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Trefoil Knots With Tritangent Planes

Dedicated to the memory of Bernulf Weißbach

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Abstract. As we know, there are examples of trefoil knots without tritangent planes. Here we show that trefoil knots of the familiar shape always have such planes. We relate this result to the search for the minimum number of vertices.

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1. Introduction

Freedman [3] conjectured that any regular closed knotted curve in 3-space has a tritangent plane, i.e. a plane which is tangent at three points. But Montesinos Amilibia and Nuño Ballesteros [7] gave an example of a trefoil knot without tritangent planes, and Morton [8] found a similar trefoil knot where this property is easy to show. Both examples are $(3,2)$-torus curves, so at first sight they do not look like trefoil knots. In this paper we will show that trefoil knots of the familiar shape always have at least two supporting tritangent planes.

If a wire model of such a knot touches a plane at two points, then one may turn it until it rests on a third point. But it seems difficult to make this idea precise. Instead we approximate the knot C by a polygon and apply Sperner's Lemma to the boundary of its convex hull, more precisely to its upper and lower parts so that we get two suitable triangles in the boundary of the convex hull. If we approximate C, their planes converge to planes supporting C at 3 points.

In Section 3 we relate this result to the search for minimum numbers of vertices and twistings.

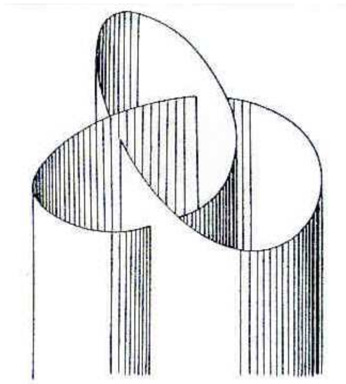
2. Existence of tritangent planes

Definition. *We call a trefoil knot C a special trefoil knot if it is a continuous injective image of S^1 in $\mathbb{R}^3$ with the following properties:*

(a) *there is a projection Π into a plane such that ΠC has rotation number 2 (i.e. ΠC moves around some point in the plane twice);*

(b) *the restriction of Π_C to C is injective except for exactly three double points;*

(c) *ΠC is locally convex, i.e. each point of C has a relative neighborhood which is supported by a straight line.*



A special trefoil knot and the projection as shown by Artin [1]

Instead of (c), we will work in the proof of Theorem 1 with the more technical condition (c*) below which holds for special trefoil knots as can easily be seen. We think of the projection plane as horizontal. The lower pre-images $L_i, i = 1, 2, 3$, of the double points divide C into closed subarcs C_i.

(c*) $$\Pi C_i \cap \partial \operatorname{conv} \Pi C \neq \emptyset$$

(Likewise we can use the upper pre-images.)

Besides Sperner's Lemma we will use a simple Lemma which we need not prove here.

Lemma. *Assume that K_j, $j = 1, 2, \ldots$, are uniformly bounded convex sets in 3-space with $K_j \subset K_{j+1}$. If E_j is a supporting plane of K_j and if the E_j converge to a plane E, then E supports the closure of $\cup_j K_j$.*

Sperner's Lemma in 3-space**.** (see e.g. Henle [6] p. 38) *Let a triangle with vertices P_1, P_2, P_3 and a triangulation be given and let the vertices be labelled in the following manner: P_i gets label i, a vertex on the edge P_iP_j gets label i or j, and the interior vertices are labelled arbitrarily. Then there is at least one subtriangle with labels 1, 2, 3.*

Theorem 1. *A special trefoil knot C has at least two planes which support C at three or more points. (Of course they are tangent planes if C is a regular C^1-curve.)*

Proof. By (c*) we may choose a point P_i on each of the arcs C_i such that

$$\Pi P_i \in \partial \operatorname{conv} \Pi C \qquad (i = 1, 2, 3).$$

We approximate C by a sequence of polygons $Q_j\,(j = 1, 2, \ldots)$ starting with the triangle P_1, P_2, P_3 and adding more and more points of C such that the maximum edge length tends to zero. Because of their choice the P_i are vertices of conv Q_j for all j. If necessary the facets of the polytope conv Q_j are subdivided into triangles. We now consider the upper part of ∂ conv Q_j. It consists of the facets with normals pointing upwards, the projection plane again being regarded as horizontal. This part may be considered as a triangulated triangle with vertices P_1, P_2, P_3. We get a Sperner labelling in the following manner: all vertices of Q_j which lie on C_i get label i, especially P_i gets label i. (If $C_1 \cap C_2$ is a vertex it gets label 1 or 2. There are no vertices of ∂ conv Q_j which do not lie on one of the C_i.) Sperner's Lemma gives us a triangle labelled 1, 2, 3. Since it is part of ∂ conv Q_j, the plane containing Δ_j supports Q_j in at least 3 points, one on each of the C_i, $i = 1, 2, 3$. Since $C_1 \times C_2 \times C_3$ is compact, there is a subsequence such that the vertices of the Δ_j converge to three points. We have to show that no two of them coincide. By the Lemma the planes of the Δ_j converge to a supporting plane of conv C. Since the plane of Δ_j supports the upper part of ∂ conv Q_j, the limit plane supports the upper part of ∂ conv C or its relative boundary. So it does not meet one of the points L_1, L_2, L_3, which lie on the lower side or in the interior of conv C. Thus, the limit plane supports C at three distinct points, each lying in the relative interior of the arcs C_i. In the same way we obtain a plane supporting C from below at three distinct points. □

Remark. As noted above we used the weaker condition (c*) instead of (c). We could further relax (b). But this seems not to be appropriate if we have in mind the questions described in Section 3.

3. Vertices of trefoil knots

In this section *vertex* of a C^3-space curve means a point where the torsion changes sign. Sometimes it is also called *flattening point,* in German *Henkelpunkt*. Barner [2], p. 209, describes a method how to find vertices: Let a plane tangent at two points roll along the curve until both points coincide. This method, ascribed by him to Blaschke, is also the essence of Sedykh's proof of the four-vertex theorem for closed curves lying on the boundary of their convex hull, see [10].

The attempt to use this method together with Theorem 1 in order to find vertices of special trefoil knots was not successful, at least not without further assumptions about the position of contact points with a tritangent plane. Nevertheless we express the following

Conjecture. *A special trefoil knot in the sense of Section 2 with nonvanishing curvature has at least four (or even six) vertices.*

Weiner [11] gave an example of an unknotted closed curve looking similar to a special trefoil knot. It has constant torsion and therefore no vertices. This may be a hint that even the existence of two vertices may be difficult to show for special trefoil knots.

Consider the unit tangent vectors of a space curve C as a spherical curve $\hat{C}$. The geodesic curvature of $\hat{C}$ is τ/κ where τ is the torsion and $\kappa(\neq 0)$ the curvature of C. The vertices of $\hat{C}$, the extrema of τ/κ, are the points where the Darboux vector of C changes its sense of rotation. Therefore they were called *Darboux-vertices* of C in [5]. In [9] they were called *twistings*. Theorem 1 implies that $\hat{C}$ crosses some great circle at least six times. With the method used in [4] we can prove the following theorem.

Theorem 2. *A special trefoil knot with nonvanishing curvature has at least four twistings.*

It is not known if the number four can be attained. Since the proof is rather lengthy and also since the conjecture above would imply Theorem 2 immediately, we will not present the proof here.

Problems. Find the minimum numbers of vertices and twistings for special trefoil knots.

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