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Ball Versus Distance Convexity of Metric Spaces

Thomas Foertsch

Institute for Mathematics, University of Zurich
Winterthurerstrasse 190, 8057 Zurich, Switzerland
e-mail: foertsch@math.unizh.ch

Abstract. We consider two different notions of convexity of metric spaces, namely (strict/uniform) ball convexity and (strict/uniform) distance convexity. Our main theorem states that (strict/uniform) distance convexity is preserved under a fairly general product construction, whereas we provide an example which shows that the same does not hold for (strict/uniform) ball convexity, not even when considering the Euclidean product.

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1. Introduction

In this paper we consider geodesic metric spaces. Recall that a metric space (X,d) is said to be geodesic if any two points $x,y \in X$ can be joined by a geodesic path, i.e. to any two points $x,y \in X$ there exists an isometric embedding γ of an interval $[a,b]$ of the Euclidean line into (X,d) such that $\gamma(a)=x$ and $\gamma(b)=y$. In presence of completeness this is equivalent to the existence of a midpoint map for (X,d), i.e. a map $m : X \times X \longrightarrow X$ satisfying

$$d\Big(m(x,y),x\Big) = \frac{1}{2}d(x,y) = d\Big(m(x,y),x\Big) \quad \forall x,y \in X.$$

In fact we are going to state and prove our results in a slightly more general context, by merely assuming the existence of a midpoint map for (X,d).

The goal of this paper is to compare two different notions of convexity of metric spaces, which we will refer to as *ball and distance convexity*:

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Definition 1. *Let (X,d) be a metric space admitting a midpoint map. (X,d) is called*

(i) *ball convex if*

$$d\Big(m(x,y),z\Big) \leq \max\Big\{d(x,z),d(y,z)\Big\} \quad \forall x,y,z \in X \tag{1}$$

for any midpoint map m of (X,d). It is called strictly ball convex if the inequality in (1) is strict whenever $x \neq y$.

(ii) *distance convex if*

$$d\Big(m(x,y),z\Big) \leq \frac{1}{2}\Big[d(x,z)+d(y,z)\Big] \quad \forall x,y,z \in X \tag{2}$$

for any midpoint map m of (X,d). It is called strictly distance convex if the inequality (2) is strict for all $x,y,z \in X$ satisfying $d(x,y) > |d(x,z)-d(y,z)|$.

Note that condition (2) implies condition (1) and already condition (1) implies the uniqueness of a midpoint map (geodesics in the complete case).

Obviously condition (1) implies that all metric balls are convex. Indeed, as we will see in Lemma 2 it is easy to prove that condition (1) is equivalent to the convexity of arbitrary metric balls in (X,d), which justifies the terminology of ball convexity.

Although this equivalence might not surprise the reader at all, it seems to show the essential difference between conditions (1) and (2): While condition (1) holds for all $x,y,z \in X$ if and only if it holds for such $x,y,z \in X$ satisfying $d(x,z)=d(y,z)$, condition (2) might not hold for all $x,y,z \in X$ when it holds for all $x,y,z \in X$ satisfying $d(x,z)=d(y,z)$!

For geodesic metric spaces condition (2) can be phrased as follows: A geodesic metric space is distance convex if and only if for all $p \in X$ the distance function $d_p := d(p,\cdot)$ is convex, where, as per usual, the convexity of d_p means that the restriction of d_p to every geodesic segment is a convex function.

Note that the restriction of d_p to a geodesic segment containing p never is strictly convex. It is for this reason that in the notion of strict distance convexity one only considers points $x,y,z \in X$ satisfying $d(x,y) > |d(x,z)-d(y,z)|$. Thus one can say that a geodesic metric space is strictly distance convex if and only if all its distance functions d_p, $p \in X$, are as strictly convex as possible.

In Section 2 we prove the equivalence of the two definitions for globally non positively Busemann curved metric spaces:

Proposition 1. *A global non positively Busemann curved geodesic metric space is (strictly/uniformly) ball convex if and only if it is (strictly/uniformly) distance convex.*

For the definitions of uniform distance/ball convexity see Definition 2.

We also give a (well-known) counterexample (see Example 1) for the general equivalence which goes back to Busemann and Phadke [4]. This counterexample, a two-dimensional space endowed with a suitable Hilbert metric, which is strictly ball convex, is shown to be not even distance convex.

In Section 3 we further provide an example of a ball convex Banach space, which is neither strictly ball convex nor distance convex. On the other hand we show that for Banach spaces distance convexity already implies strict distance convexity (Proposition 4).

In Section 4 we give a necessary and sufficient condition for a complete Riemannian manifold to be (strictly) distance/ball convex and show that in this setting all these properties are mutually equivalent (see Theorem 2).

Note that one can also define the notions of uniform ball and uniform distance convexity in a similar way (compare Definition 2) and once again uniform ball convexity is weaker than uniform distance convexity (compare Proposition 3). It is surely worthwhile to point out that this weaker condition is sufficient to prove e.g. the unique existence of circumcenters of bounded subsets or the unique existence of projections onto closed convex sets.

However, the situation changes when turning ones interest to products of convex, strictly convex or uniformly convex spaces: In Section 6 we provide an example of a Euclidean product which is not ball convex although its factors are. In contrast to this we show that (strict/uniform) distance convexity is preserved by a fairly general class of metric products. In order to state the corresponding theorem let (X_i, d_i), $i = 1, \dots, n$, be metric spaces and denote the product set by $X = \Pi_{i=1}^n X_i$. It is natural to define a metric product d on X of the form $d = d_\Phi$,

$$d_\Phi\Big((x_1, \dots, x_n), (y_1, \dots, y_n)\Big) = \Phi\Big(d_1(x_1, y_1), \dots, d_n(x_n, y_n)\Big),$$

where $\Phi : Q^n \longrightarrow [0, \infty)$ is a function defined on the quadrant $Q^n = [0, \infty)^n$. For a detailed analysis of this type of products see [1].

The function Φ has to satisfy certain natural conditions ((A) and (B) in Lemma 3) in order that d_Φ is a metric. These conditions still allow strange metrics on the product (even the trivial product when $n = 1$). In particular, Φ does not have to be continuous.

However, once we require for example that $\Phi : Q^n \longrightarrow [0, \infty)$ yields a metric space (X, d_Φ) admitting a midpoint map for all possible choices of metric spaces (X_i, d_i) admitting midpoint maps, the conditions on Φ become very rigid. In fact, those conditions imply that Φ now has to be continuous.

In order to state the corresponding theorem we consider the function

$$\Psi : \mathbb{R}^n \longrightarrow [0, \infty), \qquad \Psi\Big(\sum_{i=1}^n x_i\, e_i\Big) \;:=\; \Phi\Big(\sum_{i=1}^n |x_i|\, e_i\Big)$$

and say that Φ is induced by a norm if and only if the function Ψ is a norm.

With this terminology we obtain the

Proposition 2. *Let $\Phi : Q^k \longrightarrow \mathbb{R}^+$ be a function. Then $(X = \prod_{i=1}^k X_i, d_\Phi)$ admits the product midpoint map $m = (m_1, \dots, m_k)$ for all choices of metric spaces (X_i, d_i) admitting midpoint maps m_i, if and only if Φ is induced by a norm.*

In the light of this proposition it is natural to ask, which features such a product space inherits from its factors. The following theorem answers this for (strict/uniform) distance convexity:

Theorem 1. *Let (X_i, d_i), $i = 1, \dots, k$, be (strictly/uniformly) distance convex metric spaces and $\Phi : Q^k \longrightarrow \mathbb{R}^+$ be induced by a (strictly/uniformly) convex norm. Then (X, d_Φ) is also strictly (uniformly) distance convex.*

Note that strict and uniform convexity of Φ are equivalent, since distance balls in a finite dimensional normed space are compact.

2. Relations of ball and distance convexity

Let us first formally justify the terminology of ball convexity.

Lemma 1. *For geodesic metric spaces the condition* (1) *for (strictly) ball convexity holds for all $x, y, z \in X$ if and only if it holds for all $x, y, z \in X$ satisfying $d(x, z) = d(y, z)$.*

Proof. Here we consider ball convex metric spaces. The proof for strictly ball convex metric spaces goes along the same lines.
All we need to show is the "if part" of the lemma: Suppose that there exist $x, y, z \in X$ such that $d(m, z) > d(x, z) > d(y, z)$, where m denotes the midpoint of x and y on a geodesic segment $\overline{xy}$ connecting x to y. Let $\overline{xm}$ and $\overline{ym}$ denote the geodesic segments connecting x to m and y to m the union of which is $\overline{xy}$. Let further $u \in \overline{ym}$ and $v \in \overline{xm}$ be those points on $\overline{ym}$ and $\overline{xm}$ which are the closest to m satisfying $d(u, z) = d(x, z) = d(v, z)$. Such points exist due to the continuity of $d_z := d(z, \cdot) : X \longrightarrow \mathbb{R}_0^+$ along geodesics. Then for the midpoint $\tilde{m}$ of u and v on $\overline{uv} \subset \overline{xy}$ we obviously get $d(\tilde{m}, z) > d(x, z)$; a contradiction. $\square$

The following lemma, which is an easy consequence of Lemma 1, yields the desired justification:

Lemma 2. *A geodesic metric space is ball convex if and only if all its distance balls are convex.*

Here convexity of the distance balls means that, given two points in a distance ball, all geodesic segments connecting these points are itself contained in the distance ball. In the case of Banach spaces, this is in general not equivalent to the convexity of distance balls in the sense that with two points a distance ball also contains their connecting straight line!

We now give definitions of uniform ball and uniform distance convexity of metric spaces:

Definition 2. *Let (X, d) be a metric space admitting a midpoint map. (X, d) is called*

(i) *uniformly ball convex if for all $\epsilon > 0$ there exists a $\rho_b(\epsilon) > 0$ such that for all $x, y, z \in X$ satisfying $d(x, y) > \epsilon \max\{d(x, z), d(y, z)\}$ it holds*

$$d\Big(m(x, y), z\Big) \leq \Big(1 - \rho_b(\epsilon)\Big) \max\Big\{d(x, z), d(y, z)\Big\} \tag{3}$$

for the (unique) midpoint map m.

(ii) *uniformly distance convex if for all $\epsilon > 0$ there exists a $\rho_d(\epsilon) > 0$ such that for all $x, y, z \in X$ satisfying $d(x,y) > |d(x,z) - d(y,z)| + \epsilon[d(x,z) + d(y,z)]$ it holds*

$$d\Big(m(x,y), z\Big) \le \Big(1 - \rho_d(\epsilon)\Big)\frac{1}{2}\Big[d(x,z) + d(y,z)\Big] \tag{4}$$

for the (unique) midpoint map m.

The notion of uniform convexity plays an important role in the theory of Banach spaces. As we will see in Proposition 1, when restricted to Banach spaces, both the notions above coincide with the usual definition of uniform convexity. This already implies that, in general, uniform ball (distance, resp.) convexity is strictly stronger than strict ball (distance, resp.) convexity (see e.g. [6]). On the other hand it is clear that in compact situations uniform ball (distance, resp.) and strict ball (distance, resp.) convexity coincide.

It is worth mentioning already at this point that Clarkson introduced uniform convexity in [6], where he already proved a certain general product theorem for uniformly convex Banach spaces. As we will see in Section 6 such a product theorem can only be generalized to metric spaces when considering uniformly distance convex metric spaces but not when focusing on uniformly ball convex ones.

Obviously (strict) distance convexity implies (strict) ball convexity of metric spaces, which immediately follows from their definitions, and it is not surprising that the same statement holds for uniform distance/ball convexity. However, since in the definition of uniform distance convexity less points are required to satisfy a stronger condition than in the definition of uniform ball convexity, we need to convince ourself that those points, which are not considered in the definition of uniform distance convexity, already satisfy the weaker condition of uniform ball convexity. By doing that we prove the

Proposition 3. *Every (strictly/uniformly) distance convex metric space is (strictly/uniformly) ball convex.*

Proof. Let (X,d) be a uniformly distance convex metric space with some $\rho_d(\epsilon)$ as in equation (4). We show that (X,d) is uniformly ball convex with

$$\rho_b(\epsilon) \ := \min\{\frac{\epsilon}{12}, \rho_d(\frac{\epsilon}{6})\} \tag{5}$$

as in equation (3). Let therefore $x, y, z \in X$ such that $d(x,z) \ge d(y,z)$ and

$$d(x,y) \ < \ d(x,z) \ - \ d(y,z) \ + \ \frac{\epsilon}{6}d(x,z).$$

Then we find

$$\begin{aligned} d(m(x,y),z) \ &\le \ \frac{1}{2}d(x,y) \ + \ d(y,z) \\ &< \ \frac{1}{2}\Big(d(x,z) + d(y,z)\Big) + \frac{\epsilon}{12}d(x,z) \\ &= \ d(x,z) - \frac{1}{2}\Big(d(x,z) - d(y,z)\Big) + \frac{\epsilon}{12}d(x,z). \end{aligned}$$

a) Assume now $d(x,z)-d(y,z)\geq \frac{1}{3}\epsilon[d(x,z)+d(y,z)]$. Then it follows

$$\begin{aligned} d(m(x,y),z) &< d(x,z) \;-\; \frac{\epsilon}{12}[d(x,z)+d(y,z)] \\ &\leq (1-\frac{\epsilon}{12})\,[d(x,z)+d(y,z)]. \end{aligned}$$

b) For $d(x,z)-d(y,z)<\frac{1}{3}\epsilon[d(x,z)+d(y,z)]$ we find

$$\begin{aligned} d(x,y) &< \frac{1}{2}\epsilon[d(x,z)+d(y,z)] \\ &\leq \epsilon \max\{d(x,z),d(y,z)\}. \end{aligned}$$

Thus it indeed follows that (X,d) is uniformly ball convex with $\rho_b(\epsilon)$ as in equation (5). $\square$

Let us now emphasize that the converse of Proposition 3 is not true, i.e., there are indeed metric spaces which are (strictly) ball convex but fail to be (strictly) distance convex. The following example is due to Busemann and Phadke ([4]):

Example 1. Consider an affine plane A^2 and points a_1,a_2,c,d on a straight line such that a_1 and a_2 lie between c and d. With $a_i=(1-\tau_i)c+\tau_i d$, $0<\tau_i<1$, $i=1,2$, the cross ratio $R(a_1,a_2,d,c)$ of a_1,a_2,d and c is defined via

$$R(a_1,a_2,d,c) \;:=\; \frac{1-\tau_1}{1-\tau_2}\frac{\tau_2}{\tau_1} \;>\; 1, \quad \text{if } \tau_2>\tau_1.$$

Now let $D=D_C$ be the interior of a closed convex curve C in A^2. Given any two points $a_1,a_2\in D$ they define a straight line $L(a_1,a_2)$ which intersects C in two points c and d. The so called Hilbert metric d_H on D is now defined via

$$d_H(a_1,a_2) \;:=\; \left\{ \begin{array}{cl} |logR(a_1,a_2,d,c)| & \text{for} \quad a_1\neq a_2 \\ 0 & \text{for} \quad a_1=a_2 \end{array} \right\}.$$

This indeed defines a topological metric on D. It can be shown that intersections of straight lines in A^2 with D are geodesics in (D,d_H), thus (D,d_H) is a geodesic space. Furthermore (D,d_H) is uniquely geodesic if and only if C does not contain two non-colinear segments.

In such a two-dimensional Hilbert geometry the ellipse E with foci p and q and eccentricity $\epsilon<1$ is defined as

$$E \;=\; \{x\in D \mid d_H(p,x)+d_H(q,x) \;=\; \frac{1}{\epsilon}d_H(p,q)\}.$$

Now the distance convexity of (D,d_H) implies that all ellipses in (D,d_H) are convex in the sense that with each two points $x,y\in E$ the straight line segment $LS_{x,y}$ connecting x to y lies in the interior of E.

Taking, for instance, a square Q as convex curve $C\subset A^2$, Busemann and Phadke showed that the corresponding Hilbert geometry (D_Q,d_H) admits non convex ellipses. Approximating Q through a sequence $\{C_n\}_{n\in\mathbb{N}}$ of curves C_n in A^2 that do not contain two non-collinear segments, for some $n\in\mathbb{N}$ the space (D_{C_n},d_H) is a (strictly) convex metric space (18.8 in [3]) admitting itself a non convex ellipse. Hence (D_{C_n},d_H) is not distance convex.

Thus in general (strict/uniform) ball convexity is not equivalent to (strict/uniform) distance convexity, but in two special cases it is (see Proposition 1 in the Introduction as well as Theorem 2 in Section 4). Before we prove Proposition 1, let us recall that a geodesic metric space (X,d) is said to be globally non positively Busemann curved (NPBC) if for any three points $x,y,z\in X$ and midpoints $m(x,z)$ and $m(y,z)$ it holds

$$d\Big(m(x,z),m(y,z)\Big)\ \le\ \frac{1}{2}\,d(x,y). \tag{6}$$

Proof of Proposition 1. Once again we restrict our attention to the case of uniform ball/ distance convexity and emphasize that the proofs in the other cases go just along the same lines and even become simpler:

The "if part" follows from Proposition 3. In order to prove the "only if part", suppose that (X,d) is not uniformly distance convex. Then there exist $\epsilon>0$ and sequences $\{x_i\}_{i\in\mathbb{N}}$, $\{y_i\}_{i\in\mathbb{N}}$, $\{z_i\}_{i\in\mathbb{N}}$ of points in X such that for all $i\in\mathbb{N}$:

(a) $d(x_i,y_i)\ge d(x_i,z_i)-d(y_i,z_i)+\epsilon[d(x_i,z_i)+d(y_i,z_i)]$ and

(b) $d(m(x_i,y_i),z_i)\overset{i\to\infty}{\longrightarrow}\frac{1}{2}(d(x_i,z_i)+d(y_i,z_i))$.

Now define $\tilde{x}_i\in X$ via $\tilde{x}_i\in\overline{x_iz_i}$ and $d(\tilde{x}_i,z_i)=d(y_i,z_i)$ for all $i\in\mathbb{N}$. Then we find

$$\begin{aligned}
d(m(\tilde{x}_i,y_i),z_i) &\ \ge\ d(m(x_i,y_i),z_i)-d(m(x_i,y_i),m(\tilde{x}_i,y_i))\\
&\ \ge\ d(m(x_i,y_i),z_i)-\frac{1}{2}\,d(x_i,\tilde{x}_i)\\
&\overset{i\to\infty}{\longrightarrow}\ \frac{1}{2}d(x_i,z_i)+\frac{1}{2}d(y_i,z_i)-\frac{1}{2}d(x_i,\tilde{x}_i)\\
&\ =\ \frac{1}{2}d(\tilde{x}_i,z_i)+\frac{1}{2}d(y_i,z_i)\\
&\ =\ \max\{d(\tilde{x}_i,z_i),d(y_i,z_i)\}
\end{aligned}$$

and

$$\begin{aligned}
d(\tilde{x}_i,y_i) &\ \ge\ d(x_i,y_i)-d(x_i,\tilde{x}_i)\\
&\ \ge\ \epsilon[d(x_i,z_i)+d(y_i,z_i)]\\
&\ \ge\ \frac{\epsilon}{2}\max\{d(\tilde{x}_i,z_i),d(y_i,z_i)\}.
\end{aligned}$$

Thus (X,d) is not uniformly ball convex. □

3. Ball and distance convexity in Banach spaces

From Proposition 1 it follows immediately that for Banach spaces the notions of strict (uniform) ball convexity and strict (uniform) distance convexity coincide:

Corollary 1. *A Banach space $(V,||\cdot||)$ is strictly (uniformly) ball convex if and only if it is strictly (uniformly) distance convex.*

Proof. A Banach space which is strictly or uniformly ball/distance convex is uniquely geodesic. Those unique geodesics are the straight lines. Thus we even have equality in the inequality 6 and therefore the space is globally NPBC. □

Note that a general Banach space is not even ball convex. Here the ball convexity means that not only the particular straight line geodesics connecting two points p, q in a distance ball B must lie within B (which is always true), but all possible geodesic segments connecting p to q must have this feature. An obvious example of a not even ball convex Banach space is $\mathbb{R}^2$ endowed with the maximum metric $d_m : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}_0^+$, $d_m((x,x'),(y,y')) := \max\{|x-y|, |x'-y'|\}$ for all $(x,x'),(y,y') \in \mathbb{R}^2$.

At this point it seems worthwhile to mention the possibility of defining the notions of m-ball and m-distance convexity with respect to particular midpoint maps m. By this we mean that the inequality (1) ((2) resp.) is only assumed to hold for a certain midpoint map m rather than for all possible midpoint maps. In Banach spaces $(V, ||\cdot||)$, for example, we may consider the midpoint map associated with the linear structure of V, i.e. the map

$$m : V \times V \longrightarrow V, \qquad (x,y) \longmapsto \frac{x+y}{2}.$$

It is obvious that, for this midpoint map m, $(V, ||\cdot||)$ is m-distance/ball convex and that $(V, ||\cdot||)$ is strictly m-distance/ball convex if and only if the norm is strictly convex in the usual sense. In the following we therefore refer to this particular (strict) m-distance/ball convexity of a Banach space just as (strict) convexity.

The proof of Corollary 3 made essential use of the fact that strict ball convexity implies the uniqueness of geodesics, which, for Banach spaces, implies the NPBC condition to hold. In fact the analogue of Corollary 3 for merely ball/distance convexity does not hold. In order to see this we provide an example of a ball convex Banach space which is not distance convex:

Example 2. Consider $\mathbb{R}^2$ with a norm $||\cdot||$ determined by its unit sphere which is given through a regular hexagon with the origin as center (compare Figure 1).

This example already shows that, in the case of Banach spaces, ball convexity does not imply strict ball convexity.

We want to point out that for straight G-spaces Busemann proved that ball/distance convexity implies strict ball/distance convexity (see [3] and [4]). Here we do not want to provide the definition of (straight) G-spaces, which the reader finds for instance in [3], but it is worth mentioning that this class of spaces contains the complete Riemannian manifolds (compare Section 4). In the light of the next proposition it also seems important to note that the class of straight G-spaces does not contain general Banach spaces, as here geodesic segments can split into different extensions. However, the uniqueness of a geodesic extension played an important role when proving that distance/ball convexity of straight G-spaces implies their strict distance/ball convexity.
Assuming now that a given Banach space is distance convex, we a priori do not know that its geodesics cannot split. Thus we have to use different arguments in order to prove the

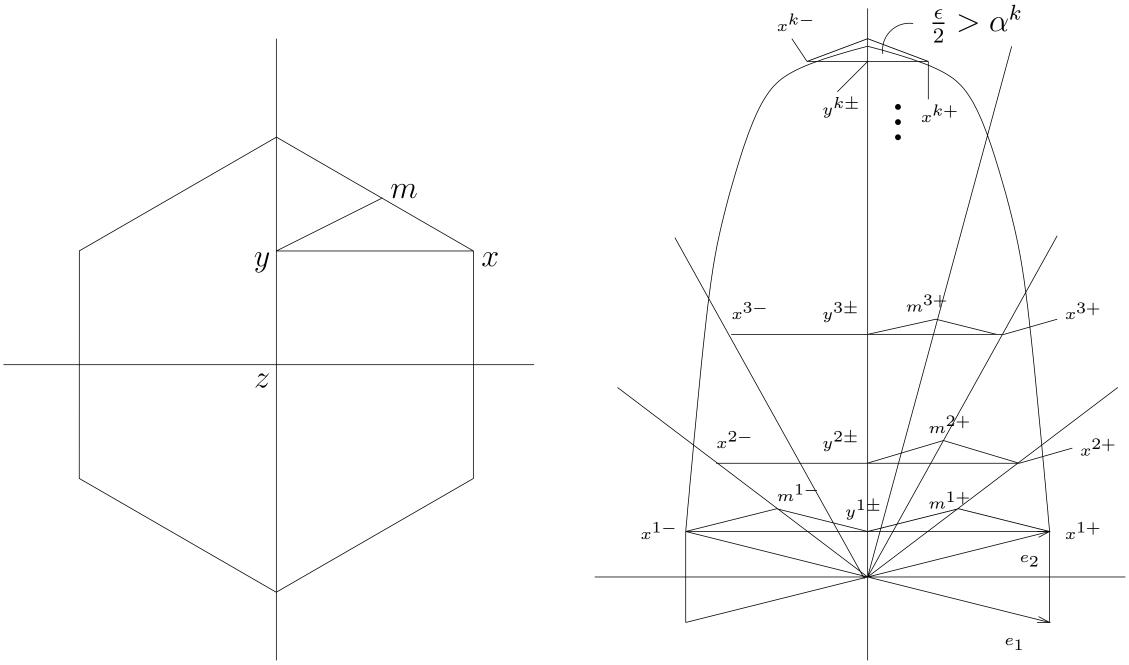

Figure 1. The figure on the left hand side shows the unit sphere of a ball convex but not distance convex Banach space. On the right hand side the construction method of the proof of Proposition 4 is illustrated.

Proposition 4. *If the Banach space* $(V, ||\cdot||)$ *is distance convex, then it already is strictly distance convex.*

Proof. Suppose that $(V, ||\cdot||)$ is not strictly convex. Then there exists a straight line segment $L \subset V$ such that $||l|| = 1$ for all $l \in L$. Denote by $e_1, e_2 \in V$ the vectors pointing to the endpoints of L and consider $V_2 := span\{e_1, e_2\} \subset V$. We show that there exist $u, v \in V_2$ with midpoint m such that

$$||m|| \;>\; \frac{1}{2}\Big(||u|| + ||v||\Big). \tag{7}$$

The idea how to obtain these points is rather simple: Given $x, y \in V_2$ such that $x - y \in span\{e_1 + e_2\}$, we can find a midpoint $m(x, y)$ of x and y such that $m(x, y) - y \in span\{e_2\}$ and $m(x, y) - x \in span\{e_1\}$. We show that if x and y with $||x||, ||y|| \leq 1$ approach $\frac{e_2 - e_1}{||e_2 - e_1||}$ this midpoint eventually lies outside the unit ball; a contradiction to the distance convexity of $(V, ||\cdot||)$ (compare to Figure 1).

In the following we write all vectors in V_2 in coordinates with respect to the basis $\{\frac{e_1+e_2}{2}, \frac{e_2-e_1}{||e_2-e_1||}\}$ of V_2. Let ϵ denote the length of the segment L.

(1) We start with $x^{1+} := (1, \frac{\epsilon}{2})$, $y^{1+} := (0, \frac{\epsilon}{2})$ and the midpoint $m^{1+} = m^{1+}(x^{1+}, y^{1+}) := (\frac{1}{2}, \frac{3\epsilon}{4})$. In order that $(V_2, ||\cdot||)$ is distance convex we must have

$$||m^{1+}|| \leq \frac{1}{2}\Big(||x^{1+}|| + ||y^{1+}||\Big) = \frac{1}{2}(1 + \frac{\epsilon}{2}) = \frac{1}{2} + \frac{\epsilon}{4}.$$

For $\lambda^{1+} \in \mathbb{R}^+$ with $||\lambda^{1+} m^{1+}|| = 1$ we find $\lambda^{1+} \geq \frac{4}{2+\epsilon}$. Hence setting $\lambda_0^{1+} := \frac{4}{2+\epsilon}$ we ensure that $||\lambda_0^{1+} m^{1+}|| \leq 1$.
Furthermore we calculate $\alpha^{1+} := \frac{1-x_2^{1+}}{x_1^{1+}} = 1 - \frac{\epsilon}{2}$.

(2) In the second step we set $x^{2+} := \lambda_0^{1+} m^{1+} = (\frac{2}{2+\epsilon}, \frac{3\epsilon}{2+\epsilon})$, $y^{2+} := (0, \frac{3\epsilon}{2+\epsilon})$ and consider the midpoint $m^{2+} = m^{2+}(x^{2+}, y^{2+}) := (\frac{2}{2+\epsilon}, \frac{\frac{7}{2}\epsilon}{2+\epsilon})$. In order that $(V_2, ||\cdot||_2)$ is distance convex we must have

$$||m^{2+}|| \leq \frac{1}{2}\Big(||x^{2+}|| + ||y^{2+}||\Big) \leq \frac{1+2\epsilon}{2+\epsilon}.$$

For $\lambda^{2+} \in \mathbb{R}^+$ with $||\lambda^{2+} m^{2+}|| = 1$ we find $\lambda^{2+} \geq \frac{2+\epsilon}{1+2\epsilon}$. Hence setting $\lambda_0^{2+} := \frac{2+\epsilon}{1+2\epsilon}$ we ensure that $||\lambda_0^{2+} m^{2+}|| \leq 1$.
Once again we calculate $\alpha^{2+} := \frac{1-x_2^{2+}}{x_1^{2+}} = 1 - \epsilon$.

(3) Repeating this process, it is easy to prove that at the kth step we have

$$\begin{aligned}
x^{k+} &= \frac{1}{1 + \sum_{i=1}^{k} (2^{i-1} - 1)\frac{\epsilon}{2}} \Big(1, \sum_{i=1}^{k} 2^{i-1}\frac{\epsilon}{2}\Big) \\
y^{k+} &= \frac{1}{1 + \sum_{i=1}^{k} (2^{i-1} - 1)\frac{\epsilon}{2}} \Big(0, \sum_{i=1}^{k} 2^{i-1}\frac{\epsilon}{2}\Big) \\
m^{k+} &= \frac{1}{2 + \sum_{i=1}^{k} (2^{i-1} - 1)\epsilon} \Big(1, \sum_{i=1}^{k+1} 2^{i-1}\frac{\epsilon}{2}\Big) \\
\lambda_0^{k+} &= \frac{1 + \sum_{i=1}^{k} (2^{i-1} - 1)\frac{\epsilon}{2}}{\frac{1}{2} + \sum_{i=1}^{k} (2^{i} - 1)\frac{\epsilon}{4}} \\
\alpha^{k+} &= 1 - k\frac{\epsilon}{2}.
\end{aligned}$$

Thus for $k > \frac{2}{\epsilon} - 1$ we find

$$\alpha^{k+} \ = \ 1 - k\frac{\epsilon}{2} \ < \ \frac{\epsilon}{2}.$$

(4) The same construction on "the other side" of $span\{e_2 - e_1\}$ in V_2 yields correspondingly

$$x^{k-} = \frac{1}{1 + \sum_{i=1}^{k} (2^{i-1} - 1)\frac{\epsilon}{2}} \Big(-1, \sum_{i=1}^{k} 2^{i-1}\frac{\epsilon}{2}\Big)$$

$$\begin{aligned}
y^{k-} &= \frac{1}{1+\sum\limits_{i=1}^{k}(2^{i-1}-1)\frac{\epsilon}{2}}\Big(0,\sum_{i=1}^{k}2^{i-1}\frac{\epsilon}{2}\Big)\\
m^{k-} &= \frac{1}{2+\sum\limits_{i=1}^{k}(2^{i-1}-1)\epsilon}\Big(-1,\sum_{i=1}^{k+1}2^{i-1}\frac{\epsilon}{2}\Big)\\
\lambda_0^{k-} &= \frac{1+\sum\limits_{i=1}^{k}(2^{i-1}-1)\frac{\epsilon}{2}}{\frac{1}{2}+\sum\limits_{i=1}^{k}(2^{i}-1)\frac{\epsilon}{4}}\\
\alpha^{k-} &= 1-k\frac{\epsilon}{2},
\end{aligned}$$

where $\alpha^{k-} := \frac{x_2^{k-}-1}{x_1^{k-}}$.

(5) Let $\frac{2}{\epsilon} \notin \mathbb{N}$. Then it is not hard to see that when setting $k_0 := [\frac{2}{\epsilon}]$, i.e. k_0 being the largest integer less or equal to $\frac{2}{\epsilon}$, the points $u := x^{k_0+}$ and $v := x^{k_0-}$ admit a midpoint $m = m(u,v)$ with $||m|| > 1$.

(6) For $\frac{2}{\epsilon} \in \mathbb{N}$ we set $k_0 := \frac{2}{\epsilon} - 1$. Then it is once again not hard to see that $u := x^{k_0+}$, $v := y^{k_0+}$ and their midpoint $m \in span\{y^{k_0+}+e_2\} \cap span\{x^{k_0+}+e_1\}$ or $u := x^{k_0-}$, $v := y^{k_0-}$ and their midpoint $m \in span\{y^{k_0-}+e_1\} \cap span\{x^{k_0-}+e_2\}$ satisfy the inequality (7), or there exists $\mu \in \mathbb{R}^+$ such that $u := \mu x^{k_0+}$, $v := \mu x^{k_0-}$ and their midpoint $m \in span\{\mu x^{k_0+}+e_1\} \cap span\{\mu x^{k_0-}+e_2\}$ satisfy the inequality (7), which completes the proof. $\square$

Note that Proposition 4 becomes essential when proving Theorem 1 in the merely distance convex case.

4. Ball and distance convexity in Riemannian manifolds

A Riemannian metric g on a differentiable manifold M induces the Riemannian distance function $d : M \times M \longrightarrow \mathbb{R}_0^+$ on M. Due to the Hopf Rinow Theorem the metric space (M,d) is geodesic if (M,g) is complete, i.e., all maximal geodesics in (M,g) are defined for all parameter values in $\mathbb{R}$.
Recall that (M,g) is said to have no focal points if every geodesic $\gamma : (-\infty,\infty) \longrightarrow M$ has no focal points as a 1-dimensional submanifold of M.
The goal of this section is, on the one hand, to prove the equivalence of ball and distance convexity in complete Riemannian manifolds and on the other hand to characterize them by giving a necessary and sufficient condition for a complete Riemannian manifold to be (strictly) ball/distance convex.
Before we state and prove the corresponding theorem, we want to emphasize that its content is probably well known to most differential geometers. However, for the sake of completeness we provide a short proof.

Theorem 2. *Let (M, g) be a complete Riemannian manifold with induced Riemannian distance function d. Then the following conditions are mutually equivalent:*

(a) *(M, d) is distance convex,*

(b) *(M, d) is strictly distance convex,*

(c) *(M, d) is ball convex,*

(d) *(M, d) is strictly ball convex,*

(e) *(M, d) is simply connected and without focal points.*

Proof. (1) In the first step we prove that if (M, g) is distance/ball convex, then all geodesic segments $\gamma : [a, b] \longrightarrow M$ in (M, g) must minimize the distance between its endpoints.
This is clear for distance convex Riemannian manifolds, for if a geodesic segment $\gamma : [a, b] \longrightarrow M$ is not minimizing the distance, take the last point $\gamma(t_a)$ along γ such that the length of $\gamma|_{[a,t_m]}$ equals $d(m, \gamma(a))$. Let $\epsilon > 0$ be sufficiently small such that γ is minimizing along $(t_m - \epsilon, t_m + \epsilon)$ and set $z := \gamma(a)$, $x := \gamma(t_m - \frac{\epsilon}{2})$ and $y := \gamma(t_m + \frac{\epsilon}{2})$. Then it is easy to see that

$$2d(m, z) \ > \ d(x, z) \ + \ d(y, z).$$

To make such an argument precise in the case of ball convexity of (M, g), we need somewhat more sophisticated arguments: First of all note that if (M, g) is ball convex, there cannot be different minimizing geodesic segments connecting two points p and q in M. Suppose on the contrary that γ and γ' would be two such geodesics. Take two points x and y on γ and γ', respectively, which lie in a strictly ball convex neighborhood around p, satisfying $d(x, p) = d(y, p)$. Then one finds

$$2d\Big(m(x, y), q\Big) \ > \ d(x, q) \ = \ d(y, q),$$

which contradicts the ball convexity of (M, g). Thus (M, g) has no conjugate points either, since given a point $p \in M$ the set of those points in M where at least two minimal geodesics from p intersect is dense in the cut locus $Cu(p)$ of p (see Lemma 2 in [13]). From this we conclude that arbitrary geodesic segments in (M, g) indeed are minimizing.

(2) In a complete Riemannian manifold, such that arbitrary geodesic segments are minimizing, distance convexity is equivalent to strict distance convexity (compare Section 4 in [4]) and the same is true for ball and strict ball convexity (compare 20.9 in [3]). This proves $(a) \Longleftrightarrow (b)$ and $(c) \Longleftrightarrow (d)$.

(3) Due to (1) every ball/distance convex Riemannian manifold is simply connected, which follows from the existence of closed geodesics in each non trivial homotopy class of a complete Riemannian manifold and the fact that such a closed geodesic is not minimizing. For complete, simply connected Riemannian manifolds without conjugate points, however, $(a) \Longleftrightarrow (c)$ has been proved in [7].

(4) Thus it remains to show $(a) \Longleftrightarrow (e)$: $(e) \Longrightarrow (a)$ has also been proved in [7], while $(a) \Longrightarrow (e)$ for instance follows from Lemma 2.11 in [11]. ☐

5. Relations of (strict) distance and (strict) ball convexity revisited

Let us now quickly summarize the results of the preceding three sections.

General metric spaces: For general metric spaces we have the following relations:

$$\begin{array}{ccc}
\text{strictly distance convex} & \begin{array}{c}\not\Longleftarrow \\ \Longrightarrow\end{array} & \text{distance convex} \\
\Downarrow\not\Uparrow & & \Downarrow\not\Uparrow \\
\text{strictly ball convex} & \begin{array}{c}\not\Longleftarrow \\ \Longrightarrow\end{array} & \text{ball convex}
\end{array}$$

Whereas all the other implications have been proven above, we still have to give an example of a distance convex metric space which is not strictly distance convex. Therefore consider the following piece of the sphere in $\mathbb{R}^3$ given by the restrictions of the spherical coordinates via

$$r = 1, \qquad 0 \leq \varphi \leq frac\pi 2, \qquad 0 \leq \theta \leq \frac{\pi}{2}.$$

Banach spaces: As we have seen in Section 3 for Banach spaces it holds

$$\begin{array}{ccc}
\text{strictly distance convex} & \Longleftrightarrow & \text{distance convex} \\
\Updownarrow & & \Downarrow\not\Uparrow \\
\text{strictly ball convex} & \begin{array}{c}\not\Longleftarrow \\ \Longrightarrow\end{array} & \text{ball convex}
\end{array}$$

Complete Riemannian manifolds: Finally for complete Riemannian manifolds Theorem 2 gives

$$\begin{array}{ccc}
\text{strictly distance convex} & \Longleftrightarrow & \text{distance convex} \\
\Updownarrow & & \Updownarrow \\
\text{strictly ball convex} & \Longleftrightarrow & \text{ball convex}
\end{array}$$

6. A product theorem

We start this section by emphasizing that the Euclidean product of general (strictly) ball convex metric spaces is not necessarily (strictly) ball convex.

Example 3. Here we consider the Euclidean product $(C_n \times C_n, d_e)$ of two copies of the strictly ball convex but not strictly distance convex metric space (C_n, d_H) as described in the Example 1. Since in (C_n, d_H) maximal geodesics are defined on $\mathbb{R}$, we conclude that there exist points $x, y, z \in C_n$ such that

$$2d_H^2(z, m) > d_H^2(z, x) + d_H^2(z, y),$$

where $m = m(x, y)$ denotes a midpoint of x and y in (C_n, d_H) (see Section 4 in [4]).

Let now $X, Y, Z \in C_n \times C_n$ be defined as

$$X := (x, y), \quad Y := (y, x) \quad \text{and} \quad Z := (z, z).$$

Then $M := (m, m)$ is a midpoint of X and Y in $(C_n \times C_n, d_e)$ and we find

$$\begin{aligned} 2d_e(M, Z) &= 2\sqrt{2}\, d_H(m, z) \\ &> 2\sqrt{d_H^2(x, z) + d_H^2(y, z)} \\ &= d_e(X, Z) \;+\; d_e(Y, Z). \end{aligned}$$

Hence $(C_n \times C_n, d_e)$ is not ball convex.

In the light of this example it seems to be an interesting fact that the Euclidean product of finitely many (strictly/uniformly) distance convex metric spaces indeed is (strictly/uniformly) distance convex. In fact this holds for more general metric products than merely the Euclidean one. Before we prove the corresponding Theorem 1, we first properly discuss the metric products to be considered (compare to [1]):

On Q^n we define a partial ordering $\leq$ in the following way: if $q^1 = (q_1^1, \ldots, q_n^1)$ and $q^2 = (q_1^2, \ldots, q_n^2)$ then

$$q^1 \leq q^2 \quad :\Longleftrightarrow \quad q_i^1 \leq q_i^2 \;\; \forall i \in \{1, 2, \ldots, n\}.$$

Let $\Phi : Q^n \longrightarrow [0, \infty)$ be a function and consider the function $d_\Phi : X \times X \longrightarrow [0, \infty)$,

$$d_\Phi\Big((x_1, \ldots, x_n), (y_1, \ldots, y_n)\Big) \;=\; \Phi\Big(d_1(x_1, y_1), \ldots, d_n(x_n, y_n)\Big),$$

where (X_i, d_i) are metric spaces and X denotes the topological product $X = \Pi_{i=1}^n X_i$. In order to obtain a metric space (X, d_Φ) for all possible choices of metric spaces as factors, the function Φ has to fulfill two necessary and sufficient conditions, which are given in the

Lemma 3. (Lemma 1 in [1]) *Let $\Phi : Q^n \longrightarrow [0, \infty)$ be a function. Then d_Φ is a metric on X for all possible choices of metric spaces (X_i, d_i), $i = 1, \ldots, n$, if and only if Φ satisfies*

(A) $\Phi(q) \geq 0 \; \forall q \in Q$ *and* $\Phi(q) = 0 \;\Leftrightarrow\; q = 0$ *and*

(B) *for all points $q^1, q^2, q^3 \in Q^n$ with $q^j \leq q^k + q^l$ we have*

$$\Phi(q^j) \;\leq\; \Phi(q^k) \;+\; \Phi(q^l).$$

Assuming further that (X, d_Φ) admits the product midpoint map $m = (m_1, \ldots, m_k)$ for all possible choices of geodesic factors (X_i, d_i), $i = 1, \ldots, n$, admitting midpoint maps $m_1, \ldots, m_k$, the conditions on Φ become quite restrictive. Now Φ has to be induced by a norm. This is the content of Proposition 2 (see the Introduction) which we are going to prove now:

Proof of Proposition 2: The if part is obvious. For the only if part we restrict on the case where all the factors are the reals endowed with the Euclidean metric:

First of all note that Φ has to be continuous at 0. Suppose it was not, then due to (B) the restriction of Φ to one of the coordinate axis (say the first one) is discontinuous at 0. Thus from $\Phi(2q) \leq 2\Phi(q)$ it follows that there exists $\epsilon > 0$ with $\Phi(\lambda e_1)$ for all $\lambda > 0$ and any two points $(x_1, x_2, \ldots, x_k), (y_1, x_2, \ldots, x_k) \in X$ with $2\epsilon > d_1(x_1, y_1)$ do not admit a midpoint with respect to the product midpoint map m.

Now the continuity of Φ at 0 and (B) imply that Φ is continuous everywhere.
Furthermore we have for all $x, y \in X$:

$$\begin{aligned}
\frac{1}{2}\Phi\Big(d_1(x_1,y_1),\ldots,d_k(x_k,y_k)\Big) &= \frac{1}{2}d_\Phi(x,y) \;=\; d_\Phi\Big(m(x,y),x\Big)\\
&= d_\Phi\Big(d_1(m_1(x_1,y_1),x_1),\ldots,d_1(m_k(x_k,y_k),x_k)\Big)\\
&= \Phi\Big(\frac{1}{2}d_1(x_1,y_1),\ldots,\frac{1}{2}d_k(x_k,y_k)\Big).
\end{aligned}$$

Together with the subadditivity of Φ this implies $\Phi(\lambda q) = \lambda\Phi(q)$ for all dyadic numbers and by continuity for all $\lambda \geq 0$. It now follows that Φ is a convex function on Q^k (see e.g. Theorem 5.4.6 in [12]) and hence its level sets are convex.

Next we want to ensure that the level sets of Ψ also are convex. This easily follows from the convexity of the Φ-level sets, the definition of Ψ and the following observation:
Let $p \in \mathbb{R}^+$ with $p_i = 0$. Then

$$\Phi(p+\lambda e_i) \;\geq\; \Phi(p) \qquad \forall \lambda \geq 0,\;\; i=1,\ldots,k.$$

In order to see that, assume that for some $i \in \{1,\ldots,k\}$ it holds $\Phi(p+\lambda e_i) < \Phi(p)$. Then we find

$$\begin{aligned}
\Phi(2p) &= \Phi(2p_1,\ldots,2p_k) = d_\Phi\Big((0,\ldots,0),(2p_1,\ldots,2p_k)\Big)\\
&\leq d_\Phi\Big((0,\ldots,0),(p_1,\ldots,p_{i-1},\epsilon,\ldots,p_k)\Big)+d_\Phi\Big((p_1,\ldots,p_{i-1},\epsilon,\ldots,p_k),(2p_1,\ldots,2p_k)\Big)\\
&= 2\Phi(p_1,\ldots,p_{i-1},\epsilon,\ldots,p_k) < 2\Phi(p),
\end{aligned}$$

which contradicts the positive homogeniety of Φ. Thus Ψ has convex level sets and, being positively homogeneous, it itself is convex (see e.g. Theorem 5.4.7 in [12]). Thus it follows with Theorem 5.4.6 in [12] that Ψ is subadditive and therefore a norm. $\square$

Proposition 2 ensures that product maps of midpoint maps are once again midpoint maps. However, as we need to control all possible midpoint maps in the product, it is desirable that there are no other midpoint maps in the product than those arising by midpoint maps in the factors. The next lemma says that this indeed is the case whenever Φ is strictly convex.

Lemma 4. *If Φ is induced by a strictly convex norm Ψ, each midpoint $m = m(x,y)$ of points $x, y \in X$ with respect to d_Φ projects to midpoints m_i of $x_i, y_i \in X_i$ with respect to d_i, $i = 1,\ldots,k$.*

Proof. Let $m = (m_1,\ldots,m_k)$ be an arbitrary midpoint of $x = (x_1,\ldots,x_k)$ and $y = (y_1,\ldots,y_k)$ in (X, d_Φ). Then

$$\begin{aligned}
&\Phi\Big(d_1(m_1,x_1)+d_1(m_1,y_1),\ldots,d_k(m_k,x_k)+d_k(m_k,y_k)\Big)\\
\leq\;& \Phi\Big(d_1(m_1,x_1),\ldots,d_k(m_k,x_k)\Big)+\Phi\Big(d_1(m_1,y_1),\ldots,d_k(m_k,y_k)\Big) \qquad (8)\\
=\;& d_\Phi(m,x) \;+\; d_\Phi(m,y)\\
=\;& d_\Phi(x,y)\\
=\;& \Phi\Big(d_1(x_1,y_1),\ldots,d_k(x_k,y_k)\Big).
\end{aligned}$$

Since due to the monotonicity of Φ the opposite inequality also holds, we deduce from the fact that Φ is strictly convex that

$$d_l(x_l, y_l) = d_l(m_l, x_l) + d_l(m_l, y_l) \quad \forall l = 1, \dots, k. \tag{9}$$

From the strict convexity of Φ and the inequality opposite to inequality (8) we find that there exists $\lambda \in \mathbb{R}^+$ such that $d_l(m_l, x_l) = \lambda d_l(m_l, y_l)$ for all $l = 1, \dots, k$, which, together with

$$\frac{1}{2} d_\Phi(x, y) = \Phi\Big(d_1(m_1, x_1), \dots, d_k(m_k, x_k)\Big) = \Phi\Big(d_1(m_1, y_1), \dots, d_k(m_k, y_k)\Big),$$

yields $\lambda = 1$ and thus

$$\frac{1}{2} d_l(x_l, y_l) = d_l(m_l, x_l) = d_l(m_l, y_l) \quad \forall l = 1, \dots, k. \qquad \square$$

We are now ready to give the

Proof of Theorem 1: (A) *the distance convex case*: In this case the statement simply follows from Proposition 4, Lemma 4 and the inequalities in (10).

(B) *the strictly distance convex case*: Due to Lemma 4 a midpoint $m = (m_1, \dots, m_k)$ of $x = (x_1, \dots, x_k)$ and $y = (y_1, \dots, y_k)$ in (X, d_Φ) projects to midpoints m_i of x_i and y_i in (X_i, d_i), $i = 1, \dots, k$. We have

$$\begin{aligned} 2d_\Phi(m, z) &= \Phi\Big(2d_1(m_1, z_1), \dots, 2d_k(m_k, z_k)\Big) \\ &\leq \Phi\Big(d_1(x_1, z_1) + d_1(y_1, z_1), \dots, d_k(x_k, z_k) + d_k(y_k, z_k)\Big) \\ &\leq \Phi\Big(d_1(x_1, z_1), \dots, d_k(x_k, z_k)\Big) + \Big(d_1(y_1, z_1), \dots, d_k(y_k, z_k)\Big) \\ &= d_\Phi(x, z) + d_\Phi(y, z). \end{aligned} \tag{10}$$

Assume that $2d_\Phi(m, z) = d_\Phi(x, z) + d_\Phi(y, z)$ then all the inequalities above become equalities and the strict convexity of Φ together with the strict distance convexity of the (X_i, d_i), $i = 1, \dots, k$, yields

$$d_l(x_l, y_l) \;=\; |d_l(x_l, z_l) - d_l(y_l, z_l)| \qquad \forall l = 1, \dots, k \quad \text{as well as}$$

$$\exists \lambda \in \mathbb{R}_0^+ \text{ such that } d_l(x_l, z_l) = \lambda d_l(y_l, z_l) \quad \forall l = 1, \dots, k.$$

Without loss of generality we can assume that $\lambda \leq 1$ and find

$$d_l(x_l, y_l) \;=\; (1 - \lambda) d_l(x_l, z_l) \quad \forall l = 1, \dots, k,$$

such that

$$\begin{aligned} d_\Phi(x, y) &= \Phi\Big(d_1(x_1, y_1), \dots, d_k(x_k, y_k)\Big) \\ &= (1 - \lambda)\, \Phi\Big(d_1(x_1, z_1), \dots, d_k(x_k, z_k)\Big) \\ &= |d_\Phi(x, z) \;-\; d_\Phi(y, z)|. \end{aligned}$$

Thus (X, d_Φ) indeed turns out to be strictly distance convex.

(C) *the uniformly distance convex case*: For the sake of simplicity we only consider the product (X, d_Φ) of two factors (X_1, d_1) and (X_2, d_2):
Since Φ is induced by a uniformly distance convex norm Ψ, for all $\epsilon > 0$ there exists $\rho_\Phi(\epsilon)$ such that

$$\begin{aligned} & \Psi(a-b) \geq && |\Phi(a)-\Phi(b)| + \epsilon[\Phi(a)+\Phi(b)] \\ \Longrightarrow\quad & \Phi(a+b) \leq && [1-\rho_\Phi(\epsilon)][\Phi(a)+\Phi(b)]. \end{aligned}$$

Note that for all $\epsilon > 0$ there exists $d(\epsilon) > 0$ such that

$$\begin{aligned} &\Phi\Big(t, \tfrac{\epsilon}{3}\Big) < \Phi\Big(0, \tfrac{2}{3}\epsilon\Big) \quad \forall t \in [0, d] \quad \text{and} \\ &\Phi\Big(\tfrac{\epsilon}{3}, t\Big) < \Phi\Big(\tfrac{2}{3}\epsilon, 0\Big) \quad \forall t \in [0, d]. \end{aligned} \tag{11}$$

Since the (X_i, d_i), $i = 1, 2$, are uniformly distance convex, for all $\epsilon > 0$ exists $\rho_i(\epsilon)$ such that

$$\begin{aligned} & d_i(x_i, y_i) && \geq && |d_i(x_i, z_i) - d_i(y_i, z_i)| + \epsilon[d_i(x_i, z_i) + d_i(y_i, z_i)] \\ \Longrightarrow\quad & 2d_i(m_i, z_i) && \leq && [1-\rho_i(\epsilon)][d_i(x_i, z_i) + d_i(y_i, z_i)], \quad i = 1, 2, \end{aligned}$$

where m_i denotes a midpoint of x_i and y_i in (X_i, d_i), $i = 1, 2$.
We write $\rho(\epsilon) := \min\{\rho_1(\epsilon), \rho_2(\epsilon)\}$ and define

$$\tilde{\rho}(\epsilon) \ := \ -\max_{t\in[0, \frac{1}{d(\epsilon)}]} \Big\{ \frac{\Phi(1-\rho(\epsilon), t) - \Phi(1, t)}{\Phi(1, t)}, \frac{\Phi(t, 1-\rho(\epsilon)) - \Phi(t, 1)}{\Phi(t, 1)} \Big\}. \tag{12}$$

We have to show that for all $\epsilon > 0$ there exists $\rho_{d_\Phi}(\epsilon) > 0$ such that

$$\begin{aligned} & d_\Phi(x, y) && \geq && |d_\Phi(x, z) - d_\Phi(y, z)| + \epsilon[d_\Phi(x, z) + d_\Phi(y, z)] \\ \Longrightarrow\quad & 2d_\Phi(m, z) && \leq && [1-\rho_{d_\Phi}][d_\Phi(x, z) + d_\Phi(y, z)] \end{aligned}$$

for all midpoints m of x and y in (X, d_Φ), and we claim that we may set

$$\rho_{d_\Phi}(\epsilon) := \ \min\Big\{\rho_\Phi(\frac{\epsilon}{3}), \tilde{\rho}(\frac{\epsilon}{3}), \rho(\frac{\epsilon}{3})\Big\}.$$

Due to Lemma 4 all midpoints $m = (m_1, m_2)$ of $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in (X, d_Φ) project to midpoints m_i of x_i and y_i in (X_i, d_i), $i = 1, 2$.

(1) With $a := \Big(d_1(x_1, z_1), d_2(x_2, z_2)\Big)$ and $b := \Big(d_1(y_1, z_1), d_2(y_2, z_2)\Big)$ we find

$$\begin{aligned} & \Psi(a-b) && \geq && |\Phi(a)-\Phi(b)| + \frac{\epsilon}{3}[\Phi(a)+\Phi(b)] \\ \Longrightarrow\quad & 2d_\Phi(m, z) && = && \Phi\Big(2d_1(m_1, z_1), 2d_2(m_2, z_2)\Big) \\ & && < && \Phi\Big(d_1(x_1, z_1) + d_1(y_1, z_1), d_2(x_2, z_2) + d_2(y_2, z_2)\Big) \\ & && = && \Phi(a+b) \\ & && \leq && [1-\rho_\Phi(\frac{\epsilon}{3})][\Phi(a)+\Phi(b)] \\ & && = && [1-\rho_\Phi(\frac{\epsilon}{3})][d_\Phi(x, z) + d_\Phi(y, z)]. \end{aligned}$$

Thus we can assume without loss of generality that

$$\Psi(a-b) < |\Phi(a)-\Phi(b)| + \frac{\epsilon}{3}[\Phi(a)+\Phi(b)]. \tag{13}$$

(2) Suppose now that

$$d_l(x_l,y_l) < |d_l(x_l,z_l)-d_l(y_l,z_l)| + \frac{\epsilon}{3}|d_l(x_l,z_l)+d_l(y_l,z_l)|, \quad l=1,2. \tag{14}$$

Then we obtain

$$\begin{aligned} d_\Phi(x,y) &= \Phi\Big(d_1(x_1,y_1), d_2(x_2,y_2)\Big) \\ &< \Psi(a-b) + \frac{\epsilon}{3}[\Phi(a)+\Phi(b)] \\ &< |d_\Phi(x,z)-d_\Phi(y,z)| + \frac{2}{3}\epsilon[d_\Phi(x,z)+d_\Phi(y,z)], \end{aligned}$$

where the last inequality is due to inequality (14).
Thus we can assume without loss of generality that there exists $l \in \{1,2\}$ such that

$$d_l(x_l,y_l) \geq |d_l(x_l,z_l)-d_l(y_l,z_l)| + \frac{\epsilon}{3}|d_l(x_l,z_l)+d_l(y_l,z_l)|. \tag{15}$$

On the other hand, if the inequality (15) holds for $l=1$ and $l=2$, we find

$$2d_\Phi(m,z) = \Phi\Big(2d_1(m_1,z_1), 2d_2(m_2,z_2)\Big) \leq [1-\rho(\frac{\epsilon}{3})][d_\Phi(x,z)+d_\Phi(y,z)].$$

Thus we can assume without loss of generality that inequality (15) holds for $l=1$, while inequality (14) holds for $l=2$.
(3) Setting $u := d_1(x_1,z_1)+d_1(y_1,z_1)$ and $v := d_2(x_2,z_2)+d_2(y_2+z_2)$, we find

$$2d_\Phi(m,z) \leq \Phi\Big([1-\rho(\frac{\epsilon}{3})]u, v\Big). \tag{16}$$

We now consider two different cases; namely (i) $\frac{v}{u} \leq \frac{1}{d(\epsilon)}$ and (ii) $\frac{v}{u} > \frac{1}{d(\epsilon)}$:
(i) From the equation (12) we deduce

$$\frac{\Phi\Big(1-\rho(\frac{\epsilon}{3},\frac{v}{u})\Big) - \Phi\Big(1,\frac{v}{u}\Big)}{\Phi\Big(1,\frac{v}{u}\Big)} \leq -\tilde{\rho}(\frac{\epsilon}{3})$$

and thus

$$\Phi\Big([1-\rho(\frac{\epsilon}{3})]u, v\Big) \leq [1-\tilde{\rho}(\frac{\epsilon}{3})]\Phi(u,v),$$

from which we conclude with inequality (16):

$$2d_\Phi(m,z) \leq [1-\tilde{\rho}(\frac{\epsilon}{3})][d_\Phi(x,z)+d_\Phi(y,z)].$$

(ii) Finally it follows from inequality (14) for $l = 2$, the inequalities (11) as well as inequality (13) that

$$\begin{aligned} d_\Phi(x,y) &= \Phi\Big(d_1(x_1,y_1), d_2(x_2,y_2)\Big) \\ &\leq \Phi\Big(d_1(x_1,z_1) + d_1(y_1,z_1), |d_2(x_2,z_2) - d_2(y_2,z_2)| + \frac{\epsilon}{3}[d_2(x_2,z_2) + d_2(y_2,z_2)]\Big) \\ &\leq \Psi(a-b) + \Phi(u, \frac{\epsilon}{3}v) \\ &< \Psi(a-b) + \frac{2}{3}\epsilon\Phi(0,v) \\ &\leq \Psi(a-b) + \frac{2}{3}\epsilon\Phi(u,v) \\ &< |d_\Phi(x,z) - d_\Phi(y,z)| + \epsilon[d_\Phi(x,z) + d_\Phi(y,z)], \end{aligned}$$

which completes the proof. □

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