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Near Hexagons and Triality

Dedicated to the memory of Bernulf Weißbach

Hans Cuypers **Anja Steinbach**

Department of Mathematics, Eindhoven University of Technology
P.O. BOX 513, 5600 MB Eindhoven, The Netherlands
e-mail: hansc@win.tue.nl

Mathematisches Institut, Justus-Liebig-Universität Gießen
Arndtstraße 2, D-35392 Gießen, Germany
e-mail: Anja.Steinbach@math.uni-giessen.de

Abstract. We give a geometric characterization of generalized hexagons and near hexagons naturally embedded into a polar space of type D_4 and related to triality.

1. Introduction

Among polar spaces those of type D_4 are of special interest as they admit triality automorphisms. In this paper we investigate certain subgeometries of these polar spaces of type D_4 related to triality. Such investigations have already been started by Tits in his paper [9]. In that paper Tits introduces the notion of a generalized polygon and constructs various generalized hexagons inside a D_4 polar space. In this paper we will provide a geometric characterization of these hexagons as well as of some near hexagons as subgeometries embedded in the D_4 polar space.

In the following we consider two classes of partial linear spaces, polar spaces and near hexagons. For the latter, in particular two special types, the generalized hexagons and the dual polar spaces, are of interest. For the basic notation and definitions on these spaces, as well as for embeddings of one partial linear space in another, we refer to Section 2.

Let $\mathcal{D}$ be a non-degenerate polar space of type D_4. So, $\mathcal{D}$ can be identified with the polar space of an 8-dimensional orthogonal space of hyperbolic (+)-type. The triality automorphisms of $\mathcal{D}$ give rise to several classical (near) hexagons embedded into $\mathcal{D}$ and into its

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subspaces of type B_3. For the generalized hexagons in question, we refer again to Tits [9]. Embeddings of dual polar spaces of rank 3 into $\mathcal{D}$ obtained by applying a triality automorphism to a (polar) subspace of rank 3 (over a subfield) will be described in Section 3. We give the following characterization of these embedded near hexagons.

1.1. Theorem. *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ be a non-degenerate orthogonal polar space of rank ≥ 3. Assume $\Pi = (P, L)$ is a thick partial linear space laxly embedded into $\mathcal{S}$ satisfying the following:*

(PL) *Let $p \in P$ and $l \in L$. Then $|p^\perp \cap l| \geq 1$. Moreover, there is a point $q \in l$ collinear (in Π) to p if and only if $p \perp l$.*

(PP) *Suppose $l, m \in L$ are in a singular plane π of $\mathcal{S}$. Then l and m intersect in a point p of P. This point p is collinear to all points of P in π. Moreover, for all other points $q \in P$ we have $|q^\perp \cap \pi \cap P| \geq 2$.*

(Q) *If Π contains a quadrangle, then any pair $p, q \in P$, at distance 2 in the collinearity graph of Π, is in at least two quadrangles.*

(ND) *If $p, q \in \Pi$ are at distance 2 in the collinearity graph of Π, then there is a line l in L on p not contained in $q^\perp$.*

Then Π and its embedding into $\mathcal{S}$ is (up to isomorphism) one of the following:

(i) *A split Cayley generalized hexagon or a mixed generalized hexagon (related to the groups of type G_2 and to the mixed groups of type G_2 in characteristic 3 only, respectively) naturally embedded into a 7-dimensional orthogonal polar space of rank 3;*

(ii) *a triality generalized hexagon (related to the groups of type ${}^3D_4$ or ${}^6D_4$) naturally embedded into an 8-dimensional orthogonal polar space of rank 4, or possibly 3 for type ${}^6D_4$;*

(iii) *a dual polar space arising from a non-degenerate rank 3 polar space $\mathcal{B}$ laxly embedded into a subspace of $\mathcal{S}$ with the extra property that two points of $\mathcal{B}$ are collinear in $\mathcal{B}$ if and only if they are collinear in $\mathcal{S}$. The embedding is into an 8-dimensional orthogonal polar space of rank 4 obtained by applying a triality automorphism to $\mathcal{B}$.*

In Theorem 1.1, the $\perp$-relation always refers to the polar space $\mathcal{S}$. Collinearity, however, refers to Π.

We remark that condition (Q) above avoids thin polar spaces and condition (ND) avoids degenerate examples (as we will see in Section 6). For further details on the embeddings of the generalized hexagons in question, we refer to Section 5.

The motivation for giving the characterization in Theorem 1.1 comes from group theory. Let (V, q) be a non-degenerate orthogonal space of rank at least 2. A *Siegel transformation* in the orthogonal group $O(V, q)$ acting on its natural orthogonal module (V, q) is an element $t \in O(V, q)$ with $[V, t]$ being an isotropic 2-space contained in $C_V(t)$. A *Siegel transformation subgroup* is some subgroup of the orthogonal group generated by all Siegel transformations with the same commutator space. For finite dimensional orthogonal spaces (V, q), all Siegel transformations in the orthogonal group generate the derived subgroup $\Omega(V, q)$. But there are more classes of quasi-simple irreducible subgroups of the orthogonal group generated by Siegel transformation subgroups. As examples we mention subgroups of type $G_2(k)$ inside

$O_7(k)$, and subgroups of type $B_3(k)$ in their spin representation inside $O_8^+(k)$. These exceptional cases are geometrically characterized by Theorem 1.1. Subgroups of orthogonal groups generated by Siegel transformations will be studied in detail in [4].

The next section is used to fix the notation and give the basic definitions used throughout the paper. In Section 3 we describe the embeddings of near hexagons appearing in case (iii) of Theorem 1.1. The remaining sections are devoted to a proof of Theorem 1.1. In Section 4, we show that a partial linear space Π satisfying the hypothesis of Theorem 1.1 is a near hexagon. For this we do not need the assumption that the polar space $\mathcal{S}$ is orthogonal. In the last two sections we consider the case of a generalized hexagon and of a near hexagon with quads, respectively.

2. Notation and definitions

In this section we introduce the notation used throughout this paper.

2.1. Partial linear spaces. A *partial linear space* $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ is a pair consisting of a set $\mathcal{P}$ of *points* together with a set $\mathcal{L}$ of subsets of $\mathcal{P}$, called *lines*, such that every line contains at least two points and any two points are in at most one line. Two points are called *collinear* if there is a line containing them. For each point $x \in \mathcal{P}$, we denote by $x^\perp$ the set of all points, including x, collinear to x. If $X \subseteq \mathcal{P}$, then $X^\perp$ denotes the intersection of $x^\perp$, for all $x \in X$. The space $\mathcal{S}$ is called *thick*, if every point is on at least three lines and every line contains at least three points.

A subset X of $\mathcal{P}$ is called a *subspace*, if every line meeting X in more than one point is contained in X. The lines of $\mathcal{L}$ contained in a subspace X define a partial linear space on X. A subspace is often identified with this induced partial linear space. The intersection of any set of subspaces is again a subspace. This implies that one can consider for any subset Y of $\mathcal{P}$ the subspace *generated* by Y to be the smallest subspace containing Y. The subspace in $\mathcal{S}$ generated by a set Y will be denoted by $\langle Y \rangle_{\mathcal{S}}$ or $\langle Y \rangle$ if it is clear that we consider it as a subspace of $\mathcal{S}$.

2.2. Polar spaces. A partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ is called a *polar space* if it satisfies the 'one or all' or Buekenhout-Shult [1] axiom:

For every point $p \in \mathcal{P}$ and line $l \in \mathcal{L}$, the point p is collinear to one or all points on l.

In the polar space $\mathcal{S}$ we denote collinearity by the perp relation $\perp$. So, for $p, q \in \mathcal{P}$, we write $p \perp q$ if and only if $p = q$ or p and q are on a common line. The polar space is called *non-degenerate* if there is no point p with $p^\perp = \mathcal{P}$. Subspaces of polar spaces in which any two points are collinear, are called *singular* subspaces. Singular subspaces of non-degenerate polar spaces are projective spaces, see [1] or [3]. For a natural number n, a polar space has *rank* n, if its maximal singular subspaces have projective dimension $n-1$.

For any vector space, say V, endowed with a quadratic form of Witt index ≥ 2, we define the associated *orthogonal polar space* as the isotropic points and lines (i.e., 1- and 2-dimensional subspaces, where the form vanishes). When V has dimension n, the orthogonal polar space is called *n-dimensional*. We refer to the projective space $P(V)$ as the *universal embedding space* of the orthogonal polar space.

2.3. D_4-polar spaces and triality. By a polar space, say $\mathcal{D}$, of type D_4 over some field k, we mean an orthogonal polar space arising from an 8-dimensional vector space over k endowed with a quadratic form of Witt index 4 (also called of hyperbolic type). The points of $\mathcal{D}$ are the isotropic points (with respect to the quadratic form).

The set of maximal singular subspaces of $\mathcal{D}$ can be partitioned into two parts $\mathcal{M}_1$ and $\mathcal{M}_2$, such that two elements from different parts intersect in either a point or a singular plane. D_4-polar spaces admit so-called *triality automorphisms*. Under a triality automorphism, say τ, points are mapped to maximal singular subspaces in one part, say $\mathcal{M}_1$, these are mapped to maximal singular subspaces in $\mathcal{M}_2$ and the latter are mapped to points. Furthermore, lines are mapped to lines, incidence is preserved and τ^3 is the identity.

2.4. Near hexagons. A partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ is called a *near polygon* if it satisfies the axiom:

For every point $p \in \mathcal{P}$ and line $l \in \mathcal{L}$, there is a unique point on l closest to p.

Here the distance between two points is the usual graph distance in the collinearity graph of $\mathcal{S}$, i.e., the graph with vertex set P, two vertices being adjacent if and only if they are collinear. A near polygon is called a *near hexagon* if its collinearity graph has diameter 3. A *quadrangle* of $\mathcal{S}$ is a set of four points q_1, q_2, q_3 and q_4, such that q_i and q_{i+1} are collinear, but q_i and q_{i+2} not. Here indices are taken modulo 4. A *quad* of $\mathcal{S}$ is a geodesically closed subspace of diameter 2.

A special type of near hexagon is a *generalized hexagon.* Here every point is on at least two lines, for every point p there is a point q at distance 3 of p, and there are no quadrangles in the collinearity graph of $\mathcal{S}$.

Another special type of near hexagon is a *dual polar space* of rank 3, defined as follows: Let $\mathcal{S}$ be a non-degenerate polar space of rank 3. The associated dual polar space is the partial linear space with as points the planes of $\mathcal{S}$, as lines the lines of $\mathcal{S}$ and induced incidence.

2.5. Embeddings of a partial linear space in another. For partial linear spaces $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ and $\Pi = (P, L)$, we say that Π is *(laxly) embedded* in $\mathcal{S}$, provided that P is a subset of $\mathcal{P}$ generating $\mathcal{S}$ and for each $l \in L$ there exists a (unique) $\hat{l} \in \mathcal{L}$ containing l. We call Π an *embedded subspace* of $\mathcal{S}$.

3. Some embeddings of near hexagons in polar spaces

In this section we describe the embedded near hexagons ocurring in case (iii) of Theorem 1.1.

3.1. Description. Let $\mathcal{D}$ be a polar space of type D_4 over some field k as defined in (2.3). By $\mathcal{B}$ we denote a non-degenerate rank 3 polar space laxly embedded into a subspace of $\mathcal{D}$ with the extra property that two points p and q of $\mathcal{B}$ are collinear in $\mathcal{B}$ if and only if they are collinear in $\mathcal{D}$. Then any singular plane of $\mathcal{B}$ spans a singular plane of $\mathcal{D}$. Indeed, if π is such a singular plane of $\mathcal{B}$, then we can find a point p in $\mathcal{B}$ such that $p^\perp \cap \pi$ is a line. So inside the subspace of $\mathcal{D}$ spanned by π we find at least two points collinear to p and two points not collinear to p. This clearly implies that π is not contained in a line of $\mathcal{D}$.

Hence every maximal singular subspace M of $\mathcal{B}$ is contained in a unique maximal singular subspace in $\mathcal{M}_1$, denoted by $\widehat{M}$. Notice that for two maximal singular subspaces M_1, M_2 of $\mathcal{B}$ intersecting in a line l of $\mathcal{B}$, the intersection of $\widehat{M_1} \cap \widehat{M_2}$ is also a line.

Denote by $\mathcal{B}^*$ the near hexagon on the set $\mathcal{M}_{\mathcal{B}}$ of planes and $\mathcal{L}_{\mathcal{B}}$ of lines of $\mathcal{B}$. Fix a triality automorphism τ of $\mathcal{D}$ mapping the elements of $\mathcal{M}_1$ to points of $\mathcal{D}$. We define $\tau_{\mathcal{B}^*} : \mathcal{M}_{\mathcal{B}} \to \mathcal{P}$ by

$$\tau_{\mathcal{B}^*}(M) = \tau(\widehat{M})$$

for all $M \in \mathcal{M}_{\mathcal{B}}$.

By the above we see that $\tau_{\mathcal{B}^*}$ induces a (lax) embedding of $\mathcal{B}^*$ into the subspace of $\mathcal{D}$ generated by the image of $\tau_{\mathcal{B}^*}$. However, as for each $M \in \mathcal{M}_{\mathcal{B}}$ there is an $M' \in \mathcal{M}_{\mathcal{B}}$ with $M \cap M' = \emptyset$, we see that the image of $\tau_{\mathcal{B}^*}$ is not contained in any degenerate hyperplane of $\mathcal{D}$ (as the disjoint maximal subspaces $\widehat{M}$ and $\widehat{M'}$ are mapped under τ to opposite points). Moreover, the image of a quad of $\mathcal{B}^*$ generates a maximal singular subspace of $\mathcal{D}$. This implies, that the image of $\tau_{\mathcal{B}^*}$ is also not contained in a non-degenerate subspace of $\mathcal{D}$, as such a subspace does not contain maximal singular subspaces of $\mathcal{D}$. As each subspace of rank at least 2 is contained in a hyperplane, this implies that the image of $\tau_{\mathcal{B}^*}$ generates all of $\mathcal{D}$.

For the polar spaces $\mathcal{B}$ occuring in the description above, we have the following classification result:

3.2. Theorem. *Let $\mathcal{D}$ be a polar space of type D_4 with underlying vector space V over k. Suppose that $\mathcal{B}$ is a thick non-degenerate rank 3 polar space with any line contained in at least three planes, which is laxly embedded into a subspace of $\mathcal{D}$ such that two points of $\mathcal{B}$ are collinear in $\mathcal{B}$ if and only if they are collinear in $\mathcal{D}$.*

Then $\mathcal{B}$ is isomorphic to an orthogonal polar space with associated vector space W over a subfield k_0 of k endowed with a quadratic form q of Witt index 3. The isomorphism is induced by an injective linear mapping $\varphi : W \to V$.

The dimension of W over k_0 is at least 7. If the radical, say R, of the symmetric bilinear form associated to q has dimension at most 1 (which happens for instance whenever char $k \neq 2$), then a k_0-basis of W is mapped under φ to a k-basis of $\langle \mathcal{B} \rangle$.

When char $k = 2$, then there is no restriction on the dimension over k_0 of the radical R. But the image under φ of any non-zero R spans a 1-dimensional nonisotropic k-subspace of V (and the dimension over k of $\langle \mathcal{B} \rangle$ is 7 in this case).

Proof. The singular planes of $\mathcal{B}$ are Desarguesian, as this holds for $\mathcal{D}$. Thus [10, (8.22)] (which is part of the classification of polar spaces of rank 3 due to Tits) yields that $\mathcal{B}$ is a classical polar space. By the classification of generalized quadrangles which admit a so-called weak embedding of degree 2, given in Steinbach and Van Maldeghem [7], $\mathcal{B}$ is isomorphic to an orthogonal space, associated to W, k_0 and q as desired. The isomorphism is induced by a linear mapping $\varphi : W \to V$ by Steinbach and Van Maldeghem [6, (5.1.1)]. By Steinbach and Van Maldeghem [7, (5.1)] φ is injective.

The dimension of W over k_0 is at least 7 (as otherwise a line of $\mathcal{B}$ is contained in only two planes). The next assertion in the theorem is proved in Section 5 of Steinbach and Van Maldeghem [7] and the last assertion may be proved with similar arguments. □

Applying the construction described above to any of the polar spaces $\mathcal{B}$ yields a near hexagon laxly embedded into $\mathcal{D}$. We remark that if $\mathcal{B}$ is isomorphic to an orthogonal polar space $O_6^+(k_0)$, then the resulting near hexagon is thin, i.e., lines contain only two points.

Thin polar spaces $\mathcal{B}$ inside $\mathcal{D}$ also yield examples of near hexagons inside $\mathcal{D}$. These, however, fail to satisfy condition (Q) of Theorem 1.1. Indeed, if $\mathcal{B}$ is thin, then a pair of points of the near hexagon at distance 2 is in exactly one quadrangle. Examples of thin polar spaces $\mathcal{B}$ inside $\mathcal{D}$ can be obtained as follows.

Suppose k has characteristic 2. Let $e_1, f_1, e_2, f_2, e_3, f_3, e_4, f_4$ be a hyperbolic basis for V and let X_i, for $i = 1, 2$ or 3, be a subset of size at least 3 of the set of isotropic points in $\langle e_i, f_i, e_4 + f_4\rangle$. Then each point $x_i \in X_i$ is perpendicular to all points of X_j, $j \neq i$, but no point in X_i distinct from x_i. So, $X_1 \cup X_2 \cup X_3$ together with all the pairs of points from distinct sets X_i and X_j is a thin polar space embedded in $\mathcal{D}$.

4. Reduction to near hexagons

Suppose $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ is a non-degenerate polar space and $\Pi = (P, L)$ is a partial linear space laxly embedded in $\mathcal{S}$ satisfying the hypotheses of Theorem 1.1. In this and the next sections $\perp$ refers to the polar space $\mathcal{S}$. Collinearity, however, refers to Π, unless stated otherwise; distance is also measured in Π, it is the usual graph distance on the collinearity graph of Π. Sometimes we use conditions (PL) and (PP) of Theorem 1.1 without reference.

4.1. *Let $p \in P$, then there is some $q \in P$ with $p \not\perp q$.*

Proof. Since P generates $\mathcal{S}$ and $\mathcal{S}$ is non-degenerate, P is not contained in the hyperplane $p^\perp$ of $\mathcal{S}$. □

4.2. *If $l \in L$ is contained in $\hat{l} \in \mathcal{L}$, then $l = \hat{l} \cap P$.*

Proof. Suppose $l \in L$ is contained in $\hat{l} \in \mathcal{L}$. We fix $p \in \hat{l} \cap P$. Then $p \perp l$ and by condition (PL) of Theorem 1.1, there is a point $q \in l$ collinear to p. Let $m \in L$ be the line through p and q. Now fix a point $r \in P \setminus q^\perp$. Then, by condition (PL) of Theorem 1.1, $r^\perp \cap l$ contains a single point in P just as $r^\perp \cap m$. So the unique point t in $\hat{l} \cap r^\perp$ is in $l \cap m$. In particular, l and m intersect in the distinct points q and t. Thus $l = m$ and $p \in l$, as desired. □

4.3. *If $p, q \in P$ are at distance 1 or 2, then $p \perp q$.*

Proof. Let p and q be collinear and suppose l is a line on p not through q. Then $p \in l \subseteq q^\perp$, by condition (PL) of Theorem 1.1.

If p and q are at distance 2, then there are lines $l, m \in L$ on p and q respectively, meeting in a point $r \in P$. But that implies that $p \perp m$ and thus $p \perp q$. □

4.4. *Let $p \in P$ and $l \in L$, $p \notin l$. If $l \subseteq p^\perp$, then there is a unique point $q \in P$ on l collinear with p.*

Proof. By condition (PL) of Theorem 1.1, there exists a point as desired. We prove its uniqueness. Suppose there are distinct lines $m, n \in L$ on $t \in P$ meeting a third line $l \in L$ in points $p, q \in P$, respectively. Notice that $l \perp m \perp n \perp l$, so that the three lines are in a singular plane, say π, of $\mathcal{S}$. Then condition (PP) implies that a point of $\pi \cap P$ not on l is collinear to both p and q, and thus the intersection point of two lines in L. Therefore it is collinear to every point of P inside π. From this we deduce that any two points of $\pi \cap P$ are collinear.

By (4.1) there exists a point r not in $p^\perp$. As we saw in the previous paragraph, there is a line in L on the unique points on m and l, respectively, perpendicular to r. Now r is collinear to a point s on this line (condition (PL) of Theorem 1.1). But then consider the line $k \in L$ through p and s. The point r is collinear with the point s on k, but $k \not\subseteq r^\perp$. This contradicts condition (PL) of Theorem 1.1. $\square$

Notice that (4.4) implies that there are no proper triangles in Π, i.e. there are no three distinct lines in L pairwise meeting at a point, but not all three containing the same point.

4.5. *Let $p \in P$ and $l \in L$, $p \notin l$. If $l \not\subseteq p^\perp$, then the unique point $q \in p^\perp \cap l$ is at distance 2 from p, all other points on l are at distance 3 from p.*

Proof. Let $p \in P$ and $l \in L$, $p \notin l$. Suppose $l \not\subseteq p^\perp$, and let q be the unique point on l perpendicular to p. Then by (4.3) all points on l distinct from q are at distance ≥ 3 from p. Hence it remains to prove that q is at distance 2 from p.

Let m be a second line on q. Then $l \perp m$ and the lines l and m span a singular plane, say π. Moreover, all points of P inside this plane are collinear to q. But $p^\perp \cap \pi$ contains not only q, but also a second point, say q' (condition (PP) of Theorem 1.1). The line $k \in L$ on q and q' is contained inside $p^\perp$. Hence p is collinear with some point r on k, and (p, r, q) is a path of length 2 from p to q. $\square$

Combining (4.4) and (4.5) we obtain the following:

4.6. Proposition. Π *is a near hexagon.*

5. Generalized hexagons

In this section we assume that Π and $\mathcal{S}$ are as in the hypothesis of Theorem 1.1, but that there is no quadrangle in Π.

5.1. Π *is a thick generalized hexagon.*

Proof. Let p be a point of Π. By (4.1) there is a point $q \in P \setminus p^\perp$. By (4.5) this point is (inside Π) at distance 3 from p. This implies that Π is a generalized hexagon. Thickness follows by assumption. $\square$

5.2. *Let p be a point in P. All lines of L on p are inside a singular plane of $\mathcal{S}$.*

Proof. Suppose p is a point of Π such that the lines of L on p are not inside a singular plane of $\mathcal{S}$. Then there are three lines l_1, l_2 and l_3 on p which are not coplanar.

Let q be a point not in $p^\perp$. As Π is a near hexagon, there is a path p, p_1, q_1, q of collinear points from p to q with $p_1 \in l_1$. Now consider $q_1^\perp \cap \pi$, where π is the singular plane spanned by l_2 and l_3. This intersection contains a line l on p. Notice that $l \neq l_1$. Since $q_1 \perp l$, the point q_1 is collinear with a point $r \in l$ distinct from p. So, p, p_1, q_1, r is a quadrangle. Moreover, as there are no triangles, this is a proper quadrangle, which contradicts our assumptions. $\square$

Let the projective space $\mathbb{P}$ be the universal embedding space of $\mathcal{S}$. The (lax) embedding of the generalized hexagon Π in $\mathbb{P}$ has the following properties: The point set P generates $\mathcal{S}$ and whence also $\mathbb{P}$. As singular planes of $\mathcal{S}$ are planes of $\mathbb{P}$, (5.2) implies that the lines on a point of Π are inside a plane of $\mathbb{P}$. Moreover, all points at distance ≤ 2 from a fixed point p of Π are inside $p^{\perp}$, whence in a hyperplane of $\mathbb{P}$.

This means that the embedding of Π in $\mathbb{P}$ is a so-called *regular embedding*. We have thus obtained the following:

5.3. Proposition. *The generalized hexagon* Π *embeds regularly into the universal embedding space of* $\mathcal{S}$. □

From the classification of regular embeddings of generalized hexagons by Steinbach and Van Maldeghem [8] we now can deduce the following result:

5.4. Theorem. *For the generalized hexagon* Π *and its embedding into the non-degenerate orthogonal polar space* $\mathcal{S}$ *one of cases* (i) *or* (ii) *of Theorem* 1.1 *holds.*

Proof. Regular embeddings of generalized hexagons have been classified by Steinbach and Van Maldeghem [8]. We use the notation introduced there. By [8] Π is a so-called Moufang hexagon. For the theory of these, we refer to Tits und Weiss [11]. In [8] is it shown that only the generalized hexagons occurring in cases (i) or (ii) of Theorem 1.1 admit a regular embedding.

For split Cayley hexagons, $\mathbb{P}$ is the natural 7-dimensional orthogonal module or the natural 6-dimensional symplectic module in characteristic 2 by [8, Theorem 6.2] (possibly with scalars extended). Under the additional assumption that $\mathbb{P}$ is the universal embedding space of a non-degenerate orthogonal polar space, the symplectic module does not arise.

For triality hexagons, $\mathbb{P}$ is the natural 8-dimensional orthogonal module (as given in [8, (4.3)]) of Witt index 4 or 3, depending on whether Π has type ${}^3D_4$ or ${}^6D_4$ (possibly with scalars extended). Because of the possible extension of scalars, the rank of $\mathcal{S}$ can be 3 or 4 for Π of type ${}^6D_4$.

For mixed hexagons, the regular embeddings in a projective space occur in possibly unbounded dimension. They are all quotients of some universal embedding, see [8, Theorem 9.7]. From [8, Theorem 9.4] we deduce that the vector space, say V, underlying $\mathbb{P}$ may be endowed with a quadratic form of Witt index 3 such that each point of Π is isotropic with respect to this form. The vector space V is the orthogonal sum of a 7-dimensional non-degenerate orthogonal vector space of Witt index 3 and the radical, say R, of the associated symmetric bilinear form. (Note that the characteristic is 3 as we deal with mixed hexagons). Under the additional assumption that $\mathbb{P}$ is the universal embedding space of a non-degenerate orthogonal polar space, a non-trivial radical R does not arise. Thus only the natural 7-dimensional orthogonal module (possibly with scalars extended) remains. This proves Theorem 5.4. □

We remark that the field underlying the natural 8-dimensional orthogonal module for a triality hexagon is the separable cubic extension field, say J, (of the ground field coordinatizing the long root subgroups) which coordinatizes the short root subgroups. For triality hexagons of type ${}^3D_4$, this cubic extension is a Galois extension and the hexagon is obtained as the set of absolute points and lines of a triality automorphism of a D_4-polar space over J. For triality hexagons of type ${}^6D_4$ however, the cubic extension in question is not Galois and the hexagon is obtained as the set of points and lines absolute for both a triality automorphism of a D_4-polar space over the Galois closure of J, and an automorphism of order 2 of the D_4-polar space (resulting in a ${}^2D_4$-polar space). This indicates why the natural module has only Witt index 3 for the ${}^6D_4$-hexagons.

6. Near hexagons with quads

In this last section we consider the case that Π is a partial linear space embedded into a polar space $\mathcal{S}$ as in the hypothesis of Theorem 1.1 containing quadrangles. By assumption (Q) in Theorem 1.1 we can assume that every pair of non-collinear points p, q in Π with $p \perp q$ is inside a quadrangle.

A *quad* of Π is a geodesically closed subspace of diameter 2. Any such subspace is a generalized quadrangle. Yanushka's Lemma, see [5], implies that every quadrangle of Π is contained in a quad. This quad is uniquely determined by any two non-collinear points contained in it.

6.1. *Let $\mathcal{Q}$ be a quad of Π. Then $\mathcal{Q}$ is the intersection of P with a maximal singular subspace of $\mathcal{S}$.*

Proof. If x, y are two points of $\mathcal{Q}$, then their distance is at most 2. Hence $x \perp y$ by (4.3). So $\mathcal{Q}$ is contained in a singular subspace of $\mathcal{S}$. Now suppose z is a point of P inside a maximal singular subspace M containing $\mathcal{Q}$. We may assume that inside $\mathcal{Q}$ there is a point u not collinear to z. But z is perpendicular to all lines of $\mathcal{Q}$ on u. Hence the point z is collinear to at least two points of $\mathcal{Q}$ on distinct lines through u. These two points are at mutual distance 2. As $\mathcal{Q}$ is geodesically closed, the point z has to be in $\mathcal{Q}$. □

6.2. *Let $\mathcal{Q}$ be a quad of Π. The lines of $\mathcal{Q}$ through a fixed point are contained in a singular plane of $\mathcal{S}$.*

Proof. Let x be a point of $\mathcal{Q}$ and assume that the lines in $\mathcal{Q}$ on x are not inside a singular plane. Let l, m and n be three lines on x inside $\mathcal{Q}$ but not in a singular plane. Suppose $z \in P$ is not perpendicular to x. Then z is at distance 3 from x. There exists a point y collinear to z and to a point $v \in P$ on l.

The point y is perpendicular to x by (4.3). Hence by condition (PP) of Theorem 1.1, there is a second point, say u, inside the plane spanned by m and n, perpendicular to y. The point u is in $\mathcal{Q}$, see (6.1), and collinear to x by condition (PP). Since y is perpendicular to both x and u, there is a point u' on the line through x and u which is collinear to y. Hence y is collinear to both u' and v. As the point u' is inside the plane spanned by n and m, it is not on l and hence certainly distinct from v. Moreover, u' is not collinear to v, as there are no triangles in Π. Since $\mathcal{Q}$ is geodesically closed, we find $y \in \mathcal{Q}$.

The points on m and n collinear to y are the unique points of these lines in $z^\perp$. Now let u be a third point on the line through y and v. Then u is collinear to points w_1 on m and w_2 on n which are not in $z^\perp$. Condition (PP) of Theorem 1.1 implies that there is a point w in the plane generated by u, w_1 and w_2 which is also in $z^\perp$. The point w is collinear to u. So, as $u, w \perp z$, the point z is collinear to some point t on the line through u and w. But that implies that z is collinear to both y and t, contradicting that $\mathcal{Q}$ is geodesically closed. $\square$

6.3. *Suppose $\mathcal{Q}$ is a quad and $p \in P$ a point not in $\mathcal{Q}$. Then p is collinear to a (unique) point of $\mathcal{Q}$.*

Proof. Suppose $\mathcal{Q}$ is a quad and $p \in P$ a point not in $\mathcal{Q}$. Assume that the pair $(p, \mathcal{Q})$ is not classical. Then the points in $\mathcal{O} := p^{\perp} \cap \mathcal{Q}$ form an ovoid in $\mathcal{Q}$ (i.e., $\mathcal{O}$ is a set of points of the generalized quadrangle $\mathcal{Q}$ such that each line of $\mathcal{Q}$ is incident with a unique point of $\mathcal{O}$) and are all at distance 2 from p. Let (p, q, r) be a path of collinear points, with r in $\mathcal{Q}$. Now consider two lines l and m on r inside $\mathcal{Q}$. As $p \perp r$, there is a second point in the singular plane spanned by l and m that is perpendicular to p. As this point is collinear to r, we have found a line inside $p^{\perp} \cap \mathcal{Q}$, contradicting this to be an ovoid. $\square$

By $\mathcal{S}(\Pi)$ we denote the space with as points the quads of Π and as lines the lines in L, a quad being incident with a line if and only if the line is a line of the quad. We will see below that two quads are collinear points of $\mathcal{S}(\Pi)$ precisely when they intersect non-trivially (whence in a line of Π). Thus two disjoint quads are opposite points of $\mathcal{S}(\Pi)$ and conversely.

6.4. Proposition. *The space $\mathcal{S}(\Pi)$ is a non-degenerate rank 3 polar space. Two quads $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are collinear in $\mathcal{S}(\Pi)$ if and only if they intersect nontrivially.*

Proof. This follows by (6.3) and Cameron's characterization of dual polar spaces, see [2]. We just have to check that the conditions A1–A5 of Theorem 1 of [2] are satisfied. Since Π is a near hexagon, A1 and A2 are fulfilled. As all lines in Π have size at least 3, Yanushka's Lemma [5] implies condition A3. From (6.3) we deduce as follows that A4 holds: The only non-trivial case is $d = 2$ and $z \notin \mathcal{Q}$, where $\mathcal{Q}$ is the quad on x and y. By (6.3) there is a point $w \in \mathcal{Q}$ collinear to z. Necessarily, w is collinear to x (and y), as required by A4. Indeed, otherwise x is at distance 2 from both w and z and hence collinear to some point v on the line through z and v. But since $\mathcal{Q}$ is geodesically closed, v is in $\mathcal{Q}$ and thus also z, a contradiction.

So, A5 remains. However, this follows for $d = 3$ from condition (ND) of Theorem 1.1 as we show below. (For $d = 2$, we use that a quad is a generalized quadrangle.) Indeed, if x, y and z are three points in Π with $d(x, y) = 3$, $d(x, z) = 2$ and $d(y, z) = 1$. Then by condition (ND) there is a line l on x not in $z^{\perp}$. The unique point $w \in l \cap y^{\perp}$ is at distance 1 from x, distance 3 from z and distance 2 from y. So w is the point required by A5 of Theorem 1 of [2]. Thus, by applying Theorem 1 of [2], we have obtained that $\mathcal{S}(\Pi)$ is a non-degenerate rank 3 polar space. $\square$

Condition (Q) of Theorem 1.1 implies that the polar space $\mathcal{S}(\Pi)$ is thick, see the observation just before Theorem 2 at page 80 of [2]. Furthermore, any line of $\mathcal{S}(\Pi)$ is in at least three planes, as any line of Π has at least three points.

6.5. *Every quad $\mathcal{Q}$ of Π is contained in a singular subspace of $\mathcal{S}$ which is a projective space of (vector space) dimension 4.*

Proof. The singular subspaces of $\mathcal{S}$ are projective spaces, see [1] or [3]. It is well-known that a generalized quadrangle is generated by a point p, all the points collinear to p and a single point not collinear to p. Thus, by (6.2), we find that the singular subspace generated by a quad has (vector space) dimension at most 4.

Given a quad $\mathcal{Q}$, there exists a point $x \in P$ outside $\mathcal{Q}$. The points in $x^\perp \cap \mathcal{Q}$ are all the points collinear to the unique point y of $\mathcal{Q}$ which is collinear to x. So, the singular plane spanned by the lines in $\mathcal{Q}$ on y is a proper subspace of the singular space spanned by $\mathcal{Q}$. Hence this latter space has dimension 4. □

6.6. *Suppose $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are two disjoint quads of Π. Then the polar space $\mathcal{S}$ can be generated by* 4 *points from $\mathcal{Q}_1$ together with* 4 *points from $\mathcal{Q}_2$. In particular, the universal embedding space of $\mathcal{S}$ has (vector space) dimension at most* 8.

Proof. Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be two disjoint quads from Π. (Notice that such quads exist.) Let M_1 and M_2 be the maximal singular subspaces of $\mathcal{S}$ containing $\mathcal{Q}_1$ and $\mathcal{Q}_2$ respectively. Both M_1 and M_2 can be generated by four points from Π, see (6.5). Now suppose p is a point of Π not in $\mathcal{Q}_1$ or $\mathcal{Q}_2$. By (6.3), the point p is collinear with a point $q_1 \in \mathcal{Q}_1$. The point q_1 is collinear to a point $q_2 \in \mathcal{Q}_2$. Let l be the line through q_1 and q_2 and m the line through q_1 and p. If $l = m$, then p is in the subspace generated by both $\mathcal{Q}_1$ and $\mathcal{Q}_2$. If $l \neq m$, then l and m are in a unique quad $\mathcal{Q}_3$. As $\mathcal{Q}_3$ meets $\mathcal{Q}_1$ nontrivially, they intersect in a line, say n, through q_1. The lines l, m and n are all on q_1 inside the quad $\mathcal{Q}_3$. Hence by (6.2), the three lines l, m and n and thus also the point p are inside a singular plane of $\mathcal{S}$. As $l \neq n$, this plane is spanned by l and n. This implies that p is inside the subspace of $\mathcal{S}$ spanned by M_1 and M_2. As P generates the whole space $\mathcal{S}$, we find $\mathcal{S}$ to be generated by any 4 points generating M_1 together with 4 points generating M_2. This shows that the universal embedding of $\mathcal{S}$ is spanned by at most 8 points. □

6.7. Proposition. *$\mathcal{S}$ is a polar space of type D_4.*

Proof. By assumption $\mathcal{S}$ is an orthogonal polar space. By (6.5) $\mathcal{S}$ has rank at least 4, but as its universal embedding space has dimension at most 8, see (6.6), the rank of $\mathcal{S}$ is at most 4. But then $\mathcal{S}$ is of type D_4. □

The set of maximal singular subspaces of the polar space $\mathcal{S}$ can be partitioned into two parts $\mathcal{M}_1$ and $\mathcal{M}_2$, such that two elements from different parts intersect in either a point or a singular plane of $\mathcal{S}$. Fix a quad $\mathcal{Q}_1$ of Π. Let M_1 be the maximal singular subspace of $\mathcal{S}$ containing $\mathcal{Q}_1$. Without loss of generality, we can assume $M_1 \in \mathcal{M}_1$.

6.8. *Suppose $\mathcal{Q}$ is a quad of Π. Let M be the maximal singular subspace of $\mathcal{S}$ containing $\mathcal{Q}$. Then $M \in \mathcal{M}_1$.*

Proof. Let $\mathcal{Q}$ be a quad of Π. If $\mathcal{Q}$ is a quad disjoint from $\mathcal{Q}_1$, then the maximal singular space M containing $\mathcal{Q}$ generates together with M_1 the whole space $\mathcal{S}$. Hence $M \cap M_1$ has to be empty and $M \in \mathcal{M}_1$.

Since the graph with vertex set, the set of all quads of Π, two quads being adjacent if and only if their intersection is empty, is connected (as this is the perp relation in $\mathcal{S}(\Pi)$), we find that $M \in \mathcal{M}_1$ also if $\mathcal{Q}$ and $\mathcal{Q}_1$ have a non-empty intersection. □

6.9. Theorem. *There exists a rank* 3 *polar space* $\mathcal{B}$ *laxly embedded into a subspace of* $\mathcal{S}$ *with two points of* $\mathcal{B}$ *collinear in* $\mathcal{B}$ *if and only if they are collinear in* $\mathcal{S}$*, such that* Π *can be obtained by applying a triality automorphism to* $\mathcal{B}$ *as described in Section* 3.

Proof. Let τ be a triality automorphism of $\mathcal{S}$ mapping the members of $\mathcal{M}_1$ to points of $\mathcal{S}$. Then τ induces a map from the set of quads of Π into the point set of $\mathcal{S}$ mapping a quad $\mathcal{Q}$ to the point $\tau(\langle\mathcal{Q}\rangle)$. As τ maps lines to lines, it induces an embedding of the polar space $\mathcal{S}(\Pi)$ into a subspace of $\mathcal{S}$. The induced image of $\mathcal{S}(\Pi)$ under τ is denoted by $\mathcal{B}$. Notice that all the quads on a point p of Π are mapped into a singular plane of $\mathcal{B}$. The inverse map of τ applied to $\mathcal{B}$ induces now the embedding of Π into $\mathcal{S}$. □

The possible polar spaces $\mathcal{B}$ have been classified in Theorem 3.2. The main result Theorem 1.1 follows from Theorem 5.4 and Theorem 6.9.

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