,x_{k+2},\dots,x_n)\\
&\quad\times\Delta_k(z_1,\dots,z_{k+1})^{d-k}\exp\left(-\sum_{i=1}^{k+1}\|z_i\|^2-(k+1)\|y\|^2-\sum_{i=k+2}^{n}\|x_i\|^2\right)\\
&\quad\times\lambda_L(dz_1)\dots\lambda_L(dz_{k+1})\lambda_{L^\perp}(dy)\nu_k(dL)\lambda_d(dx_{k+2})\dots\lambda_d(dx_n),
\end{aligned}
$$

where

$$c_4(n,k,d) := \binom{n}{k+1}\pi^{-\frac{d}{2}n}c_{dk}(k!)^{d-k}.$$

Assume that $z_1,\dots,z_{k+1}\in L$ are affinely independent, let $z_{k+2},\dots,z_n\in L$ and $y\in L^\perp$. Then, for $\lambda_{L^\perp}$-almost all $y_{k+2},\dots,y_n\in L^\perp$,

$$[z_1+y,\dots,z_{k+1}+y]\in\mathcal{F}_k([z_1+y,\dots,z_{k+1}+y,z_{k+2}+y_{k+2},\dots,z_n+y_n])$$

if and only if

$$y\notin[y_{k+2},\dots,y_n].$$

Hence, defining

$$g(y,y_{k+2},\dots,y_n) := \mathbf{1}\{y\notin[y_{k+2},\dots,y_n]\},$$

for $y,y_{k+2},\dots,y_n\in\mathbb{R}^d$, we obtain

$$
\begin{aligned}
&\mathbb{E}f_k([X_1,\dots,X_n])\\
&= c_5(n,k,d)\int\limits_{\mathcal{L}_k^d}\left[\int\limits_{L}\cdots\int\limits_{L}\Delta_k(z_1,\dots,z_{k+1})^{d-k}\exp\left(-\sum_{i=1}^{k+1}\|z_i\|^2\right)\lambda_L(dz_1)\dots\lambda_L(dz_{k+1})\right]\\
&\quad\times\int\limits_{L^\perp}\cdots\int\limits_{L^\perp} g(y,y_{k+2},\dots,y_n)\exp\left(-\sum_{i=k+2}^{n}\|y_i\|^2-(k+1)\|y\|^2\right)\\
&\quad\times\lambda_{L^\perp}(dy_{k+2})\dots\lambda_{L^\perp}(dy_n)\lambda_{L^\perp}(dy)\nu_k(dL),
\end{aligned}
$$

where

$$c_5(n,k,d) := c_4(n,k,d)\pi^{\frac{1}{2}k(n-k-1)}.$$

Thus, by Lemma 3.1 and the rotation invariance of the integrand, it follows that

$$\begin{aligned}
& \mathbb{E} f_k([X_1, \ldots, X_n]) \\
= \; & c_5(n,k,d)\pi^{\frac{k}{2}(k+1)} M(k,k,d-k) \\
& \times \int_{\mathbb{R}^{d-k}} \cdots \int_{\mathbb{R}^{d-k}} g(y, y_{k+2}, \ldots, y_n) \exp\left(- \sum_{i=k+2}^{n} \|y_i\|^2 - (k+1)\|y\|^2 \right) \\
& \times \lambda_{d-k}(dy_{k+2}) \ldots \lambda_{d-k}(dy_n) \lambda_{d-k}(dy).
\end{aligned}$$

Applying Lemma 3.1 and simplifying the constants, we obtain the assertion of the theorem.

□

In the remainder of this section, we will explain how the preceding argument can be modified to yield a similar relation in the centrally symmetric case. Moreover, the approach will be extended to cover more general functionals.

3.1. The centrally symmetric case

Let $X_1, \ldots, X_n$ be $n \geq d$ independent random points in $\mathbb{R}^d$ with $X_i \stackrel{d}{=} N\left(0, \frac{1}{2} I_d\right)$. We write

$$[x_1, \ldots, x_n]_c := [x_1, -x_1, \ldots, x_n, -x_n]$$

for the (centrally) symmetric convex hull of $x_1, \ldots, x_n \in \mathbb{R}^d$. For subsets $I, J \subset \{1, \ldots, n\}$ with $|I| + |J| = k+1$, we put

$$h_{IJ}(x_1, \ldots, x_n) := \mathbf{1}\{[x_i, -x_j : i \in I, j \in J] \in \mathcal{F}_k([x_1, \ldots, x_n]_c)\}$$

and set $I_0 := \{1, \ldots, k+1\}$, $J_0 := \emptyset$. Since $[X_1, \ldots, X_n]_c$ is $\mathbb{P}$-almost surely a simplicial polytope, by symmetry and by the reflection invariance of the normal distribution, we find that

$$\begin{aligned}
\mathbb{E} f_k([X_1, \ldots, X_n]_c) &= \sum_{r=0}^{k+1} \sum_{|I|=r} \sum_{|J|=k+1-r} \int h_{IJ}(X_1, \ldots, X_n)\, d\mathbb{P} \\
&= \sum_{r=0}^{k+1} \binom{n}{r} \binom{n-r}{k+1-r} \int h_{I_0 J_0}(X_1, \ldots, X_n)\, d\mathbb{P} \\
&= 2^{k+1} \binom{n}{k+1} \int h_{I_0 J_0}(X_1, \ldots, X_n)\, d\mathbb{P}.
\end{aligned}$$

The integral thus obtained can be further simplified as shown in the next theorem.

Theorem 3.3. *Let $X_1, \ldots, X_n$ be $n \geq d$ independent random points in $\mathbb{R}^d$ with $X_i \stackrel{d}{=} N\left(0, \frac{1}{2} I_d\right)$. Then, for $k \in \{0, \ldots, d-1\}$,*

$$\mathbb{E} f_k([X_1, \ldots, X_n]_c) = 2^{k+1} \binom{n}{k+1} \mathbb{P}(Y \notin [Y_1, \ldots, Y_{n-k-1}]_c),$$

where $Y, Y_1, \ldots, Y_{n-k-1}$ are independent random points in $\mathbb{R}^{d-k}$ with $Y \stackrel{d}{=} N\left(0, \frac{1}{2(k+1)} I_{d-k}\right)$ and $Y_i \stackrel{d}{=} N\left(0, \frac{1}{2} I_{d-k}\right)$ for $i = 1, \ldots, n-k-1$.

Proof. Repeat the proof of Theorem 3.2 with g replaced by

$$g_c(y, y_{k+2}, \dots, y_n) := \mathbf{1}\{y \notin [y_{k+2}, \dots, y_n]_c\}$$

for $y, y_{k+2}, \dots, y_n \in \mathbb{R}^d$. □

3.2. A general functional

A class of functionals, which has first been introduced by Wieacker [17] and has further been studied in [1], [2], depends on two parameters. For a polytope $P \subset \mathbb{R}^d$, real numbers $a, b \geq 0$ and $k \in \{0, \dots, d-1\}$, we define

$$T_{a,b}^{d,k}(P) := \sum_{F \in \mathcal{F}_k(P)} (\eta(F))^a \, (\lambda_k(F))^b,$$

where $\lambda_k(F)$ denotes the k-dimensional Lebesgue measure of a k-dimensional face $F \in \mathcal{F}_k(P)$ calculated in the affine hull $\operatorname{aff}(F)$ of F, and $\eta(F) := \operatorname{dist}(\operatorname{aff}(F), 0)$ is defined as the distance of $\operatorname{aff}(F)$ from the origin. For $a = b = 0$, we have $T_{0,0}^{d,k} = f_k$. But already for $a = 0$, $b = 1$ we get a functional which is not affine invariant, but merely rotation invariant; in that case,

$$T_{0,1}^{d,k}(P) = \sum_{F \in \mathcal{F}_k(P)} \lambda_k(F)$$

is the total k-dimensional volume of the k-skeleton of P. In particular, $T_{0,1}^{d,d-1}(P)$ is the surface area of a d-dimensional polytope $P \subset \mathbb{R}^d$. Finally, we emphasize that $T_{1,1}^{d,d-1}(P) = d\lambda_d(P)$ if $0 \in P$.

If $X_1, \dots, X_n$ are $n \geq d+1$ independent random points in $\mathbb{R}^d$ with $X_i \overset{d}{=} N\left(0, \frac{1}{2} I_d\right)$, then $\mathbb{P}$-almost surely $[X_i : i \in I]$ is a k-dimensional polytope whenever $I \subset \{1, \dots, n\}$ with $|I| = k+1$, and we put

$$\eta_I := \eta([X_i : i \in I]), \quad \lambda_{k,I} := \lambda_k([X_i : i \in I]), \quad h_I := h_I(X_1, \dots, X_n).$$

By symmetry we thus obtain

$$\mathbb{E} T_{a,b}^{d,k}([X_1, \dots, X_n]) = \binom{n}{k+1} \int h_{I_0} \, (\eta_{I_0})^a \, (\lambda_{k,I_0})^b \, d\mathbb{P},$$

where $I_0 := \{1, \dots, k+1\}$.

Theorem 3.4. *Let $X_1, \dots, X_n$ be $n \geq d+1$ independent random points in $\mathbb{R}^d$ with $X_i \overset{d}{=} N\left(0, \frac{1}{2} I_d\right)$. Then, for $k \in \{0, \dots, d-1\}$ and $a, b \geq 0$,*

$$\mathbb{E} T_{a,b}^{d,k}([X_1, \dots, X_n]) = \binom{n}{k+1} C(b,k,d) \int \mathbf{1}\{Y \notin [Y_1, \dots, Y_{n-k-1}]\} \|Y\|^a \, d\mathbb{P},$$

where

$$C(b,k,d) := \left(\frac{\sqrt{k+1}}{k!} \right)^b \prod_{j=1}^{k} \frac{\Gamma\left(\frac{d+b+1-j}{2}\right)}{\Gamma\left(\frac{d+1-j}{2}\right)}$$

and $Y, Y_1, \ldots, Y_{n-k-1}$ *are independent random points in* $\mathbb{R}^{d-k}$ *with* $Y \stackrel{d}{=} N\left(0, \frac{1}{2(k+1)} I_{d-k}\right)$ *and* $Y_i \stackrel{d}{=} N\left(0, \frac{1}{2} I_{d-k}\right)$ *for* $i = 1, \ldots, n-k-1$.

Proof. By the same arguments as in the proof of Theorem 3.2, we get

$$\begin{aligned}
&\mathbb{E}T_{a,b}^{d,k}([X_1, \ldots, X_n]) \\
= \; & c_5(n,k,d) \int_{\mathbb{R}^k} \cdots \int_{\mathbb{R}^k} \Delta_k(z_1, \ldots, z_{k+1})^{d-k+b} \exp\left(-\sum_{i=1}^{k+1} \|z_i\|^2\right) \lambda(dz_1) \ldots \lambda(dz_{k+1}) \\
& \times \int_{\mathbb{R}^{d-k}} \cdots \int_{\mathbb{R}^{d-k}} g(y, y_{k+2}, \ldots, y_n) \|y\|^a \exp\left(-\sum_{i=k+2}^{n} \|y_i\|^2 - (k+1)\|y\|^2\right) \\
& \times \lambda_{d-k}(dy_{k+2}) \ldots \lambda_{d-k}(dy_n) \lambda_{d-k}(dy).
\end{aligned}$$

The proof is completed by using Lemma 3.1 and by simplifying the constants. $\square$

Clearly, a centrally symmetric version of Theorem 3.4 could be stated and proved in a similar way.

4. Asymptotic expansions

In Theorem 3.2 the mean number of k-faces $\mathbb{E}f_k([X_1, \ldots, X_n])$ of a Gaussian polytope in $\mathbb{R}^d$ has been expressed in terms of a basic geometric probability. In this section, we will derive the asymptotic expansion of probabilities of this type. For this purpose, let $l, m \in \mathbb{N}$ with $l \geq m+1$ and $k > -1$, and let $Y, Y_1, \ldots, Y_l$ be independent random points in $\mathbb{R}^m$ with

$$Y \stackrel{d}{=} N\left(0, \frac{1}{2(k+1)} I_m\right) \quad \text{and} \quad Y_i \stackrel{d}{=} N\left(0, \frac{1}{2} I_m\right).$$

The choice $k \in \{0, \ldots, d-1\}$, $l = n-k-1$ and $m = d-k$ then corresponds to the situation of Theorem 3.2. In order to state our result, we define constants $A(1,k) := 1$,

$$\begin{aligned}
A(m,k) \; := \; & \int_{\mathbb{R}^{m-1}} \cdots \int_{\mathbb{R}^{m-1}} \int_{[u_1, \ldots, u_m]} \Delta_{m-1}(u_1, \ldots, u_m) \\
& \times \exp\left(-\sum_{i=1}^{m} \|u_i\|^2 - (k+1)\|u\|^2\right) \lambda_{m-1}(du) \lambda_{m-1}(du_1) \ldots \lambda_{m-1}(du_m)
\end{aligned}$$

for $m \geq 2$, and

$$C(m,k) := 2^{m+k}(k+1)^{\frac{m}{2}-1} \frac{\Gamma(m+k+1)}{m\Gamma\left(\frac{m}{2}\right)} \pi^{\frac{1}{2}(k+1+m-m^2)}$$

for $m \in \mathbb{N}$. An interpretation of the numbers $A(m,k)$ in terms of interior angles of regular simplices will be given below in the case when $k \in \mathbb{N}$.

We now consider the asymptotic behaviour of the probability that a normally distributed random point is contained in a Gaussian polytope.

Theorem 4.1. *Let $l, m \in \mathbb{N}$ and $k > -1$. Let $Y, Y_1, \ldots, Y_l$ be independent random points in $\mathbb{R}^m$ with $Y \overset{d}{=} N\left(0, \frac{1}{2(k+1)} I_m\right)$ and $Y_i \overset{d}{=} N\left(0, \frac{1}{2} I_m\right)$. Then*

$$\mathbb{P}(Y \notin [Y_1, \ldots, Y_l]) \sim C(m,k) A(m,k)\, l^{-(k+1)} (\log l)^{\frac{m+k-1}{2}}$$

as $l \to \infty$.

Combining Theorem 3.2 and a special case of Theorem 4.1, we obtain the expansion (1.3), though with a different form of the constant, i.e. for $k \in \{0, \ldots, d-1\}$,

$$\mathbb{E} f_k([X_1, \ldots, X_n]) \sim \frac{C(d-k,k)}{(k+1)!} A(d-k,k) (\log n)^{\frac{d-1}{2}} \tag{4.1}$$

as $n \to \infty$, giving the constant $\bar{c}_{(k,d)}$ in (1.6). By comparison, we thus conclude that

$$A(d-k,k) = \frac{(d-k)\Gamma\left(\frac{d-k}{2}\right) \pi^{\frac{(d-k)^2}{2}-1}}{(d-k-1)!(k+1)^{\frac{d-k}{2}-1}\sqrt{d}} \beta(T^k, T^{d-1}), \tag{4.2}$$

for $k \in \{0, \ldots, d-1\}$. Relation (4.2) can be interpreted as an apparently new integral representation for the interior angles of a regular simplex. It would be nice to have a short direct proof of (4.2), possibly extending to more general parameters, if the analytic expression for $\beta(T^k, T^{d-1})$ obtained in [5, (2.3)] with $\alpha = 1/(d-k)$ and $n = d-k$ is used.

Proof of Theorem 4.1. We can assume that $l \geq m+1$ and put $A := \{Y \notin [Y_1, \ldots, Y_l]\}$. Then Wendel's theorem [16] yields that

$$\mathbb{P}(A) = \mathbb{P}(A \cap \{0 \in \operatorname{int}[Y_1, \ldots, Y_l]\}) + O\left(\frac{l^m}{2^l}\right).$$

For a set $F \subset \mathbb{R}^m$, we define

$$\operatorname{pos}_1(F) := \{\lambda x : x \in F, \lambda > 1\};$$

hence, if F is an $(m-1)$-dimensional convex set with $0 \notin F$, then $\operatorname{pos}_1(F)$ is the truncated cone generated by F. Under the assumption that the origin is an interior point of $[Y_1, \ldots, Y_l]$, we decompose the complement of $[Y_1, \ldots, Y_l]$ into the truncated cones generated by the facets of $[Y_1, \ldots, Y_l]$. Thus, again applying Wendel's theorem and by symmetry, we get

$$\mathbb{P}(A) = \binom{l}{m} \mathbb{P}\big(Y \in \operatorname{pos}_1([Y_1, \ldots, Y_m]), \operatorname{aff}\{Y_1, \ldots, Y_m\} \cap [0, Y_{m+1}, \ldots, Y_l] = \emptyset\big) + O\left(\frac{l^m}{2^l}\right).$$

Define indicator functions h_0 and h_1 by putting

$$h_0(y_1, \ldots, y_l) := \mathbf{1}\{\operatorname{aff}\{y_1, \ldots, y_m\} \cap [0, y_1, \ldots, y_l] = \emptyset\},$$

and

$$h_1(y, y_1, \ldots, y_m) := \mathbf{1}\{y \in \operatorname{pos}_1([y_1, \ldots, y_m])\},$$

where $y, y_1, \ldots, y_l \in \mathbb{R}^m$. Hence, $\mathbb{P}(A)$ can be rewritten as

$$\begin{aligned}
\mathbb{P}(A) &= \binom{l}{m} \frac{(k+1)^{\frac{m}{2}}}{\pi^{\frac{m}{2}(l+1)}} \underbrace{\int_{\mathbb{R}^m} \cdots \int_{\mathbb{R}^m}}_{l+1} h_0(y_1, \ldots, y_l) h_1(y, y_1, \ldots, y_m) \\
&\quad \times \exp\left(- \sum_{i=1}^{l} \|y_i\|^2 - (k+1)\|y\|^2 \right) \lambda_m(dy)\lambda_m(dy_1)\ldots\lambda_m(dy_l) + O\left(\frac{l^m}{2^l}\right) \\
&= \binom{l}{m} \frac{(k+1)^{\frac{m}{2}}}{\pi^{\frac{m}{2}(m+1)}} \underbrace{\int_{\mathbb{R}^m} \cdots \int_{\mathbb{R}^m}}_{m+1} \phi(\mathrm{dist}(\mathrm{aff}\{y_1, \ldots, y_m\}, 0))^{l-m} h_1(y, y_1, \ldots, y_m) \\
&\quad \times \exp\left(- \sum_{i=1}^{m} \|y_i\|^2 - (k+1)\|y\|^2 \right) \lambda_m(dy)\lambda_m(dy_1)\ldots\lambda_m(dy_m) + O\left(\frac{l^m}{2^l}\right),
\end{aligned}$$

where Fubini's theorem has been used in the second step.

Now we first consider the case $m \geq 2$. We apply the Blaschke-Petkantschin formula (2.1) and use the rotation invariance of the integrand as in the proof of Theorem 3.2. Identifying $\mathbb{R}^{m-1}$ with the orthogonal complement $e_m^\perp \subset \mathbb{R}^m$ of the unit vector e_m, we finally get

$$\mathbb{P}(A) = p(l, m, k) + O\left(\phi(1)^l\right)$$

with

$$\begin{aligned}
p(l, m, k) &:= 2\binom{l}{m} c_6(m, k) \int_1^\infty \int_{\mathbb{R}^{m-1}} \cdots \int_{\mathbb{R}^{m-1}} \int_{\mathbb{R}^m} \phi(z)^{l-m} h_1(y, u_1 + ze_m, \ldots, u_m + ze_m) \\
&\quad \times \Delta_{m-1}(u_1, \ldots, u_m) \exp\left(- \sum_{i=1}^{m} \|u_i\|^2 - mz^2 - (k+1)\|y\|^2 \right) \\
&\quad \times \lambda_m(dy)\lambda_{m-1}(du_1)\ldots\lambda_{m-1}(du_m)\lambda_1(dz)
\end{aligned}$$

and

$$c_6(m, k) := \frac{(k+1)^{\frac{m}{2}}(m-1)!}{\pi^{\frac{m^2}{2}} \Gamma\left(\frac{m}{2}\right)}.$$

It remains to evaluate the asymptotic behaviour of $p(l, m, k)$. The transformation formula for multiple integrals yields that

$$\begin{aligned}
&\int_{\mathbb{R}^m} h_1(y, u_1 + ze_m, \ldots, u_m + ze_m) \exp(-(k+1)\|y\|^2) \lambda_m(dy) \\
&= \int_1^\infty \int_{[u_1, \ldots, u_m]} z s^{m-1} \exp\left(-(k+1)s^2(\|u\|^2 + z^2)\right) \lambda_{m-1}(du)\lambda_1(ds),
\end{aligned}$$

and hence

$$\begin{aligned} p(l,m,k) &= 2\binom{l}{m} c_6(m,k) \int\limits_{\mathbb{R}^{m-1}} \cdots \int\limits_{\mathbb{R}^{m-1}} \int\limits_{[u_1,\dots,u_m]} \Delta_{m-1}(u_1,\dots,u_m) \\ &\quad \times \exp\left(-\sum_{i=1}^{m} \|u_i\|^2\right) I_0(l+k+1, m+k+1, k+1; \|u\|) \qquad (4.3) \\ &\quad \times \lambda_{m-1}(du)\lambda_{m-1}(du_1)\dots\lambda_{m-1}(du_m), \end{aligned}$$

where the functional I_a, for $a \geq 0$, was introduced in Section 2. Define $q(l,m,k)$ by the right-hand side of (4.3), but with I_0 replaced by J_0. Then Lemma 2.2 implies that

$$\mathbb{P}(A) = q(l,m,k) + O\left(l^{-(k+1)}(\log l)^{\frac{m+k-2}{2}}\right).$$

By substituting the definition of $A(m,k)$ and applying Lemma 2.1, we can complete the proof in the case $m \geq 2$. The case $m = 1$ follows easily by a direct argument specializing the preceding one. ☐

4.1. Again the centrally symmetric case

This subsection is devoted to the study of the asymptotic behaviour of the probabilities

$$\mathbb{P}\left(Y \notin [Y_1,\dots,Y_{n-k-1}]_c\right)$$

arising in Theorem 3.3. More generally, we obtain the following result by a similar reasoning as for Theorem 4.1.

Theorem 4.2. *Let $l, m \in \mathbb{N}$ and $k > -1$. Let $Y, Y_1, \dots, Y_l$ be independent random points in $\mathbb{R}^m$ with $Y \overset{d}{=} N\left(0, \frac{1}{2(k+1)} I_m\right)$ and $Y_i \overset{d}{=} N\left(0, \frac{1}{2} I_m\right)$. Then*

$$\mathbb{P}\left(Y \notin [Y_1,\dots,Y_l]_c\right) \sim 2^{-(k+1)} C(m,k) A(m,k)\, l^{-(k+1)} (\log l)^{\frac{m+k-1}{2}}$$

as $l \to \infty$.

In particular, by combining Theorems 4.1 and 4.2 we deduce the following asymptotic relation for which no direct proof seems to be known.

Corollary 4.3. *Let $l, m \in \mathbb{N}$ and $k > -1$. Let $Y, Y_1, \dots, Y_l$ be independent random points in $\mathbb{R}^m$ with $Y \overset{d}{=} N\left(0, \frac{1}{2(k+1)} I_m\right)$ and $Y_i \overset{d}{=} N\left(0, \frac{1}{2} I_m\right)$. Then*

$$\mathbb{P}\left(Y \notin [Y_1,\dots,Y_l]_c\right) \sim 2^{-(k+1)} \mathbb{P}\left(Y \notin [Y_1,\dots,Y_l]\right)$$

as $l \to \infty$.

Proof of Theorem 4.2. Assume that $l \geq m$. For $y, y_1, \ldots, y_l \in \mathbb{R}^m$, we define

$$g_c(y, y_1, \ldots, y_l) := \mathbf{1}\{y \notin [y_1, \ldots, y_l]_c\}.$$

Abbreviating $A_c := \{Y \notin [Y_1, \ldots, Y_l]^c\}$, we get

$$\begin{aligned}\mathbb{P}(A_c) &= (k+1)^{\frac{m}{2}} \pi^{-\frac{m}{2}(l+1)} \int_{\mathbb{R}^m} \cdots \int_{\mathbb{R}^m} g_c(y, y_1, \ldots, y_l) \\ &\quad \times \exp\left(-\sum_{i=1}^{l} \|y_i\|^2 - (k+1)\|y\|^2\right) \lambda_m(dy)\lambda_m(dy_1)\ldots\lambda_m(dy_l).\end{aligned}$$

Since $0 \in \operatorname{int}([Y_1, \ldots, Y_l]_c)$ holds $\mathbb{P}$-almost surely, we can decompose $\mathbb{R}^m \setminus [Y_1, \ldots, Y_l]_c$ as in the proof of Theorem 4.1. Recall that $h_1(y, y_1, \ldots, y_m) = \mathbf{1}\{y \in \operatorname{pos}_1([y_1, \ldots, y_m])\}$ and define

$$h_2(y_1, \ldots, y_l) := \mathbf{1}\left\{[y_1, \ldots, y_m] \in \mathcal{F}_{m-1}\left([y_1, \ldots, y_l]_c\right)\right\},$$

for $y, y_1, \ldots, y_l \in \mathbb{R}^m$. By symmetry and by the reflection invariance of the normal distribution, we get

$$\begin{aligned}\mathbb{P}(A_c) &= \sum_{r=0}^{m} \binom{l}{r}\binom{l-r}{m-r}(k+1)^{\frac{m}{2}} \pi^{-\frac{m}{2}(l+1)} \int_{\mathbb{R}^m} \cdots \int_{\mathbb{R}^m} h_1(y, y_1, \ldots, y_m) h_2(y_1, \ldots, y_l) \\ &\quad \times \exp\left(-\sum_{i=1}^{l} \|y_i\|^2 - (k+1)\|y\|^2\right) \lambda_m(dy)\lambda_m(dy_1)\ldots\lambda_m(dy_l) \\ &= 2^m \binom{l}{m}(k+1)^{\frac{m}{2}} \pi^{-\frac{m}{2}(m+1)} \int_{\mathbb{R}^m} \cdots \int_{\mathbb{R}^m} \left(2\phi(\operatorname{dist}(\operatorname{aff}\{y_1, \ldots, y_m\}, 0)) - 1\right)^{l-m} \\ &\quad \times h_1(y, y_1, \ldots, y_m) \exp\left(-\sum_{i=1}^{m} \|y_i\|^2 - (k+1)\|y\|^2\right) \\ &\quad \times \lambda_m(dy)\lambda_m(dy_1)\ldots\lambda_m(dy_m).\end{aligned}$$

Here we used that $[Y_1, \ldots, Y_m]$ is a facet of $[Y_1, \ldots, Y_l]_c$ if and only if the $l-m$ random points $Y_{m+1}, \ldots, Y_l$ lie between the hyperplane $\operatorname{aff}\{Y_1, \ldots, Y_m\}$ and its reflection in the origin, $\mathbb{P}$-almost surely.

By the same arguments as in the proof of Theorem 4.1, we now obtain that

$$\begin{aligned}\mathbb{P}(A_c) &= 2^{m+1} \binom{l}{m} \frac{c_6(m,k)}{2(k+1)} A(m,k) \int_{1}^{\infty} (2\phi(z) - 1)^{l-m} z^{-1} \exp\left(-(m+k+1)z^2\right) dz \\ &\quad + O\left(l^{-(k+1)} (\log l)^{\frac{m+k-2}{2}}\right).\end{aligned}$$

An application of the second part of Lemma 2.1 then yields the result. $\square$

A combination of Theorems 3.3 and 4.2 and relation (4.1) show that

$$\mathbb{E} f_k\left([X_1, \ldots, X_n]_c\right) \sim \frac{C(d-k,k)}{(k+1)!} A(d-k,k)(\log n)^{\frac{d-1}{2}} \sim \mathbb{E} f_k\left([X_1, \ldots, X_n]\right),$$

where $X_1, \ldots, X_n$ is a Gaussian sample.

4.2. The general functional

We now turn to the asymptotic expansion of the integral

$$\int \mathbf{1}\{Y \notin [Y_1, \dots, Y_{n-k-1}]\} \|Y\|^a \, d\mathbb{P},$$

which is related to the expected value $\mathbb{E}T_{a,b}^{d,k}([X_1, \dots, X_n])$ as shown in Theorem 3.4. Again we consider a more general situation.

Theorem 4.4. *Let $l, m \in \mathbb{N}$ and $k > -1$. Let $Y, Y_1, \dots, Y_l$ be independent random points in $\mathbb{R}^m$ with $Y \stackrel{d}{=} N\left(0, \frac{1}{2(k+1)} I_m\right)$ and $Y_i \stackrel{d}{=} N\left(0, \frac{1}{2} I_m\right)$. Then*

$$\int \mathbf{1}\{Y \notin [Y_1, \dots, Y_l]\} \|Y\|^a \, d\mathbb{P} \sim C(m,k) A(m,k)\, l^{-(k+1)} (\log l)^{\frac{m+k+a-1}{2}}$$

as $l \to \infty$.

Proof. We may assume that $l \geq m+1$. Following the proof of Theorem 4.1, we deduce that

$$\begin{aligned} E_a(l,m,k) &:= \int \mathbf{1}\{Y \notin [Y_1, \dots, Y_l]\} \|Y\|^a \, d\mathbb{P} \\ &= 2\binom{l}{m} c_6(m,k) \int_1^\infty \int_{\mathbb{R}^{m-1}} \cdots \int_{\mathbb{R}^{m-1}} \int_{\mathbb{R}^m} \phi(z)^{l-m} h_1(y, u_1 + ze_m, \dots, u_m + ze_m) \\ &\quad \times \Delta_{m-1}(u_1, \dots, u_m) \|y\|^a \exp\left(-\sum_{i=1}^m \|u_i\|^2 - mz^2 - (k+1)\|y\|^2\right) \\ &\quad \times \lambda_m(dy) \lambda_{m-1}(du_1) \dots \lambda_{m-1}(du_m) \lambda_1(dz) + O\left(\phi(1)^l\right). \end{aligned}$$

By the transformation formula,

$$\begin{aligned} &\int_{\mathbb{R}^m} h_1(y, u_1 + ze_m, \dots, u_m + ze_m) \|y\|^a \exp\left(-(k+1)\|y\|^2\right) \lambda_m(dy) \\ &= \int_1^\infty \int_{[u_1,\dots,u_m]} z s^{m+a-1} \left(\|u\|^2 + z^2\right)^{a/2} \exp\left(-(k+1)s^2(\|u\|^2 + z^2)\right) \lambda_{m-1}(du) \lambda_1(ds). \end{aligned}$$

Thus, by an application of Lemma 2.2 we finally get

$$\begin{aligned} E_a(l,m,k) &= \binom{l}{m} \frac{c_6(m,k)}{k+1} A(m,k) \int_1^\infty \phi(z)^{l-m} z^{a-1} \exp\left(-(k+1+m)z^2\right) dz \\ &\quad + O\left(l^{-(k+1)} (\log l)^{\frac{m+k+a-2}{2}}\right), \end{aligned}$$

from which the assertion follows by another application of Lemma 2.1. □

For $k \in \{0, \dots, d-1\}$, we can combine Theorems 3.4 and 4.4 with (4.2) to find the asymptotic expansion for the expected value of the general functional

$$\mathbb{E}T^{d,k}_{a,b}([X_1, \dots, X_n]) \sim C(b, k, d) \binom{d}{k+1} \frac{2^d}{\sqrt{d}} \beta(T^k, T^{d-1}) \pi^{\frac{d-1}{2}} (\log n)^{\frac{d+a-1}{2}}, \tag{4.4}$$

where $C(b, k, d)$ was defined in Theorem 3.4. The special case $a = 0$, $b = 1$ has already been mentioned in the Introduction; the constant $c_{(k,d)}$ in (1.5) now follows from (4.4). Moreover, we get

$$\mathbb{E}\lambda_d([X_1, \dots, X_n]) \sim \kappa_d (\log n)^{\frac{d}{2}}$$

(here Wendel's theorem is used again) and

$$\mathbb{E}V_{d-1}([X_1, \dots, X_n]) \sim \omega_d (\log n)^{\frac{d-1}{2}},$$

where $\kappa_d = \omega_d/d$ is the volume of the d-dimensional unit ball. These two special relations had previously been established in [1].

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