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Two Optimization Problems for Convex Bodies in the n-dimensional Space

Dedicated to the memory of Bernulf Weißbach

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Abstract. If K is a convex body in the Euclidean space E^n, we consider the six classic geometric functionals associated with K: its n-dimensional volume V, $(n-1)$-dimensional surface area F, diameter d, minimal width ω, circumradius R and inradius r. We prove that the n-spherical symmetric slices are the convex bodies that maximize both, the volume and the surface area, when another two geometric magnitudes are fixed, specifically, for given values of the pairs of magnitudes (ω, d) and (ω, R).
Besides, it is proved that the sets of constant width maximize the minimal width when the circumradius and the inradius are prescribed.
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1. Introduction

Let K be a convex body (i.e., a compact convex set) in the n-dimensional Euclidean space E^n. Associated with K there are $n+1$ well-known functionals: the intrinsic volumes $V_i(K)$, $i = 0, \dots, n$, which include volume, surface area, or mean width. Let us denote by F the $(n-1)$-dimensional surface area of K and by V its n-dimensional volume. Let us also denote by ω_n the $(n-1)$-dimensional surface area of the unit ball B^n, and by κ_n its n-dimensional volume. Their values are

$$\omega_n = n\kappa_n, \qquad \text{and} \qquad \kappa_n = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)},$$

where Γ represents the Euler Gamma function.

But some other functionals are also well-known and interesting, namely, the diameter of the convex body $d = d(K)$, its minimal width $\omega = \omega(K)$ (i.e., the minimum distance between two parallel support hyperplanes of K), its circumradius $R = R(K)$ and its inradius $r = r(K)$.

For many years mathematicians have been interested in inequalities involving some of these functionals; and moreover, in finding the convex bodies for which the equality sign is attained. The study of intrinsic volumes led immediately to the discovery of inequalities amongst them. Good references for the study of these functionals are [2], [7] or [10].

In this paper we are going to study inequalities involving the functionals V (the area in the planar case), F (corresponding to the perimeter in the plane), d, ω, R and r.

If two of the above quantities are considered, there are still important questions which remain open: the problems of minimizing the volume or the surface area for given values of the minimal width seem to be hard.

Of course, inequalities involving more than two of the above quantities are naturally more difficult to obtain, and one of the most studied problems is to find the convex bodies that optimize a particular functional when other two ones are fixed.

Besides, the question becomes more interesting when the equality, for a particular inequality, is not attained for a single figure, but for a continuous family of sets. In this case, that inequality is called an *optimal* inequality; optimal inequalities say which is the maximum or minimum value of the first quantity for *each pair* of possible values of the other two considered functionals (maximum or minimum which can be attained for, of course, more than one figure).

There are almost no inequalities of this type for general convex bodies in arbitrary dimension. Most of the known inequalities have, as extremal sets, a single set. For instance, in [5], the following inequality has been obtained:

$$F^{n-1} \geq n^{n-2}\kappa_{n-1} d V^{n-2}, \tag{1}$$

which is the best possible in the sense that there is a sequence of double cones with increasing diameters such that $F^{n-1}\left(n^{n-2}\kappa_{n-1} d V^{n-2}\right)^{-1}$ tends to 1. Besides, a family of Bonnesen-style inequalities, which provide sharp bounds for the inradius and circumradius with respect to the unit ball, include

$$V - rS + (n-1)r^n\kappa_n \leq 0 \tag{2}$$

and

$$(n-1)V - 2RS + (n+1)R^n\kappa_n \geq 0, \tag{3}$$

which are due to Bokowski and Diskant (inequality (2)) and Bokowski and Heil (inequality (3)); the extremal set for both inequalities is the Euclidean ball. References for these results can be found in [7] or [10].

Recently, in [6], it has been proved the following assertion: the sets that minimize the volume for given values of the $(n-1)$-dimensional surface area and the inradius are (amongst other) the cap-bodies, i.e., the convex hull of the ball $B^n(r)$ and a finite number of points such that the segment joining any two of them intersects the ball, see Figure 1 (a); this statement is equivalent to the optimal inequality

$$nV \geq rF. \tag{4}$$

In this paper we obtain the solutions for the following problems:

– Maximizing the volume and the surface area for given values of the pair (ω, d).

– Maximizing the volume and the surface area for given values of the pair (ω, R).

We prove the optimal inequalities for each of the above problems, determining also its corresponding extremal sets.

Besides, in [9], the following result is proved:

Theorem. (Scott, 1981) *Let K be a convex body in the Euclidean space E^n having diameter d and circumradius R. Then, there exists a convex body $\bar{K}$ of constant width d which contains K, and which also has circumradius R.*

Then, since for any convex body of constant width in E^n the relation $d = R + r$ holds, the above theorem assures that the inequality

$$d \geq R + r \tag{5}$$

is true for an arbitrary convex body K of the n-dimensional Euclidean space, with equality when K has constant width d.

This inequality was also deduced by Santaló in [8] for the planar case, using a well-known theorem (see [1, p. 138]) which states that every convex body with diameter d is a subset of a set of constant width d.

Here, following similar arguments to those used by Santaló in the planar case (see [8]), we solve the "dual" problem of maximizing the minimal width for given values of the pair (R, r); this is, we state the inequality

$$\omega \leq R + r.$$

2. Maximizing the volume and the $(n-1)$-dimensional surface area

In this section, we obtain the new inequalities that state which are the sets with maximum volume and $(n-1)$-dimensional surface area for each pair of possible values of another two functionals; more precisely, we study the cases (ω, d) and (ω, R).

Theorem 1. *Let K be a convex body in the Euclidean space E^n, with diameter d and minimal width ω. Then its n-dimensional volume V and $(n-1)$-dimensional surface area F verify*

$$V \leq \frac{\kappa_{n-1}}{2^{n-1}} d^n \int_0^{\arcsin\frac{\omega}{d}} \cos^n\theta\, d\theta \tag{6}$$

and

$$F \leq \frac{\kappa_{n-1}}{2^{n-2}} \left((d^2-\omega^2)^{\frac{n-1}{2}} + (n-1)d^{n-1} \int_0^{\arcsin\frac{\omega}{d}} \cos^{n-2}\theta\, d\theta \right). \tag{7}$$

In both inequalities, equality holds when and only when K is an n-spherical symmetric slice with diameter d and minimal width ω, i.e., the part of the n-ball with radius $d/2$, $B^n(d/2)$, bounded by two parallel hyperplanes equidistant from the origin, at distance apart ω.

Corollary 1. *Let K be a convex body in the Euclidean space E^n, with minimal width ω and circumradius R. Then its n-dimensional volume V and $(n-1)$-dimensional surface area F verify*

$$V \leq 2\kappa_{n-1} R^n \int_0^{\arcsin\frac{\omega}{2R}} \cos^n\theta\, d\theta \tag{8}$$

and

$$F \leq 2\kappa_{n-1} \left(\frac{(4R^2-\omega^2)^{\frac{n-1}{2}}}{2^{n-1}} + (n-1)R^{n-1} \int_0^{\arcsin\frac{\omega}{2R}} \cos^{n-2}\theta\, d\theta \right). \tag{9}$$

The equality holds, in both inequalities, when and only when K is the n-spherical symmetric slice with minimal width ω and circumradius R.

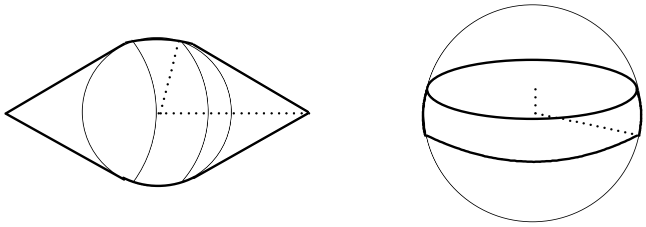
Figure 1 (b) shows the extremal sets for the above inequalities in the 3-dimensional Euclidean space.

Remark 1. In the particular case of the 3-dimensional Euclidean space, the corresponding inequalities are

$$V \leq \frac{\pi}{12}\omega(3d^2-\omega^2), \qquad F \leq \frac{\pi}{2}(d^2-\omega^2+2\omega d),$$

$$V \leq \frac{\pi}{12}\omega(12R^2-\omega^2), \qquad F \leq \frac{\pi}{2}(4R^2-\omega^2+4\omega R),$$

with extremal sets shown in Figure 1 (b).



(a) Cap-body (b) Spherical Symmetric Slice

Figure 1. Extremal sets for: (a) inequality (4) (b) Theorem 1 and Corollary 1

2.1. The case (ω, d)

In order to prove Theorem 1, let $K \subset E^n$ be an arbitrary convex body and $K^c = \frac{1}{2}(K - K)$ its central symmetrization. Then, it is known (see [1]) that K^c is a centrally symmetric convex body with

$$d(K^c) = d(K), \quad \omega(K^c) = \omega(K), \quad F(K^c) \geq F(K) \quad \text{and} \quad V(K^c) \geq V(K).$$

Hence, it suffices to consider the family of all convex bodies in E^n which present symmetry about the origin.

Thus, let $K \subset E^n$ be a centrally symmetric convex body with diameter d and minimal width ω. The properties of this kind of sets allow us to assure that the circumball of K is the n-ball $B^n(d/2)$, as well as to take the inball $B^n(\omega/2)$ so that both $B^n(d/2)$ and $B^n(\omega/2)$ have the same center: the origin of coordinates.

Besides, the inball touches the body K in two diametrically opposite points, and hence, there are two parallel hyperplanes H and H' which support both K and $B^n(\omega/2)$. Let $s(H, H')$ be the slab between H and H', and let

$$K^s = s(H, H') \cap B^n(d/2).$$

Clearly, $R(K^s) = R(K)$ and $r(K^s) = r(K)$; since $K \subset K^s$, we have $V(K) \leq V(K^s)$ and also $F(K) \leq F(K^s)$; in both cases equality holds if and only if $K = K^s$ is the n-spherical symmetric slice. In [6], the formulae for the volume and the $(n-1)$-dimensional surface area of the n-spherical symmetric slice K^s with fixed circumradius R and inradius r have been obtained. Since K^s is symmetric about the origin, $r = \omega/2$ and $R = d/2$, and then,

$$V(K^s) = \frac{\kappa_{n-1}}{2^{n-1}} d^n \int_0^{\arcsin \frac{\omega}{d}} \cos^n \theta \, d\theta \tag{10}$$

and

$$F(K^s) = \frac{\kappa_{n-1}}{2^{n-2}} \left((d^2 - \omega^2)^{\frac{n-1}{2}} + (n-1) d^{n-1} \int_0^{\arcsin \frac{\omega}{d}} \cos^{n-2} \theta \, d\theta \right). \tag{11}$$

Hence, inequalities (6) and (7) have been proved. □

2.2. The case $(\omega,\ R)$

From (10) and (11) it follows that the right-hand sides in, respectively, inequalities (6) and (7), are increasing functions in d for each fixed value of ω (since these functions represent the volume and the surface area of an n-spherical symmetric slice, which increase with the diameter). Hence, using the relation $d \leq 2R$ which holds for every convex body, the result here is a direct consequence of Theorem 1. □

3. Maximizing the minimal width

In this section we obtain the "dual" inequality to the relation (5), i.e., we prove the following theorem.

Theorem 2. *Let $K \subset E^n$ be a convex body with minimal width ω, circumradius R and inradius r. Then,*

$$\omega \leq R + r, \tag{12}$$

and equality holds when K is a constant width set.

Using similar arguments, this theorem generalizes, to arbitrary dimension, the analogous result for the planar case which was obtained by Santaló in [8].

Proof. Let $B^n(R)$ be the circumball of K, and let us denote by r^* the maximum

$$r^* = \max\bigl\{\rho \geq 0 : \ B^n(\rho) \subset K, \quad B^n(\rho) \text{ and } B^n(R) \text{ are concentric}\bigr\}.$$

As a limit case, r^* may vanish if the circumcenter lies on the boundary of the set K; hence, it holds $0 \leq r^* \leq r$.

Besides, it is clear that the ball $B^n(r^*)$ touches the boundary of K in, at least, one point; let P denote it. Then, there exists a common support hyperplane H to both K and the ball $B^n(r^*)$ through the point P.

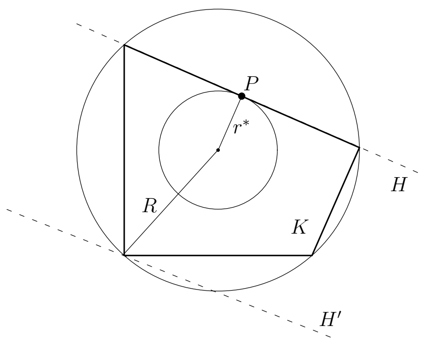


Figure 2. Proof of inequality (12)

Let H' be the support hyperplane to K which is parallel to H (see Figure 2). Clearly, H' either intersects or is tangent to the circumball $B^n(R)$, because in the opposite case, the circumradius of the set should be strictly greater than R.

Then, the width in the direction determined by the perpendicular vector to the hyperplane H, $\omega(K, H^{\perp})$, is, at most, $R + r^*$, and therefore,

$$\omega \leq \omega(K, H^{\perp}) \leq R + r^*.$$

Since $r^* \leq r$, we can conclude that

$$\omega \leq R + r.$$

Finally, a well-known property of the sets of constant width (see, for instance, [1, p. 135]) assures that the circumball and the inball (which is unique) of such a set are concentric, as well as the sum of their radii equals its minimal width, $\omega = R + r$. Some other good references for these kind of properties can be [3] or [4].

So, this family of sets verifies the equality in (12), which concludes the proof of the theorem. □

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