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Exactly Solvable and Unsolvable Shortest Network Problems in 3D-space*

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Abstract. A problem is defined to be *exactly solvable* if its solution can be obtained by solving a sequence of polynomials using radicals. Therefore, if a problem is not exactly solvable, then we have to use approximation schemes for solving the problem. It has been proved that the shortest network problem in space is not exactly solvable even if the network spans only four points and even if the topology is known. On the other hand, if the network spans three points, then obviously the problem is exactly solvable. In a previous paper we have shown that the shortest network problem for three points is still exactly solvable if only one point is constrained on a straight line but it becomes non-exactly solvable if two points are constrained on straight lines. In this paper we continue to reduce the gap between the exact solvability and non-solvability by studying the shortest networks with gradient constraints. The motivation of this study also comes from a practical network problem: designing an underground mining network so that the ore in two underground deposits can be extracted through tunnels either directly to a vertical

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shaft and then hauled up to the ground, or extracted to a tunnel in an existing underground mining network and then transported to the ground. For technical reasons the gradient of any tunnel cannot be very steep. We prove that in the former case the shortest network problem is exactly solvable, while in the latter case the exact solvability depends on edge gradients. Moreover, we show that there are good iterative approximations for the non-exactly solvable shortest network problems in space.

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1. Motivation

Given a set of n points in Euclidean 3D-space, the shortest network problem, also called the Steiner tree problem in the literature, asks for a minimum length network T interconnecting the given points, possibly with some additional points to shorten the network [4]. The given points are referred to as *terminals*, the additional points are referred to as *Steiner points*, and T is called a *Steiner minimal tree*. A basic property of Steiner minimal trees is the *angle condition*: Any angle α in a Steiner minimal tree T is at least $120°$, and $\alpha = 120°$ if α is an angle at a Steiner point. The graph structure of a Steiner minimal tree is called a *Steiner topology*. A tree with a Steiner topology is called a *Steiner tree*. The Steiner tree problem is proved to be NP-hard [4]. A main reason for this proposition is that the number of possible Steiner topologies is exponential in n. Therefore, for a Steiner tree problem with large n we have to use some heuristics to get an approximate solution. However, there is another reason for the necessity of approximation schemes, i.e. the non-existence of an exact solution to the Steiner tree problem in space. A problem is defined to be *exactly solvable* if its solution can be obtained by solving a sequence of polynomials using radicals although finding the solution may take exponential time in the size of input data.

Now we investigate the Steiner tree problem from this point of view. As is well known, when all n terminals lie in a plane and a Steiner topology is given, the Steiner tree can be constructed in a time which is linear in n either by using Melzak's algorithm improved by Hwang [4] or by using the hexagonal coordinate method [11, 5]. Therefore the Steiner tree problem in the Euclidean plane is exactly solvable although the time complexity is exponential in n. However, the situation is quite different when the terminals lie in 3D-space. If there are only 3 terminals, then the Steiner tree problem has an exact solution since the 3 terminals lie in a plane. On the other hand, it has been proved that the Steiner tree problem for 4 points in 3D-space, the simplest non-planar Steiner tree problem in space, does not have an exact solution [9, 8, 6]. Hence an interesting problem is if we can reduce the gap between the exactly solvable and non-exactly solvable Steiner tree problems in 3D-space. Since the Steiner tree problem with constraints is generally more complicated than the one without constraints, we expect that some Steiner tree problems for 3 points with constraints are still exactly solvable, and some are not. In a previous paper [1] we have shown that the Steiner tree problem for 3 points with one point being constrained on a straight line, referred to as *two-point-and-one-line Steiner tree problem* (or *2P1L Steiner tree problem* for short) is exactly solvable. In fact, in this case the length of the Steiner minimal tree can be computed

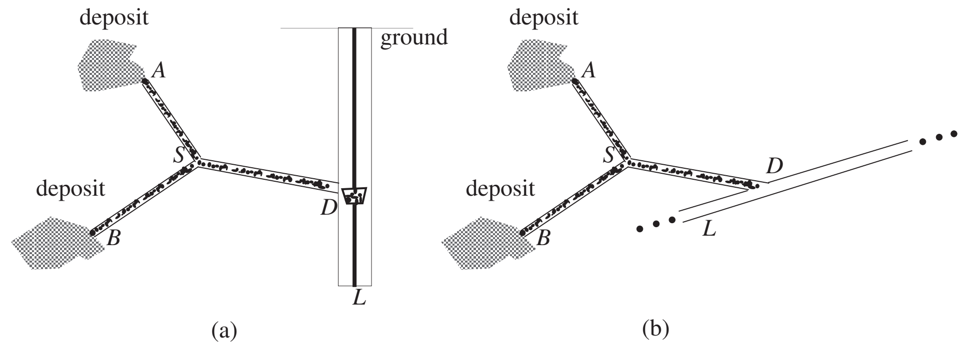


Figure 1. A mining network.

by solving a quartic equation. In this paper we continue our study in this line. We investigate two kinds of constraints:

1. some terminals lie on straight lines, and
2. the gradients of all edges have an upper bound m.

Figure 1 is a simple example showing the two constraints. In this example tunnels are to be designed so that the ore in two underground deposits can be extracted through tunnels either directly to a vertical shaft L and then hauled up to the ground (Fig. 1(a)), or extracted to a tunnel L in an existing underground mining network and then transported to the ground (Fig. 1(b)). In these figures points A and B are the prescribed access points in deposits. In practice, the gradient of any tunnel cannot be very steep. Let m be the maximal allowed gradient and let $g(e)$ be the gradient of an edge e in the underground mining network. (Typically $m \leq 1/7$). Then we need to find both the junction S and the access point D on L so that the total length of the network is minimized and that the maximal gradient constraint is satisfied, i.e. $g(SA) \leq m$, $g(SB) \leq m$ and $g(SD) \leq m$. In the case of Figure 1(b), the gradient $g(L)$ of L is no more than m since it is an existing tunnel.

This paper is organized as follows. Section 2 is an auxiliary section. To make the paper self-contained, in Section 2 we review the problem of the shortest/longest distance from a point or a straight line to an ellipse. In Section 3, using a result obtained in Section 2 we give a new proof to the 2P1L Steiner tree problem. In Section 4 we show that the *1P2L Steiner tree problem*, i.e. the problem for constructing the shortest network connecting one point and two straight lines, is not exactly solvable. The main aspect of this paper is laid on Section 5, in which, again using the results in Section 2, we study the gradient-constrained 2P1L Steiner tree problem. We show that in the case where L is a vertical shaft as shown in Figure 1(a), the problem is exactly solvable. However, if L is an existing tunnel as shown in Figure 1(b), then depending on the gradients of edges, in some cases the problem is exactly solvable but in some cases it is not. In the last section we show that the arguments for the nonexistence of exact solutions given in Sections 4 and 5, can be used to construct iterative approximation schemes for these problems. Table 1 summarizes all known exactly solvable and non-exactly solvable Steiner tree problems in space.

Steiner minimal trees for	exactly solvable
3 points	yes
2 points and 1 line	yes
2 points and 1 vertical line, gradient-constrained (Fig. 1(a))	yes
2 points and 1 non-vertical line gradient-constrained (Fig. 1(b))	depending on edge gradients
1 point and 2 lines	no
4 points	no

Table 1. A classification of exactly solvable/unsolvable cases

2. The shortest/longest distance to an ellipse

Let $\bar{R}$ be an ellipse in a horizontal plane whose canonical form is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{1}$$

We investigate the shortest/longest distance from a point D or a straight line L in space to $\bar{R}$.

(1) From a point D to $\bar{R}$.

Let D be a point in space whose projection to the plane is $\bar{D} = (u, v)$. Suppose $P = (x, y)$ is the point on $\bar{R}$ so that $DP \perp \bar{R}$. Clearly it is equivalent to $\bar{D}P \perp \bar{R}$. There is an astroid $\bar{A}$, with equation

$$(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}.$$

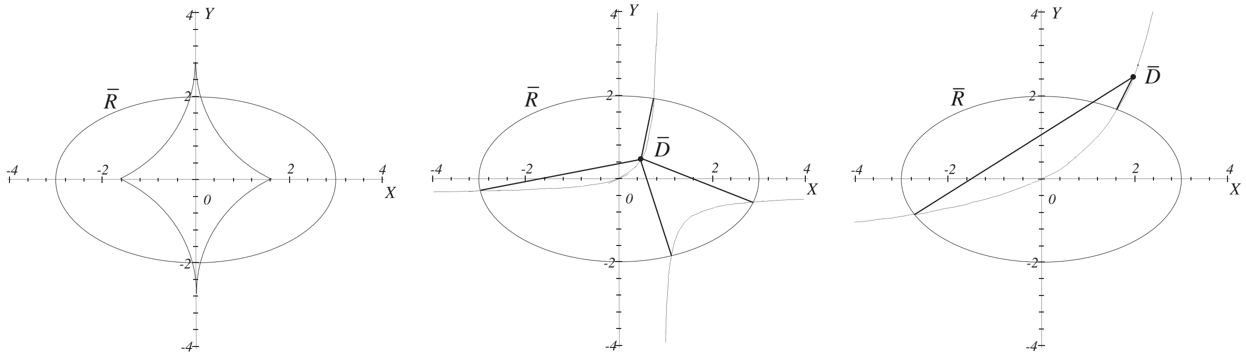
It is not difficult to see that $\bar{A}$ has the property that if $\bar{D}$ lies inside $\bar{A}$, then there are four points P on $\bar{R}$ so that $\bar{D}P \perp \bar{R}$. On the other hand, if $\bar{D}$ lies outside $\bar{A}$, then there are two points P on $\bar{R}$ so that $\bar{D}P \perp \bar{R}$ [3] (Fig. 2). In particular, if $\bar{D}$ lies outside the ellipse, then the two normals represent the shortest and the longest distance from $\bar{D}$ to $\bar{R}$. It can be shown that P is the intersection of the ellipse with the hyperbolic [10]

$$(y - v)\frac{x}{a^2} = (x - u)\frac{y}{b^2}. \tag{2}$$

That is, P is the solution of the set of equations (1) and (2), whose total degree is four. Incidentally, it can be noted that the astroid passes through the point $(0, (a^2 - b^2)/b)$. Consequently if $a^2 > 2b^2$, or equivalently, if the eccentricity of the ellipse exceeds $1/\sqrt{2}$, then part of the astroid lies outside the ellipse.

(2) From a line L to $\bar{R}$.

If L is vertical, then it becomes the case studied in (1). Hence we assume L is not vertical. Without loss of generality, assume L intersects the plane at point $P = (u, v, 0)$, and assume the direction of L is determined by (i, j, k), where $i^2 + j^2 + k^2 = 1, k \neq \infty$. Suppose the



(a) (b) (c)

Figure 2. The normals from a point to an ellipse.

shortest/longest line joining L and $\bar{R}$ is SD where S lies on $\bar{R}$ and D lies on L. Then, we may assume

$$S = (x, y, 0),\ D = (u + ti,\ v + tj,\ tk).$$

By minimality $SD \perp L$ that implies $\mathbf{SD} \cdot \mathbf{L} = 0$, i.e.

$$(u + ti - x)i + (v + tj - y)j + tk^2 = 0,$$

$$t = (x - u)i + (y - v)j.$$

Hence the coordinates of the projection $\bar{D}$ of D are

$$\bar{D} = \Big(u + (x - u)i^2 + (y - v)ij,\ v + (x - u)ij + (y - v)j^2,\ 0\Big).$$

Note $SD \perp \bar{R}$ if and only if $S\bar{D} \perp \bar{R}$. As argued in (1), S is determined by the set of equation (1) and

$$\frac{(v + (x - u)ij + (y - v)j^2 - y)\, x}{a^2} = \frac{(u + (x - u)i^2 + (y - v)ij - x)\, y}{b^2}. \tag{3}$$

The total degree of the set of Equations (1) and (3) is four. In particular, if $k = 0$, i.e. if L is parallel to the horizontal plane, then the total degree of the equation set is two.

3. A new proof of the 2P1L Steiner tree problem

Suppose two distinct points A, B and a straight line L are given, with A, B not on L. Let T be the Steiner minimal tree joining A, B, and joining L at a point D. Let S be the Steiner point in T. Because the degenerate cases are easy to deal with, we discuss only the non-degenerate case, i.e. the case in which S does not coincide with A and B, and does not lie on L either. We want to show that the problem of computing T can be turned into a problem of computing a normal of an ellipse. Therefore, the 2P1L Steiner tree problem is a quartic algebraic equation problem that is exactly solvable.

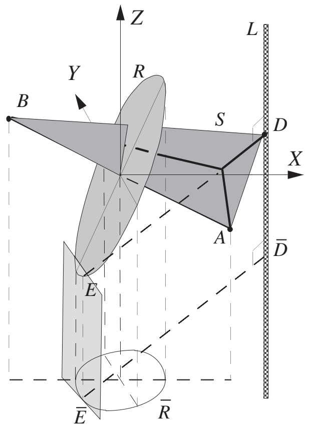


Figure 3. The Steiner minimal tree for two points and a vertical line.

Without loss of generality we may assume that after a transformation of coordinates, $A = (c, 0, h), B = (-c, 0, -h)$ with $c > 0$, $h > 0$, and L is vertical. From now on the projection of an object X to a horizontal plane is denoted by $\bar{X}$. Thus let $\bar{D} = (\bar{x}, \bar{y})$ be the projection of D to a horizontal plane, i.e. $\bar{D}$ is the intersection of L with the horizontal plane. Let $R = R_{AB}$ be the *e-circle* of AB whose center is the midpoint of AB and whose radius is $\sqrt{3}|AB|/2$ [8]. By the Melzak construction let the extension of DS meet R at the *e-point* E [8]. By the minimality of T (Figure 3)

1. $DE \perp L$, and hence, DE is parallel to the horizontal plane,
2. $DE \perp R$, and E is the farthest point from D to R since S lies in $\triangle ABD$ and since DE intersects AB.

Clearly, the projection $\bar{R}$ of R to the horizontal plane is an ellipse whose major axis (on the Y-axis) equals the diameter of R, i.e. $\sqrt{3}\sqrt{c^2 + h^2}$ and whose minor axis (on the X-axis) equals $\sqrt{3}h$. Therefore the equation for $\bar{R}$ is

$$\frac{x^2}{h} + \frac{y^2}{\sqrt{c^2 + h^2}} = \frac{\sqrt{3}}{2}.$$

Let $\bar{E} = (x, y)$ be the projection of E. Then $\bar{D}\bar{E}$ is perpendicular to $\bar{R}$ since $DE \perp L$ and $DE \perp R_{AB}$. It follows that $\bar{D}\bar{E}$ is the longest normal from $\bar{D}$ to the ellipse $\bar{R}$. By the result in Section 2(1), $\bar{E}$ can be found by solving a quartic equation. Once $\bar{E}$ is determined, then E and S can be determined by solving quadratic equations. Note $|\bar{D}\bar{E}| = |DE| = |T|$. This proves that the 2P1L Steiner tree problem is exactly solvable.

4. The 1P2L Steiner tree problem

Suppose a point A and two lines L_P, L_Q are given. Let T be the Steiner minimal tree joining A, and joining L_P at P and joining L_Q at Q. After a transformation we assume that L_Q is parallel to the x-axis and meets the z-axis at $Q_0 = (0, 0, -h)$ and assume that L_P meets z-axis at $P_0 = (0, 0, h)$ and the angle between L_P and L_Q is θ. Let $S = (x, y, z)$ be the Steiner point in T. Then the problem is an optimization problem with a strictly convex objective function $|T| = |SA| + |SP| + |SQ|$ and two linear constraints: P lies on L_P and Q lies on L_Q. Hence, the problem has a unique solution. To prove that the problem has no exact solution in general, we have an example as follows.

Example 1P2L. An example of the 1P2L Steiner tree problem.

Let $A = (2, 1, 0)$, $\theta = 90°$, $h = 1$ and let L_Q be parallel to the x-axis. By minimality $SP \perp L_P, SQ \perp L_Q$. Hence, $P = (0, y, 1)$, $Q = (x, 0, -1)$. First we show that the Steiner point $S = (x, y, z)$ is non-degenerate, i.e. $S \neq A$ and S does not lie on L_P neither on L_Q. Let P_A be the foot of the perpendicular from A to L_P, and let Q_A be the foot of the perpendicular from A to L_Q. Then, $P_A = (0, 1, 1)$, $Q_A = (2, 0, -1)$ and it is easy to show that $\angle P_A A Q_A = 108.4°$. Therefore the Steiner point $S \neq A$ by the $120°$ angle condition. Note that Q cannot lie on the left side of Q_0 since $|SP| \geq 0$ and $\angle ASP = 120°$. Computing the Steiner point S_P in the Steiner tree joining A, Q_0 and L_P by the method described in Section 3, we find that S_P does not lie on L_P. It follows that S cannot lie on L_P. Similarly, S cannot lie on L_Q.

Now we compute S. Clearly,

$$e = |SA|^2 = (2-x)^2 + (1-y)^2 + z^2,$$

$$f = |SP|^2 = x^2 + (1-z)^2,$$

$$g = |SQ|^2 = y^2 + (1+z)^2,$$

and

$$l = |T| = \sqrt{e} + \sqrt{f} + \sqrt{g}.$$

Note that $f'_y = g'_x = 0$, and S minimizing $|T|$ implies $l'_x = l'_y = l'_z = 0$. Since

$$l'_z = \frac{e'_z}{2\sqrt{e}} + \frac{f'_z}{2\sqrt{f}} + \frac{g'_z}{2\sqrt{g}} = 0,$$

we have

$$\begin{aligned} l'_x &= \frac{e'_x + e'_z z'_x}{2\sqrt{e}} + \frac{f'_x + f'_z z'_x}{2\sqrt{f}} + \frac{g'_x + g'_z z'_x}{2\sqrt{g}} \\ &= \left(\frac{e'_x}{2\sqrt{e}} + \frac{f'_x}{2\sqrt{f}}\right) + \left(\frac{e'_z}{2\sqrt{e}} + \frac{f'_z}{2\sqrt{f}} + \frac{g'_z}{2\sqrt{g}}\right) z'_x \\ &= \frac{e'_x}{2\sqrt{e}} + \frac{f'_x}{2\sqrt{f}} = 0. \end{aligned}$$

That is,

$$(e'_x)^2 f - (f'_x)^2 e = (16(1-x)z^2 - 8(x-2)^2 z + 4(4 - 4x - x^2y^2 + 2x^2y) = 0. \tag{4}$$

Similarly, from $l'_y = 0$ we obtain

$$(e'_y)^2 g - f(g'_y)^2 e = (4(1-2y)z^2 + 8(y-1)^2 z + 4(1 - 2y - 3y^2 + 4xy^2 - x^2y^2) = 0. \tag{5}$$

Because S lies on the plane determined by $\triangle APQ$, it is not hard to derive that

$$z = \frac{x - 2y}{xy - 2y - x}. \tag{6}$$

Replacing z by equation (6), equations (4) and (5) become

$$\frac{4x^2(y-1)}{(x+2y-xy)^2}F_x = 0, \qquad \frac{4y^2(x-2)}{(x+2y-xy)^2}F_y = 0,$$

where

$$F_x \stackrel{\text{def}}{=} c_2x^2 + c_1x + c_0,$$
$$F_y \stackrel{\text{def}}{=} d_3x^3 + d_2x^2 + d_1x^1 + d_0,$$
$$c_2 = (y^3 - 3y^2 + 2y + 2),\ c_1 = -4(y^3 - 2y^2 + 4)x,\ c_0 = 4(y^3 - y^2 + 14),$$
$$d_3 = (y-1)^2,\ d_2 = -2(3y^2 - 4y + 1),\ d_1 = 11y^2 - 6y,\ d_0 = -2(y^2 + 8y - 4).$$

Because $0 < x < 2$, $0 < y < 1$, and because $x + 2y - xy > 0$ when $0 < x < 2$ and $0 < y < 1$, x and y are determined by $F_x = 0, F_y = 0$. For solving y from this system, let

$$M = \begin{bmatrix} 0 & 0 & c_2 & c_1 & c_0 \\ 0 & c_2 & c_1 & c_0 & 0 \\ c_2 & c_1 & c_0 & 0 & 0 \\ 0 & d_3 & d_2 & d_1 & d_0 \\ d_3 & d_2 & d_1 & d_0 & 0 \end{bmatrix}.$$

The determinant of M is $8(y-1)(y-2)^4 f^*$, where

$$f^* \stackrel{\text{def}}{=} 2y^8 - 14y^7 + 37y^6 - 44y^5 + 11y^4 + 8y^2 + 160 - 100.$$

Again since $0 < y < 1$, $\det(M) = 0$ implies $f^* = 0$. However, f^* is a degree 8 irreducible polynomial with a non-square discriminant and its Galois group is the symmetrical group S_8. Hence, $f^* = 0$ cannot be solved by radicals, and the 1P2L Steiner tree problem is not exactly solvable.

5. The gradient-constrained 2P1L Steiner tree problem

In this section we assume T is the gradient-constrained Steiner minimal tree joining two points A, B and a straight line L. We assume L is infinitely long otherwise the access points on L may be the endpoints of L. The latter case is not difficult to deal with and will be omitted. In the first subsection we briefly review the basic properties of gradient-constrained Steiner minimal trees [2]. In the second subsection we discuss the case where L is a vertical shaft (Fig. 1(a)), and the case where L is an existing tunnel (Fig. 1(b)) is discussed in the last subsection.

5.1. Properties of gradient-constrained Steiner trees

Suppose T is a tree in which all edges have gradients no more than m. Let x_P, y_P, z_P denote the Cartesian coordinates of a point P in Euclidean space. As stated above, the gradient of a line PQ is

$$g(PQ) = \frac{|z_P - z_Q|}{\sqrt{(x_P - x_Q)^2 + (y_P - y_Q)^2}}.$$

An edge PQ in T is called an f-edge or m-edge, or b-edge, and labeled 'f' or 'm', or 'b' if $g(PQ)$ is $< m$ or $= m$, or $> m$ respectively. Clearly, if $g(PQ) \le m$, then PQ is a straight line segment and referred to as a *straight edge*, otherwise PQ can be any shortest zigzag line in which each segment has gradient equal to m. In the latter case, the non-straight edge can be represented in a canonical form which consists of two straight line segments PR, RQ. Therefore, a non-straight edge is also referred as a *bent edge* and a point R is called a *corner point* of the bent edge PQ. From another point of view, the length of PQ can be measured in a special metric, called *gradient metric* and denoted by $|PQ|_g$. It is easy to see that

$$|PQ|_g = \begin{cases} \sqrt{(x_P - x_Q)^2 + (y_P - y_Q)^2 + (z_P - z_Q)^2}, & \text{if } g(PQ) \le m; \\ (\sqrt{1 + m^{-2}}|z_P - z_Q|, & \text{if } g(PQ) \ge m. \end{cases}$$

By the definition it is easy to see that $|PQ| \le |PQ|_g$, where $|PQ|$ is the Euclidean metric, and that the gradient metric is convex though it is not strictly convex.

Suppose S is a non-degenerate degree 3 Steiner point in T, and its adjacent points are A, B and C. Let H_S be the horizontal plane through S. Then two edges of S, say SA and SB, lie on one side of H_S and the third edge, say SC, lies on the other side of H_S [2]. Let g_A, g_B, g_C denote the respective labels of these edges. Then we say the *labeling* of S is $(g_A g_B / g_C)$. The following results have also been proved [2].

Theorem 5.1. *Up to symmetry there are five feasible optimal labellings: (ff/f), (ff/m), (fm/m), (mm/m) and (mm/b). Moreover, in the last case the two m-edges lie in a vertical plane and S can be exactly determined by the two m-edges.*

Finally, for any point P let C_P be the vertical cone whose generating lines have gradient m and whose vertex is P. Let L be a straight line not containing P. Let PD be the gradient-constrained shortest line joining P and L. Then the following lemma is trivial.

Lemma 5.2. *If L does not meet C_P, then PD is an f-edge satisfying $PD \perp L$. Otherwise assume $Q'Q''$ is the part of L lying inside C_P. Then D is the one of Q', Q'' that is near to P if L is not horizontal, or D can be any point on $Q'Q''$ if L is horizontal. In the latter case if $D = Q'$ or $D = Q''$ then PD is an m-edge, otherwise PD is a b-edge.*

5.2. Vertical L

First, if both A, B are sufficiently close to L and if more than one access point on L is permitted, then T consists of two horizontal straight lines AD_A and BD_B where D_A, D_B are the perpendicular feet on L with respect to A, B separately. Trivially, this case is exactly solvable. Hence we assume that T has only one access point, say D, on L. Since L is vertical, DS must be a horizontal f-edge.

Next suppose T is degenerate, i.e. either the Steiner point S in T collapses into A or B, or S lies on L. If S collapses into A or B, say A, then $T = AB \cup AD_A$. Clearly the problem is exactly solvable in this case. If S lies on L, then AS, BS are either both f-edges or both m-edges as a special case proved in the following lemma.

Lemma 5.3. *If S does not collapse into A and B, then AS, BS are either both f-edges or both m-edges.*

Proof. Without loss of generality assume $g(AS) \leq g(BS)$. First, since DS is an f-edge, by Theorem 5.1 neither AS nor BS is a bent edge. Next, suppose $g(AS)$ is an f-edge. Let L^* be the vertical line through S. If $g(BS) = m$, then BS is a straight edge, and if $g(BS) > m$, then BS is a bent edge with a corner point R so that $g(RS) = m$. In any case the angle between BS (or RS) and L^* is strictly less than the angle between AS and L^* since $g(AS) < g(BS)$. It follows that when S is perturbed along L^* to approach B, BS shrinks faster than AS stretches by the variational argument [7]. Note that the length of SD does not change in this perturbation. Hence, T is shortened, contradicting the minimality of T. This proves that BS must be an f-edge, too. Finally, suppose $g(AS)$ is an m-edge. Then BS must be an m-edge since it is not an f-edge nor a b-edge. The proof is complete. □

Now suppose T is not degenerate. By the above lemma, T has only two possibilities. If both AS, BS are f-edges, then the problem becomes not constrained, and S can be exactly determined as described in Section 3. If both AS, BS are m-edges, we claim that S can be exactly determined, too. After a transformation we can assume that $A = (c, 0, h), B = (-c, 0, -h)$ as before. Note that $g(SA) = g(SB) = m$ and S being non-degenerate imply that $g(AB) > m$ and $h > 0$. Hence $S = (x, y, z)$ lies on the intersection $R_{AB} = C_A \cap C_B$ and satisfies

$$(x-c)^2 + y^2 = \frac{(z-h)^2}{m^2}, \ (x+c)^2 + y^2 = \frac{(z+h)^2}{m^2}. \tag{7}$$

It is easily seen [2] that the intersection R_{AB} is an ellipse lying on the plane $\tilde{P}$ determined by

$$z = m^2 \frac{c}{h} x = \frac{m^2}{k} x, \tag{8}$$

where $k = h/c = g(AB)$. The projection $\bar{R}_{AB}$ of R_{AB} on the horizontal plane (Fig. 4) is also an ellipse determined by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \ a = \frac{h^2}{m^2}, \ b = \frac{(k^2 - m^2)u^2}{m^2}.$$

Because $SD \perp R_{AB}$ and $SD \perp L$ by minimality, and because L is vertical, we conclude that SD is horizontal and $\bar{S}\bar{D}$ is a normal from the point $\bar{D}$ to the ellipse $\bar{R}_{AB}$. As proved in Section 2(1), $\bar{S}$ can be found by solving a quartic equation. This proves that the case of $g(AS) = g(BS) = m$ is also exactly solvable.

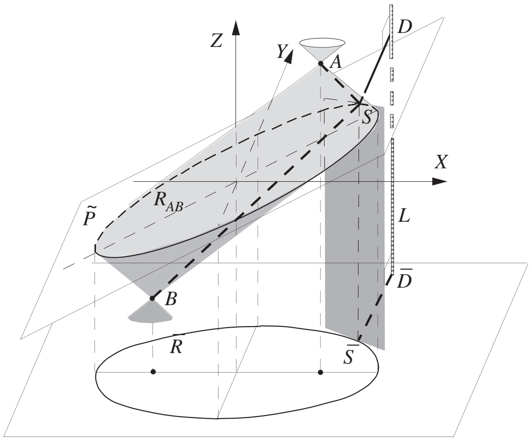


Figure 4. The gradient-constrained Steiner minimal tree for two points and a vertical line.

case	1	2	3	4	5	6	7	8
g(AS)=	f	f	f	f	m	m	m	m
g(BS)=	f	f	m	m	m	m	m	b
g(DS)=	f	m	f	m	f	m	b	m

Table 2. 8 possible labellings for non-vertical L

5.3. Non-vertical L

Now we investigate the case where L is an existing tunnel but the gradient of L is not necessarily restricted, i.e. $g(L)$ is arbitrary. We omit the two access point case and all degenerate cases since they are similar to, though a little more complicated than, the ones when L is a vertical line. Below, we assume that S is non-degenerate in the Steiner minimal tree T and assume that the access point on L is D. The type of T can be classified according to its labelling. Without loss of generality assume $G(AS) \leq g(BS)$. Note that A, B are fixed terminals but D is only constrained on L. Because of this non-symmetry between AS, BS and DS, up to symmetry there are 8 different labellings of T as shown in Table 2.

We first show 4 cases that are easily seen to be exactly solvable:

Case 1: All three edges are f-edges. In this case the problem is a 2P1L problem without gradient-constraint and is exactly solvable as proved in Section 3.

Case 5: Both AS, BS are m-edges and DS is an f-edge (Fig. 5(a)). As argued in Subsection 5.2, S lies on the ellipse R_{AB} in the plane $\tilde{P}$. After a transformation we may assume $\tilde{P}$ is the xy-plane in the new coordinate system. Then the problem becomes that of finding the shortest edge joining an ellipse on the plane and a non-vertical straight line L. Hence, by Section 2(2), the problem is exactly solvable.

Case 7: DS is a b-edge (Fig 5(b)). Then, by Theorem 5.1 both AS and BS are m-edges lying on a vertical plane. Therefore, S can be determined by this condition and the conditions $g(AS) = g(BS) = m$.

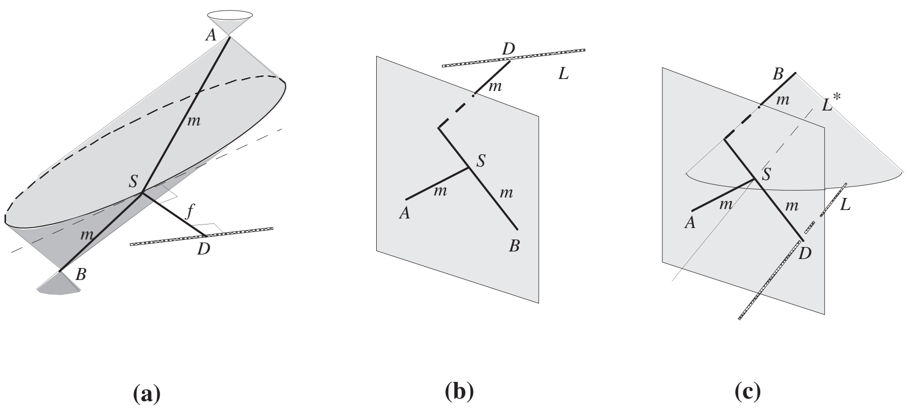


Figure 5. Exactly solvable gradient-constrained Steiner minimal trees for two points and a non-vertical line.

Case 8: One of AS or BS, say BS, is a b-edge (Fig. 5(c)). Then, again by Theorem 5.1 AS and DS are m-edges lying on a vertical plane. Let L^* be the horizontal line, through S and perpendicular to this plane. When S is perturbed along L^*, by the definition of gradient metric, the variation of $|AS|_g$ and $|BS|_g$ is zero. Hence, by minimality of T, the variation of $|DS|_g$ must be zero, too. Note that the move of S induces a move of D along L to minimize $|T|$. Hence, $SD \perp L^*$ and L/L^*. It follows that L is horizontal, and it is perpendicular to the vertical plane. This condition determines D. Then, similar to the above case S is determined by $g(BS) = g(DS) = m$.

In general, the gradient-constrained 2P1L problem is not exactly solvable. To prove this claim we need only to show that one of Cases 2,3,4 and 6 is not exactly solvable. Below is an example of Case 2 in which AS, BS are f-edges and DS is an m-edge.

Example gc2P1L. An example of gradient-constrained 2P1L Steiner tree problem.

Let $m = 1, A = (2, 5, -1)$, $B = (2, 0, 0)$, L is parallel to the y-axis and through the point $(0, 2, 0)$. Let S_0 be the Steiner point joining A, B and L without gradient constraint. Then we can compute S_0 as described in Section 3 and find $g(S_0D) > m,\ g(S_0A) < m,\ g(S_0B) < m$. It suggests that $S = (x, y, z)$ should satisfy $g(SD) = m,\ g(SA) < m,\ g(SB) < m$. In the next section we will show that a numerical calculation proves these equations and inequalities. For this labelling, the problem can be regarded as an optimization problem in the Euclidean metric with a strictly convex objective function $|T| = |SA| + |SP| + |SQ|$ and a linear constraint that S lies in the plane through L with slope m. Hence, the problem has a unique solution.

Since L is horizontal, it follows by minimality that $SD \perp L$ and $D = (0, y, 2)$. Since $g(SD) = m = 1$, we have

$$z = 2 - x. \tag{9}$$

Let

$$l = |T| = \sqrt{e} + \sqrt{f} + \sqrt{g},$$

where

$$f = |SA| = (2 - x)^2 + (5 - y)^2 + (1 + z)^2 = (2 - x)^2 + (5 - y)^2 + (3 - x)^2,$$

$$g = |SB| = (2-x)^2 + y^2 + z^2 = 2(2-x)^2 + y^2,$$
$$e = |SD| = x\sqrt{1+\frac{1}{m^2}}.$$

Then

$$l'_y = \frac{f'_y}{2\sqrt{f}} + \frac{g'_y}{2\sqrt{g}},$$
$$\sqrt{f} = \frac{-f'_y\sqrt{g}}{g'_y}. \tag{10}$$

Note f'_y, g'_y are linear in x and y. Hence, by equation (10) $l'_y = 0$ is equivalent to

$$F_y \stackrel{\text{def}}{=} g(f'_y)^2 - f(g'_y)^2 = c_2y^2 + c_1y + c_0 = 0, \tag{11}$$

where

$$c_2 = 8x - 20,\ c_1 = -80(x-2)^2,\ c_0 = -80(x-2)^2.$$

On the other hand, again by equation (10)

$$l'_x = \sqrt{1+\frac{1}{m^2}} + \frac{f'_x}{2\sqrt{f}} + \frac{g'_x}{2\sqrt{g}} = \sqrt{1+\frac{1}{m^2}} - \frac{f'_xg'_y}{2f'_y\sqrt{g}} + \frac{g'_x}{2\sqrt{g}}.$$

Hence $l'_x = 0$ is equivalent to

$$\begin{aligned} F_x &\stackrel{\text{def}}{=} \frac{g}{4}\left(1+\frac{1}{m^2}\right)(f'_y)^2 - \frac{1}{16}(f'_yg'_x - f'_xg'_y)^2 \\ &= d_4y^4 + d_3y^3 + d_2y^2 + d_1y + d_0 = 0, \end{aligned} \tag{12}$$

where

$$d_4 = 2,\ d_3 = -20,\ d_2 = 4x^2 - 16x + 65,\ d_1 = -20(x-2)(2x-5),\ d_0 = 0.$$

For solving x from the system $F_x = F_y = 0$, let

$$M = \begin{bmatrix} 0 & 0 & 0 & c_2 & c_1 & c_0 \\ 0 & 0 & c_2 & c_1 & c_0 & 0 \\ 0 & c_2 & c_1 & c_0 & 0 & 0 \\ c_2 & c_1 & c_0 & 0 & 0 & 0 \\ 0 & d_4 & d_3 & d_2 & d_1 & d_0 \\ d_4 & d_3 & d_2 & d_1 & d_0 & 0 \end{bmatrix}.$$

The determinant of M is $640000(x-2)^4f^*$, where

$$\begin{aligned} f^* \stackrel{\text{def}}{=}\ & 9792x^6 - 142080x^5 + 852864x^4 - 2706640x^3 \\ & +4775500x^2 - 4419300x + 1661375. \end{aligned} \tag{13}$$

Since $0 < x < 2$, $\det(M) = 0$ implies $f^* = 0$. However, f^* is a degree 6 irreducible polynomial with a non-square discriminant and its Galois group is the symmetrical group S_6. Hence, $f^* = 0$ cannot be solved by radicals. This proves that this example has no exact solution.

Remark 5.1. In the example we choose $m = 1$ only for simplicity. We can construct a non-exactly solvable example for m equal to $1/7$ or other values for practical mining networks but such an m leads to an irreducible polynomial of degree much higher than 6.

6. Iterative Approximations

Now we show that there exist good iterative approximation schemes for the non-exactly solvable Steiner problems discussed in Sections 4 and 5.

(1) In Example 1P2L, we have shown that the coordinates x, y of the Steiner point S are determined by

$$F_x = (y^3 - 3y^2 + 2y + 2)x^2 - 4(y^3 - 2y^2 + 4)x + 4y^3 - 4y^2 + 16 = 0,$$

$$F_y = (x^3 - 6x^2 + 11x - 2)y^2 - 2(x^3 - 4x^2 + 3x + 8)y + x^3 - 2x^2 + 8 = 0.$$

Solving $F_x = 0$ with respect to x, we have

$$x = \frac{2y^3 - 4y^2 + 8 - 2\sqrt{8 - 8y - 2y^2 + 4y^3 - y^4}}{y^3 - 3y^2 + 2y + 2}, \tag{14}$$

and

$$x^*(y) = \frac{2y^3 - 4y^2 + 8 + 2\sqrt{8 - 8y - 2y^2 + 4y^3 - y^4}}{y^3 - 3y^2 + 2y + 2}.$$

However, $x^*(y)$ is an extraneous root because $x^*(y)$ achieves the minimum when $y = 1$ and $x^*(1) = 4$ is outside the domain $0 < x < 2$. Similarly, solving $F_y = 0$, and ignoring the extraneous root we have

$$y = \frac{x^3 - 4x^2 + 3x + 8 - 2\sqrt{80 - 40x - 11x^2 + 8x^3 - x^4}}{x^3 - 6x^2 + 11x - 2}. \tag{15}$$

Hence we have the following iteration formulae:

$$x_i = \frac{2y_{i-1}^3 - 4y_{i-1}^2 + 8 - 2\sqrt{8 - 8y_{i-1} - 2y_{i-1}^2 + 4y_{i-1}^3 - y_{i-1}^4}}{y_{i-1}^3 - 3y_{i-1}^2 + 26 + 2},$$

$$y_i = \frac{x_{i-1}^3 - 4x_{i-1}^2 + 3x_{i-1} + 8 - 2\sqrt{80 - 40x_{i-1} - 11x_{i-1}^2 + 8x_{i-1}^3 - x_{i-1}^4}}{x_{i-1}^3 - 6x_{i-1}^2 + 11x_{i-1} - 2}.$$

Let $x_0 = 0$, then the sequence (x_i, y_i) converges to the solution of the example since the solution is unique as we have proved. In fact, after 10 iterations we obtain the solution

$$x = 1.467369, \ y = 0.610682,$$

with error less than 10^{-5}.

(2) Similarly in Example gc2P1L, we have shown that the coordinates x, y of the Steiner point S are determined by

$$F_y = (8x - 20)y^2 - 80(x - 2)^2 y + 200(x - 2)^2 = 0,$$

$$F_x = 2y^4 - 20y^3 + (4x^2 - 16x + 65)y^2 - 20(x - 2)(2x - 5)y = 0,$$

and the latter can be rewritten as

$$F_x = (4y^2 - 40y)x^2 + (-16y^2 + 180y)x + 2y^4 - 20y^3 + 65y^2 - 200y = 0.$$

Solving $F_x = F_y = 0$, and ignoring the extraneous roots we obtain the following iteration formulae:

$$x_i = \frac{8y_{i-1} - 90 + 2\sqrt{25 + 490y_{i-1} - 249y_{i-1}^2 + 40y_{i-1}^3 - 2y_{i-1}^4}}{4y_{i-1} - 40},$$

$$y_i = \frac{(x_{i-1} - 2)(10x_{i-1} - 20 + 5\sqrt{4x_{i-1}^2 - 20x_{i-1} + 26})}{2x_{i-1} - 5}.$$

Let $x_0 = 0$, then the sequence (x_i, y_i) converges to the solution of the example since the solution is unique as we have proved. In fact, after 5 iterations we obtain the solution with accuracy to ten digits

$$x = 1.227023368, \; y = 1.805490566.$$

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